Analytical Approaches for Rapid Prediction of Gravitational Waveforms for Relativistic Binary Systems

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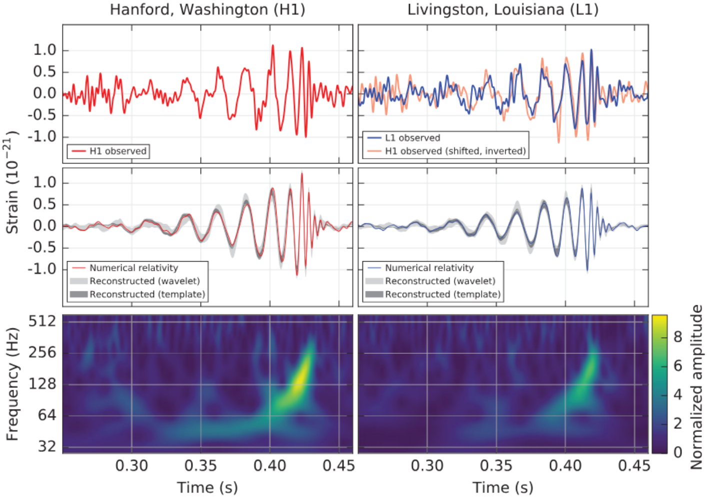
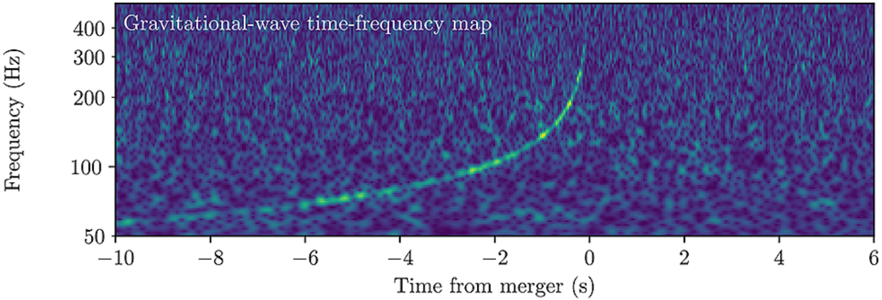
Gravitational wave signal from merging black holes (2015)



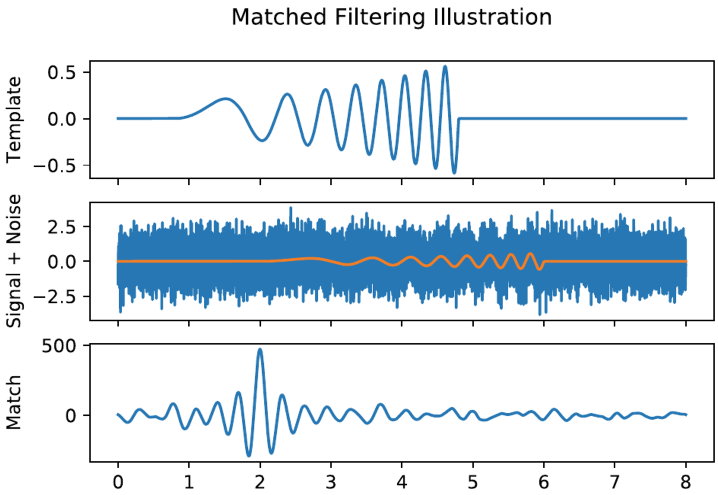
ref.: Caltech/MIT/LIGO Lab

Gravitational wave signal from merging neutron stars (2017)



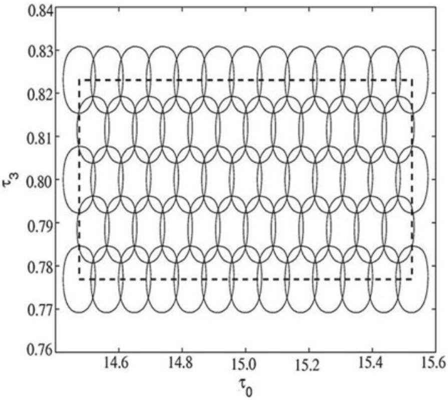
ref.: Caltech/MIT/LIGO Lab

Matched filtering illustration



ref.: Dhurandhar, 2022

Problems of matched filtering



An hexagonal packing of templates for the nonspinning binary inspiral parameter space. The parameters are the Newtonian and post-Newtonian chirp times τ_0 и τ_3.
ref.: Sengupta et al., 2002

- The templates must span the parameter space with the given minimal match, that leaves no “holes” in the parameter space;
- on the other hand, one must be able to achieve this with a minimum number of templates in order to minimise the computational cost;
- the overlaps among the neighbouring templates must be minimised

Relevance of the Work

- Matched filtering method:
  - uses the data from interferometers (LIGO, VIRGO, KAGRA);
  - uses a bank of templates to register merging objects.
- There is a necessity for faster detection methods:
  - including detection at the earliest stages

Objectives

- Obtaining a fully analytical solution for the gravitational wave shape;
    - work of Buskirk D., Babiuc-Hamilton M. C. arXiv:1609.05933.
- Comparison with the numerical solution:
    - making a table of values of the divergence between the analytical and the numerical methods for different masses of components;
    - determination of the limits of applicability of the obtained formula.
- Obtaining an analytical formula with the masses of the components as the argument

PAPER

A complete analytic gravitational wave model for undergraduates

Dillon Buskirk[2,1] and Maria C Babiuc Hamilton[3,1]

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DOI 10.1088/1361-6404/aaf81e

```
In[∘]:= (*Numerically solving a differential equation to find PN parameter x*)
```

Setting up the final integration time

```
In[∘]:= solx = NDSolve[{x'[t] - Expr6PN
            == 0, x[0] == xlow}, x, {t, 0, temp}]
```

... NDSolve: At t == 11.924777117026338`, step size is effectively zero; singularity or stiff system suspected.

Main Definitions

Post-Newtonian Parameter

$$x = \left(\frac{v}{c}\right)^2$$

v — the orbital velocity,
c — the speed of light

Its lower boundary

$$x_{\text{low}} = \left(\frac{G_N M \pi f_{\text{GW}}^{\text{low}}}{c^3}\right)^{2/3}$$

G_N — the gravitational constant,
M — the mass of the system,
$f_{\text{GW}}^{\text{low}}$ — the lower boundary of frequency, ~ 10 Hz for LIGO

Main definitions

The Gravitational Field in Vacuum:

$$g_{ik} = g_{ik}^{(0)} + h_{ik}$$

$g_{ik}^{(0)}$ — the metric tensor,
h_{ik} — the weak perturbation of the metric.

Gravitational Wave Equation:

$$\Box h_{ik} = 0$$

Main definitions

Gravitational Waveform:

$$h(t) = A(t)e^{-i2\Phi(t)}$$

$$A(t) = A_1 + iA_2$$

$$A_1 = -2\frac{M\eta}{R}\left(\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\right), \quad A_2 = -2\frac{M\eta}{R}\left(2r\dot{r}\dot{\Phi}\right)$$

A — the amplitude of gravitational wave,
$M = M_1 + M_2$ — the total mass of the system,
$\mu = M_1 M_2 / M$ — the reduced mass of the system,
$\eta = \mu / M$ — the symmetric mass ratio,
r — the orbital separation between the centers of the stars,
Φ — the phase,
R — the distance from the detector to the binary.

Main definitions

$$r(t) = M(r^{0\,\mathrm{PN}} x(t)^{-1} + r^{1\,\mathrm{PN}} + r^{2\,\mathrm{PN}} x(t) + r^{3\,\mathrm{PN}} x(t)^2)$$

$$\dot{\Phi}(x(t)) = \omega(x(t)) = \frac{x(t)^{\frac{3}{2}}}{M}$$

x — PN-parameter,
ω — the orbital angular velocity,
$\mu = M_1 M_2 / M$ — the reduced mass of the system,
$r^{i\,\mathrm{PN}}$ — Post-Newtonian coefficients from Hinder, 2010, arXiv:0806.1037.

Main definitions

Radiation energy of gravitational waves, $\mathcal{F}$ — loss power (Landau, Lifshitz, t. 2, Field Theory, §110):

$$\frac{dE}{dt} = -\mathcal{F}$$

In terms of x:

$$\frac{dx}{dt} = -\frac{\mathcal{F}}{dE/dx} = A(x) = a_5 x^5 + a_6 x^6 + \ldots + a_{11} x^{11}$$

Approximants from Ajith P. et. al, 2014, arXiv:1210.6666

$$I_A(x) \equiv \int_{x_{\rm low}}^{x} \frac{d\zeta}{A(\zeta)}, \qquad I_A(x) = t + I_A(x_{\rm low})$$

$$A(x) = a_5 x^5 + a_6 x^6 + \ldots + a_{11} x^{11}$$

Let us rewrite the expression as

$$I_A(x) = \int_{x_{\mathrm{low}}}^{x} \frac{\mathrm{d}\zeta}{a\zeta^5 \left(1 + \frac{b(\zeta)}{a\zeta^5}\right)}$$

Noting that $\zeta \ll 1$, let us expand the following expression

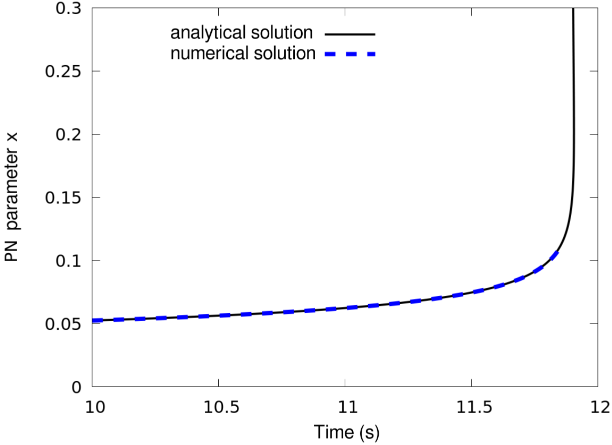
$$\frac{1}{1 + \frac{b(\zeta)}{a\zeta^5}}$$

in a Taylor series up to the sixth order:

$$I_A(x) = \int_{x_{\mathrm{low}}}^{x} \frac{b_T(\zeta)\mathrm{d}\zeta}{a\zeta^5}.$$

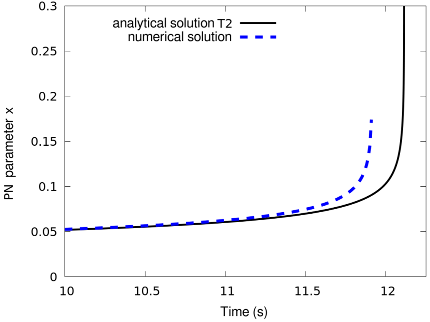
Comparison of analytical and numerical solutions

Taylor Expansion (6th order)

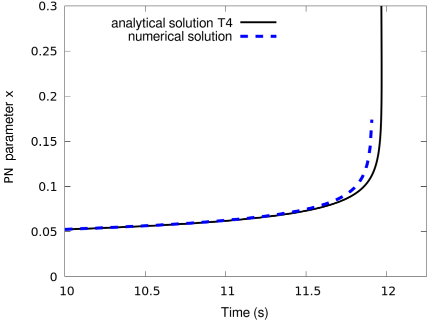


The final stages of inspiral phase for the system BH + BH (20 + 20 $M_{\odot}$)

Comparison of analytical and numerical solutions
Taylor Expansion (2nd and 4th order)



(a) Taylor Expansion (2nd order)



(b) Taylor Expansion (4th order)

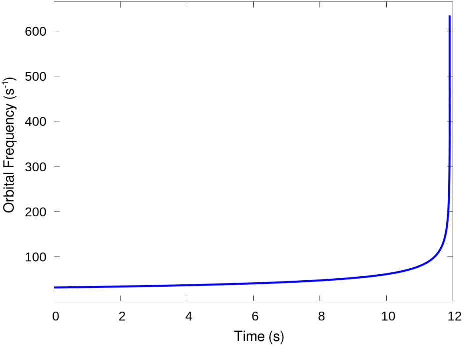
Comparison of analytical and numerical solutions

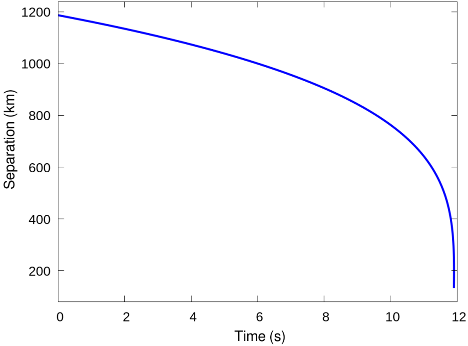
Values for different initial masses (in solar masses)

Systems BH+BH/NS					
$M_1 + M_2$	x_{low}	t_{cr}	x; t		
			$> 0.001\%$	$> 0.01\%$	$> 0.1\%$
$20 + 20$	0.0337	11.856	0.048; 9.081	0.094; 11.77	—
$15 + 1.5$	0.01868	158.76	0.066; 158.0	—	—
$10 + 10$	0.0212	38.511	0.079; 38.38	—	—
Systems NS+NS					
$M_1 + M_2$	x_{low}	t_{cr}	x; t		
			$> 10^{-5}\ \%$	$> 10^{-4}\ \%$	$> 0.001\%$
$2 + 2$	0.0073	561.884	0.031; 559.5	0.063; 561.80	—
$1.5 + 1.5$	0.0060	905.966	0.027; 903.4	0.062; 905.90	—
$1 + 1$	0.0046	1776.52	0.043; 1776.0	—	—

Evolution of the orbital frequency with time

At the inspiral phase for BH+BH system (20 + 20 $M_\odot$)



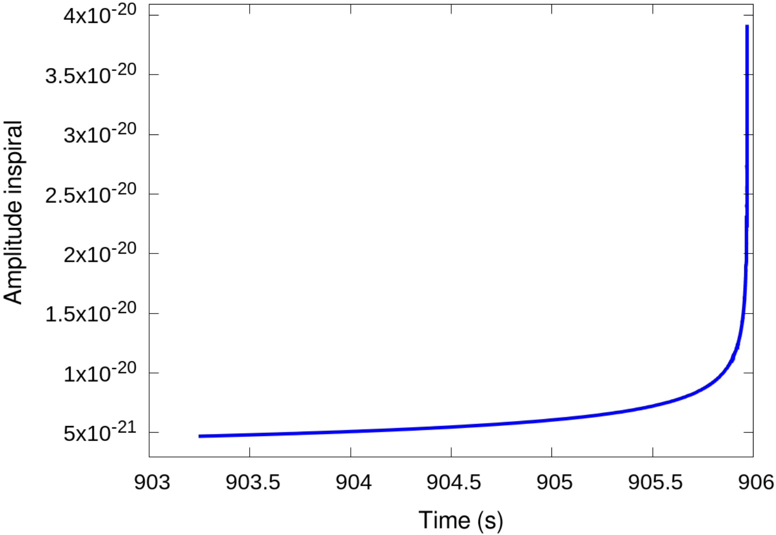


Kepler's Law, revised by Newton:

$$\omega^2 = \frac{G_N(M_1 + M_2)}{r^3}$$

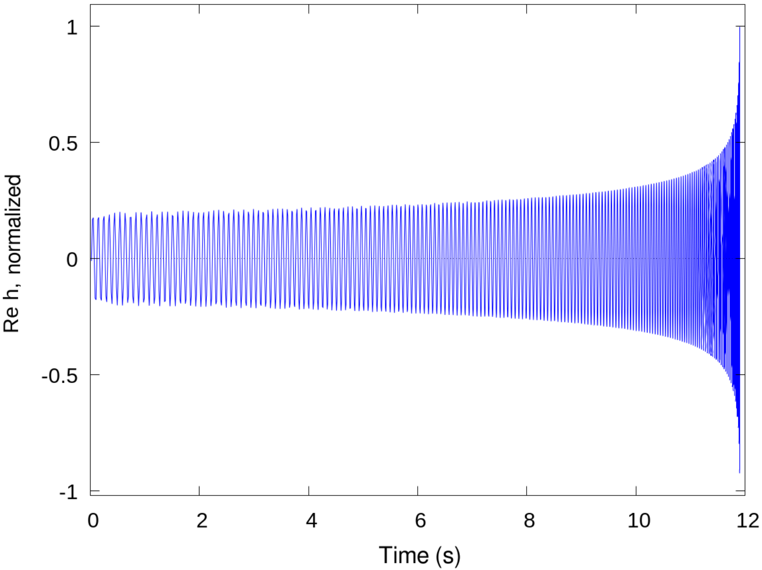
Evolution of the amplitude with time

For the system of NS + NS (1.5 + 1.5 $M_\odot$)



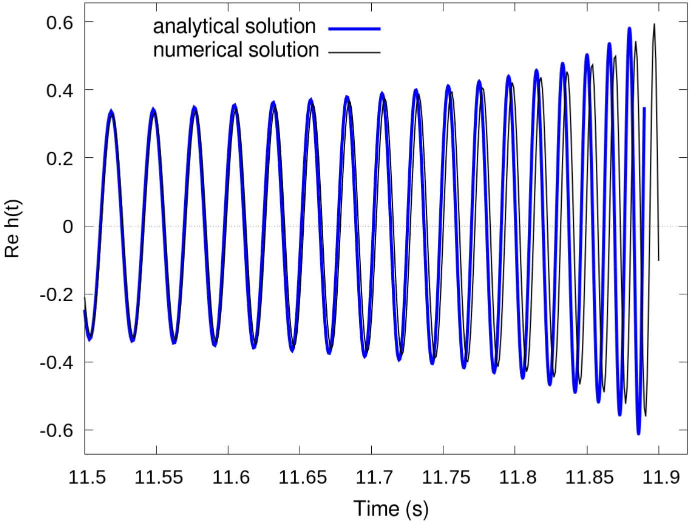
Analytical Template

For the system BH + BH (20 + 20 $M_\odot$)



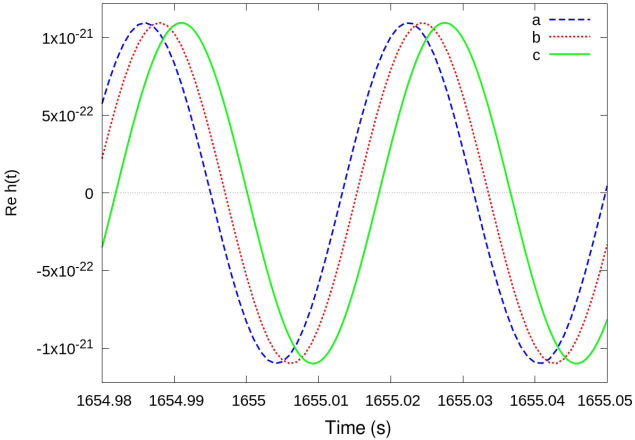
Analytical Template

Comparison of analytical and numerical solutions during the final stages of approach for the system BH + BH (20 + 20 $M_\odot$)



Analytical Template

Comparison of analytical and numerical solutions during the final stages of approach for the system NS + NS (1 + 1 $M_\odot$)



a — numerical solution (the accuracy is set by default in Mathematica), b — numerical solution (accuracy 8), c — analytical solution

Results

- A fully analytical formula for a gravitational wave shape (template) has been obtained;
  - the analytical formula will make the search for the signals faster;
  - it is possible to detect merging objects at the earliest stages of merging.
- For different masses of components, the limits of applicability of the obtained formula have been determined;
  - high accuracy for BH+BH/NS and NS+NS systems.

The final formulae are listed on GitHub. Link to the repository:

https://github.com/sblinnikov/gw-forms-analytics

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