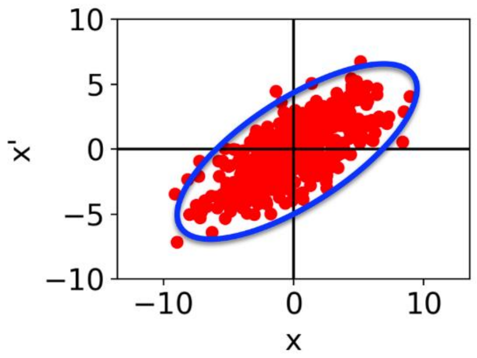
29th International Scientific Conference of Young Scientists and Specialists (AYSS-2025)
Dubna, 27–31 Oct 2025

Emittance Growth in Systems with Electron Cooling: The Impact of Optics Mismatch and X-Y Coupling

Korobitsina M, Lebedev V.A, Smirnov V.L



Emittance (ε) is the phase space volume occupied by the beam in coordinate-momentum space (x, x'). $\varepsilon\_n = \gamma \beta \varepsilon$ **ε_n is invariant of motion**

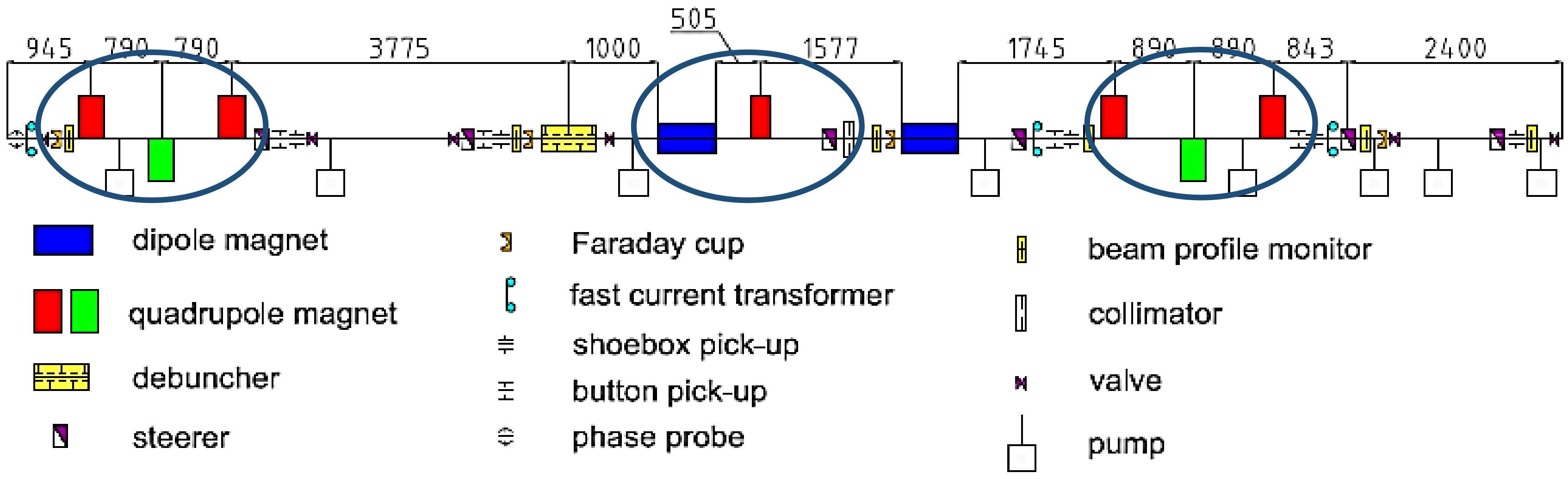
Main factors causing emittance growth during injection:

○ Optics Mismatch - non ideal matching of the beta functions (β) and alpha functions (α) between the transport line and the storage ring at the injection point.

○ Coupling - correlations between the horizontal and vertical planes of particle motion which introduced when the beam passes through the solenoids of the electron cooling system.

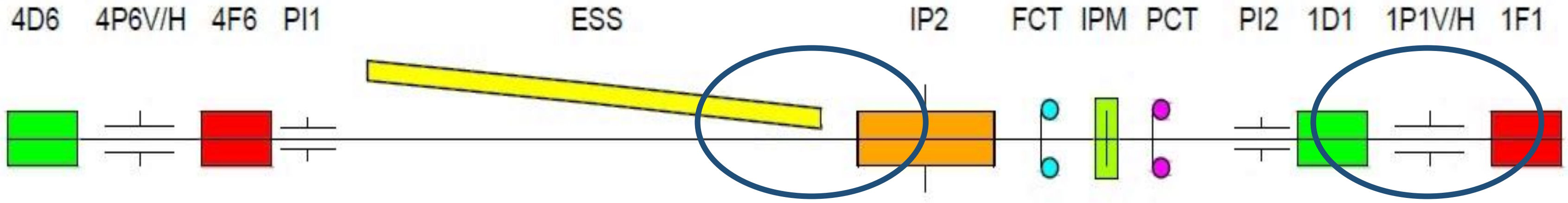
The main goal:

To investigate quantitatively the contribution of optical mismatch and transverse coupling introduced by the electron cooling system to the beam emittance growth during injection into the booster and to determine the permissible error levels.



Channel

- 7 quadrupole magnets Q1-Q7

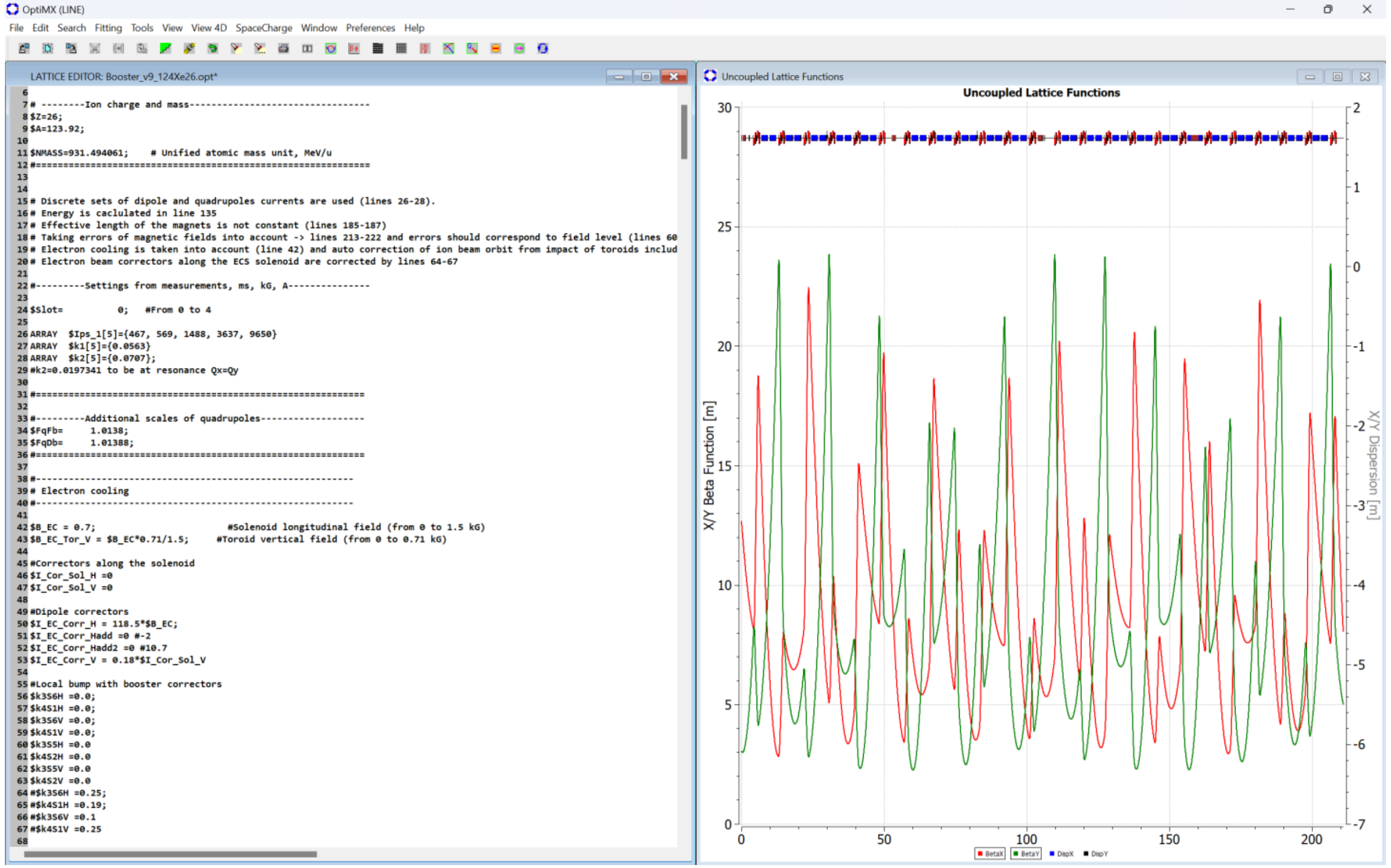


Booster

- electrostatic septum – ESS
- beam diagnostic devices (1P1V / H)

Model:

- arrangement of elements
- setting the beam parameters at the energy of injection
- tuning of the magnet-optical structure
- tuning of the operating point according to the given forces of the magnets
- control of power sources of magnet-optical elements
- solenoid of the electron cooling system
- assignment of currents in the windings of magnets



OptimX
© 2014-2024 FRA/Fermilab
Original author: val@fnal.gov

$\mathbf{x} = [x, p_x, y, p_y]^{\mathsf{T}}$ - state vector of a particle

x - the horizontal displacement from an ideal orbit, p_x is the canonical momentum in the horizontal plane

$H(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{H}\mathbf{x}$ - Hamiltonian in matrix form for 2D motion $\quad \frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i}, \quad \frac{dx_i}{ds} = \frac{\partial H}{\partial p_i}$

$\frac{d\mathbf{x}}{ds} = \mathbf{U}\mathbf{H}\mathbf{x}$ is the identity symplectic matrix: $\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

For the transition matrix M(0,s), which describes the evolution of the particle state vector from point 0 to point s $\mathbf{x} = \mathbf{M}(0, s)\mathbf{x}_0$ the symplecticity condition has the form: $\mathbf{M}^{\mathsf{T}}\mathbf{U}\mathbf{M} = \mathbf{U}$

Symplecticity ensures that our mathematical description of motion accurately preserves phase volume and energy without introducing artificial losses

For a cyclic accelerator the transition matrix M for one revolution:

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_1\lambda_2\lambda_3\lambda_4 = 1$$

eigenvectors satisfy the conditions of symplectic orthogonality:

$$\mathbf{v}_1^{+}\mathbf{U}\mathbf{v}_1 = -2i, \quad \mathbf{v}_2^{+}\mathbf{U}\mathbf{v}_2 = -2i,$$
$$\mathbf{v}_1^{T}\mathbf{U}\mathbf{v}_1 = 0, \quad \mathbf{v}_2^{T}\mathbf{U}\mathbf{v}_2 = 0,$$
$$\mathbf{v}_2^{T}\mathbf{U}\mathbf{v}_1 = 0, \quad \mathbf{v}_2^{+}\mathbf{U}\mathbf{v}_1 = 0.$$

$$\mathbf{v}_1(s) = \begin{bmatrix} \sqrt{\beta_{1x}(s)} \\ -\dfrac{iu_1(s)+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} \\ \sqrt{\beta_{1y}(s)}e^{i\nu_1(s)} \\ -\dfrac{iu_2(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)} \end{bmatrix}, \quad \mathbf{v}_2(s) = \begin{bmatrix} \sqrt{\beta_{2x}(s)}e^{i\nu_2(s)} \\ -\dfrac{iu_3(s)+\alpha_{2x}(s)}{\sqrt{\beta_{2x}(s)}}e^{i\nu_2(s)} \\ \sqrt{\beta_{2y}(s)} \\ -\dfrac{iu_4(s)+\alpha_{2y}(s)}{\sqrt{\beta_{2y}(s)}} \end{bmatrix}$$

μ1(s) and μ2(s) would be the phase advances of betatron motion
β1x(s), β1y(s), β2x(s), β2y(s) are the beta-functions
α1x(s), α1y(s), α2x(s), α2y(s) are the alphafunctions
u1(s), u2(s), u3(s), u4(s), v1(s), v2(s) are determined by the orthogonality conditions.

The turn-by-turn particle positions and angles can be represented as a linear combination of four independent solutions:

$$\mathbf{x} = \mathrm{Re}\left(A_1 e^{-i\psi_1}\mathbf{v}_1 + A_2 e^{-i\psi_2}\mathbf{v}_2\right)$$

$$\mathbf{V} = [\mathrm{Re}(\mathbf{v}_1), -\mathrm{Im}(\mathbf{v}_1), \mathrm{Re}(\mathbf{v}_2), -\mathrm{Im}(\mathbf{v}_2)]$$ is a symplectic

This allows one to rewrite in the compact form $\mathbf{x} = \mathbf{V}\mathbf{A}\boldsymbol{\xi}_A$

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \equiv \mathrm{diag}(A_1, A_1, A_2, A_2) \qquad \boldsymbol{\xi}_A = \begin{bmatrix} \cos\psi_1 \\ -\sin\psi_1 \\ \cos\psi_2 \\ -\sin\psi_2 \end{bmatrix}$$

Consider an ensemble of particles whose motion at the lattice origin is contained within a four-dimensional ellipsoid. The motion of the beam's boundary particles is given by the equation: $\mathbf{x}^{T}\Xi\mathbf{x} = 1$

It is natural to define the beam emittance as a product of the ellipsoid semiaxes so that: $\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\Xi}_{11}\hat{\Xi}_{22}\hat{\Xi}_{33}\hat{\Xi}_{44}}} = \frac{1}{\sqrt{\det(\hat{\Xi})}} = A_1^2 A_2^2$

Their product of two-dimensional emittances is equal to the total four-dimensional emittance: ε1ε2 = ε4D. Therefore, the matrix Ξ can be written as:

$$\hat{\Xi} = \mathrm{diag}\left(\varepsilon_1^{-1}, \varepsilon_1^{-1}, \varepsilon_2^{-1}, \varepsilon_2^{-1}\right)$$

$$\boxed{\varepsilon_k' = \frac{1}{2}\mathbf{v}_k'^{+}\mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Sigma}}\hat{\mathbf{V}}^{T}\mathbf{U}^{+}\mathbf{v}_k' + \frac{1}{2}\left|\mathbf{v}_k'^{+}\mathbf{U}(\mathbf{D}-\mathbf{D}')\right|^2 \sigma_p^2} \qquad k=1, 2.. \qquad [1]$$

v_1', v_2' – Eigenvectors of the Booster
Σ = diag(ε_1, ε_1, ε_2, ε_2)
V – matrix of transfer line
U – Symplectic identity matrix.
D – Dispersion vector in the channel $[D_x, D_x', D_y, D_y']^T$.
D' – Dispersion vector in the ring.
σ_p – Root-mean-square momentum spread.

If the beam and the ring optics have no mode coupling (u=0, $\beta_{1y}=\beta_{2x}=0$), and the dispersion is matched (D = D') equation simplifies to:

$$\varepsilon_x' = \frac{1}{2}\varepsilon_x\left(\frac{\beta_x}{\beta_x'}\left[1+\alpha_x'^2\right] + \frac{\beta_x'}{\beta_x}\left[1+\alpha_x^2\right] - 2\alpha_x'\alpha_x\right)$$

$$\varepsilon_y' = \frac{1}{2}\varepsilon_y\left(\frac{\beta_y}{\beta_y'}\left[1+\alpha_y'^2\right] + \frac{\beta_y'}{\beta_y}\left[1+\alpha_y^2\right] - 2\alpha_y'\alpha_y\right) \qquad [2]$$

here, βx, βy, αx, αy are the β- and α-functions of the channel, and βx′, βy′, αx′, αy′ are the β- and α-functions of the ring

A beam passing through a solenoid acquires correlations between planes (〈xy〉 ≠ 0, 〈x'y〉 ≠ 0, etc.). This effect is quantitatively described by the coupling parameter U (from 0 to 1) and coupling terms A_{12} and A_{21} appear in the emittance transformation matrix. For a coupled beam characterized by β_x, β_y, α_x, and α_y, in the absence of dispersion mismatch at the injection point that yields [2]:

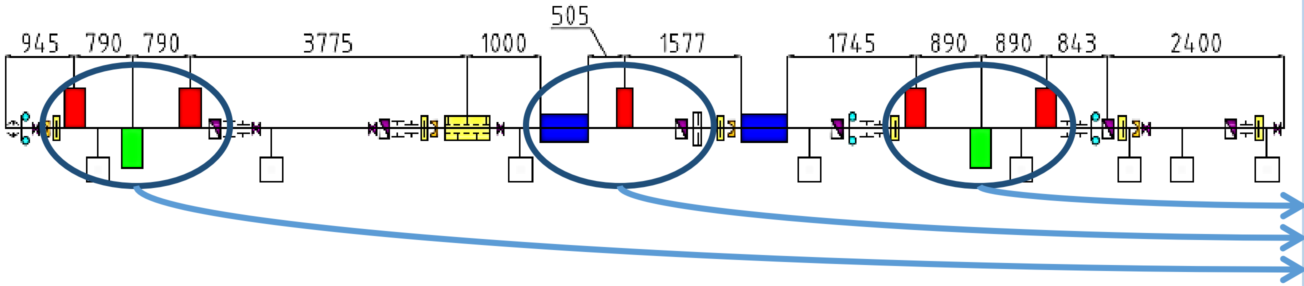
$$\varepsilon_1' = \varepsilon_1 A_{11} + \varepsilon_2 A_{12},$$
$$\varepsilon_2' = \varepsilon_1 A_{21} + \varepsilon_2 A_{22},$$

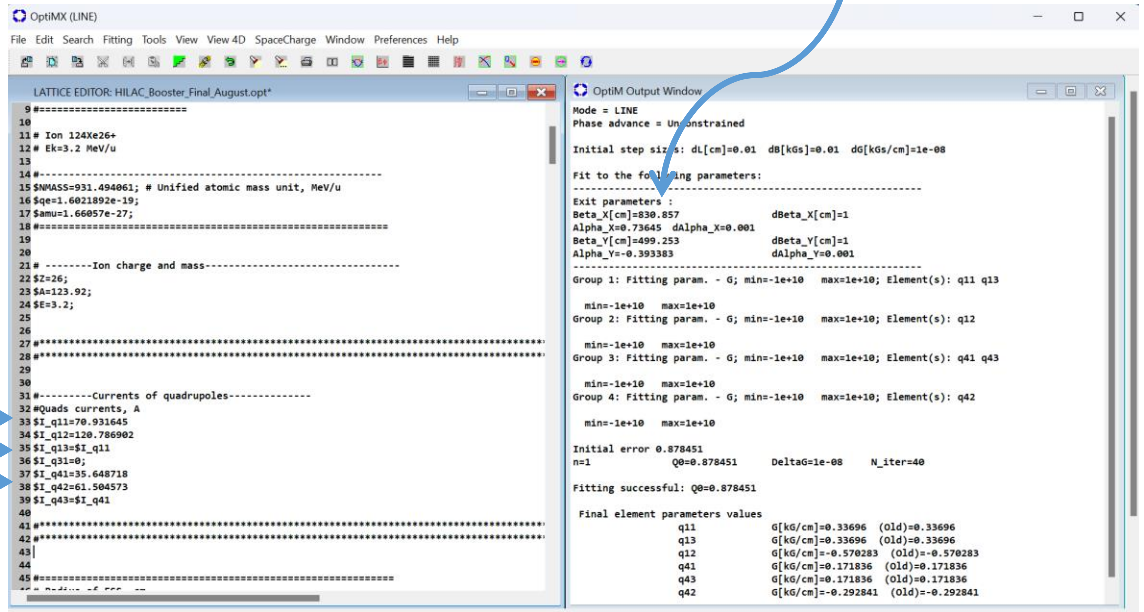
$$A_{11} = \frac{1}{2}\left(\frac{\beta_x}{\beta_{1x}}\left[(1-u)^2 + \alpha_{1x}{}^2\right] + \frac{\beta_{1x}}{\beta_x}\left[1+\alpha_x{}^2\right] - 2\alpha_{1x}\alpha_x\right),$$
$$A_{12} = \frac{1}{2}\left(\frac{\beta_y}{\beta_{1y}}\left[u^2 + \alpha_{1y}{}^2\right] + \frac{\beta_{1y}}{\beta_y}\left[1+\alpha_y{}^2\right] - 2\alpha_{1y}\alpha_y\right),$$
$$A_{22} = \frac{1}{2}\left(\frac{\beta_y}{\beta_{2y}}\left[(1-u)^2 + \alpha_{2y}{}^2\right] + \frac{\beta_{2y}}{\beta_y}\left[1+\alpha_y{}^2\right] - 2\alpha_{2y}\alpha_y\right),$$
$$A_{21} = \frac{1}{2}\left(\frac{\beta_x}{\beta_{2x}}\left[u^2 + \alpha_{2x}{}^2\right] + \frac{\beta_{2x}}{\beta_x}\left[1+\alpha_x{}^2\right] - 2\alpha_{2x}\alpha_x\right),$$

[3]

Algorithm:

1. Setting ideal currents corresponding to perfect matching with the booster optics.





2. Vary the currents in the selected group of quadrupoles by a value ΔI (for example in the range of ±3 A).

- Q1, Q3
- Q2
- Q5, Q7
- Q6

division by type of lens in the triplets

3. For each ΔI value, the OptiM calculates new optical functions of the beam at the injection point.

4. The beam emittance in the booster after injection is computed using the equations:

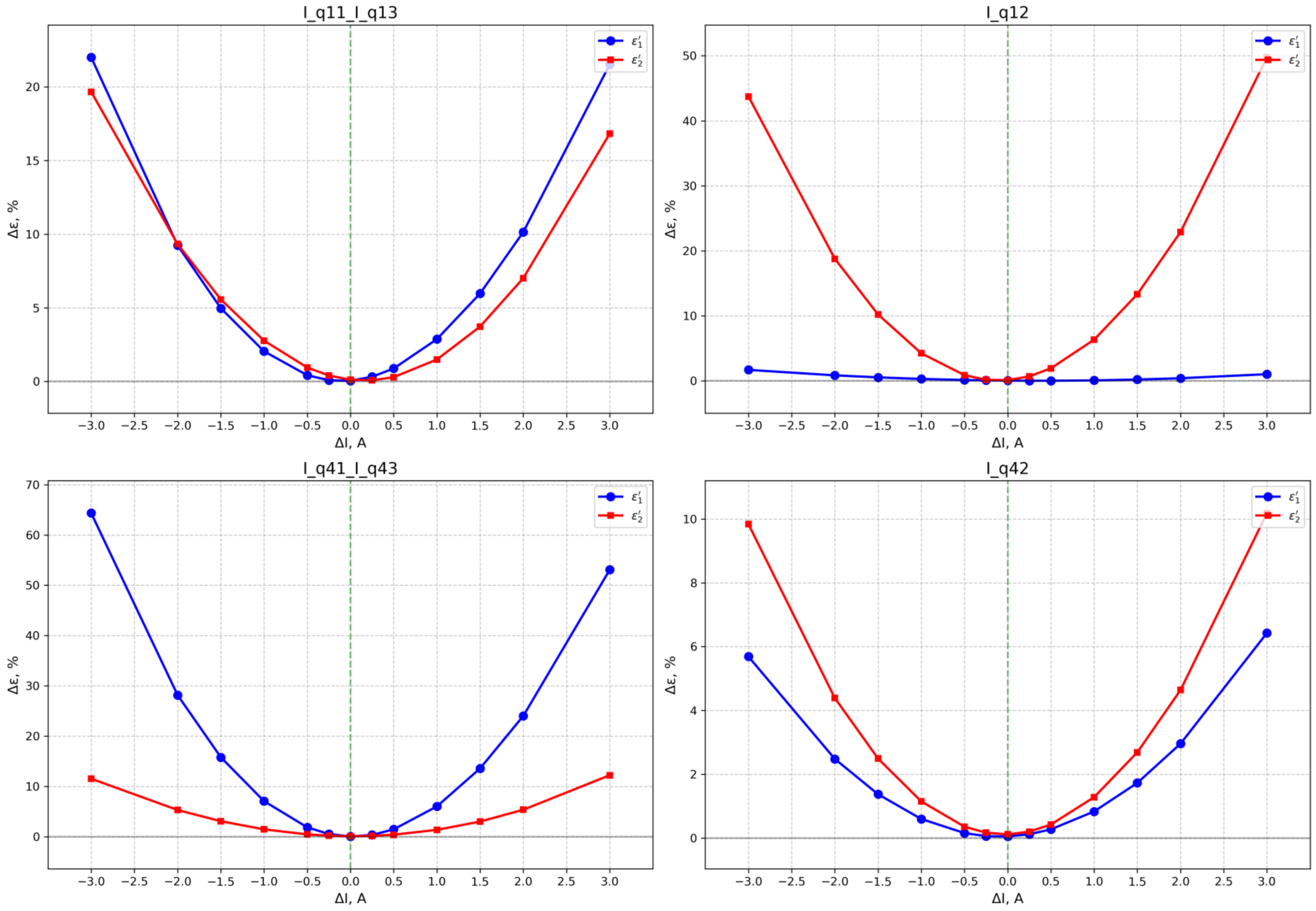
$$\varepsilon_1' = \varepsilon_1 A_{11} + \varepsilon_2 A_{12},$$
$$\varepsilon_2' = \varepsilon_1 A_{21} + \varepsilon_2 A_{22},$$

5. Obtaining plots showing the dependence of the relative emittance growth RMS on the perturbation magnitude ΔI

Optics Mismatch and No Coupling

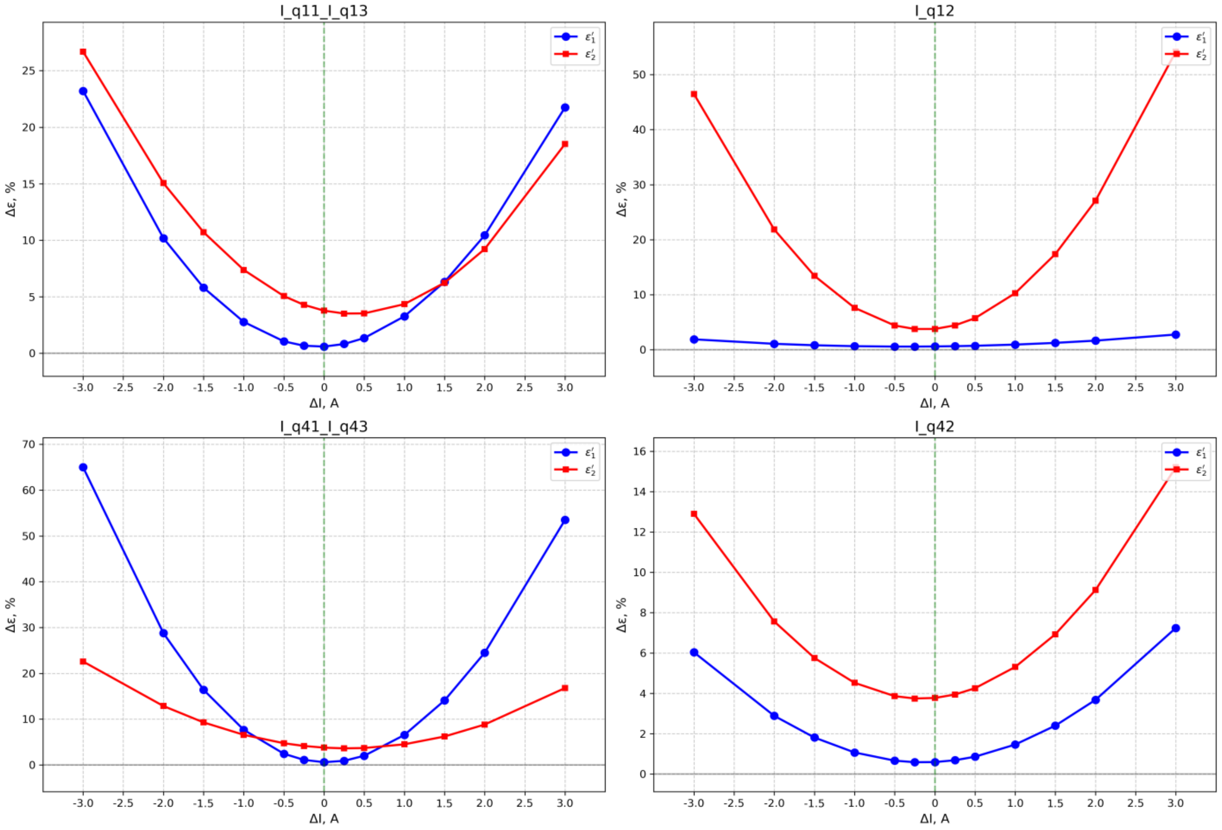
Ion 124Xe26+
Ek=3.2 MeV/u
work point Qx=4.69, Qy=5.44
B_EC = 0.7 #Solenoid longitudinal field
*EPS1 = 250 # [π*mm*mrad]*
*EPS2 = 110 # [π*mm*mrad]*

emittance increase as different current groups vary at the injection point (electrostatic septum)

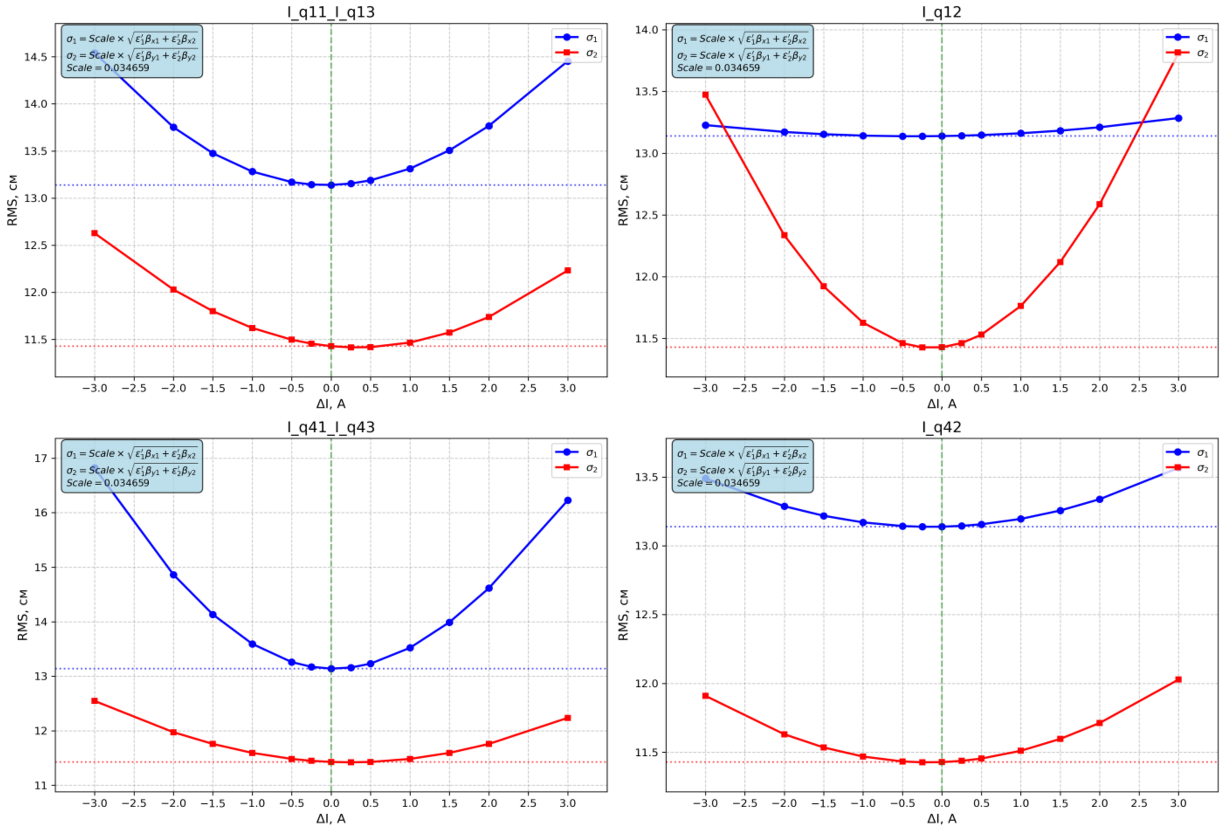


Optics Mismatch and X-Y Coupling

emittance increase as different current groups vary at the injection point (electrostatic septum)



RMS beam size at the injection profilometer



1. The simulation confirms that optics mismatch and transverse motion coupling are significant factors for emittance growth during injection.
2. The obtained dependencies allow for determining the tolerances for current settings in quadrupole lenses to minimize emittance growth and performing more fine-tuning at the injection point.

thank you for your attention