Toric Generalized Kähler Geometry in Superspace

Dmitri Bykov

with S. Kutsubin & A. Kuzovchikov

[see also DB, O. Hulik, U. Lindström, M. Roček, R.v. Unge, to appear]

Steklov Mathematical Institute
ITMP (Moscow State University)

PMMP 2026, Dubna, 16 Feb 2026

Plan

* η-DEFORMED $\mathbb{CP}^{n-1}_\eta$ SIGMA MODEL
  & ITS T-DUAL $\left(\mathbb{CP}^{n-1}_\eta\right)^V$
* GEN. KÄHLER GEOMETRY IN SUPERSPACE
* T-DUALITY IN SUPERSPACE
* NOVEL GAUGING OF GEN. KÄHLER SYMMETRIES
* EXAMPLE: $\mathbb{C}^* \times \mathbb{C}^*$

Motivation

* Integrable (?) sigma models
* Non-homogeneous (η)-deformations of homogeneous target spaces [Klimčík 2002]
* Ex. "Sausage" model [Fateev, Onofri, Zamolodchikov 1993]

$$ds^2 = \frac{|dZ|^2}{(\lambda + |Z|^2)(\lambda^{-1} + |Z|^2)}$$

$\ln \lambda$

* RG flow: $\lambda = e^t \in (0,1)$

η-deformed $\mathbb{CP}^{n-1}_\eta$

* Def. of symmetric spaces – general theory [Delduc, Magro, Vicedo '2013]

$$L = -\tfrac{1}{2}\kappa\left((g^{-1}\partial_+ g)^{(1)}, \frac{1+\eta^2}{1-\eta R_g \circ P_1}(g^{-1}\partial_- g)^{(1)}\right)$$

$g \in G/H$

↑ resembles the classical r-matrix

* Not very practical or explicit [Lukyanov 2012]
* Stable under RG flow @ 1 loop [Fateev, Litvinov 2018] [Hoare, Levine, Tsytlin 2019]

(deformed)

* $\mathbb{CP}^1$ case: relation to Gross-Neveu model [DB 2020+] [DB, Pribytok 2023]

* $\mathbb{CP}^{n-1}_{\eta}$ - generalized Kähler for $n>2$

[Demulder, Hassler Piccinini, Thompson 2020]

$\Rightarrow$ Admits $\mathcal{N}=(2,2)$ SUSY

[Lituinov 2019]

* T-dual $\left(\mathbb{CP}^{n-1}_{\eta}\right)^{V}$ - ordinary Kähler

[DB, Lüst 2020]

$$\mathcal{K}^{V} = \sum_{j=2}^{n-1} i\eta\left(z_j\overline{z_{j-1}} - \overline{z_j}z_{j-1}\right) + \frac{1}{2\eta}\sum_{j=1}^{n} P\left(2\eta\left(t_j - t_{j-1}\right)\right)$$

$$P(t) = i\left(\mathrm{Li}_2(e^{it}) - \frac{t^2}{4}\right), \qquad t_j = z_j + \overline{z_j}$$

* How to T-dualize back to $\mathbb{CP}^{n-1}_{\eta}$?

[DB 2025 Kutsubin Kuzovchikov]

Will tell you today!

$\mathcal{N}=(2,2)$ Superspace in 2D

* Dim.-reduction of $\mathcal{N}=1$ in 4D

* Kähler sigma models [Zumino '1979]

SUPERALGEBRA

Lightcone coords. $x^{\pm}$, Supercoords. $\theta^{\pm}, \overline{\theta^{\pm}}$

$$Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + i\,\overline{\theta^{\pm}}\,\frac{\partial}{\partial x^{\pm}} \qquad \overline{Q_{\pm}} = -\frac{\partial}{\partial\overline{\theta^{\pm}}} - i\,\theta^{\pm}\,\frac{\partial}{\partial x^{\pm}}$$

$$Q_{\pm}^2 = \overline{Q_{\pm}}^2 = 0 \qquad \{Q_{\pm}, \overline{Q}_{\mp}\} = 0$$

$$\{Q_{\pm}, \overline{Q_{\pm}}\} = -2i\,\frac{\partial}{\partial x^{\pm}}$$

Superfields

Superderivatives $D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm}$ $\quad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm \frac{\partial}{\partial x^\pm}$

$$D_\pm^2 = \bar{D}_\pm^2 = 0 \qquad \{D_\pm, Q_\dots\} = \{\bar{D}_\pm, Q_\dots\} = 0$$

Chiral $\quad \bar{D}_+ \mathcal{Z} = \bar{D}_- \mathcal{Z} = 0$ [Wess, Zumino 1974]

Twisted chiral $\quad \bar{D}_+ \mathcal{S} = D_- \mathcal{S} = 0$ [Gates, Hull Roček 1984]

Semi-chiral $\quad \bar{D}_+ \mathcal{X} = 0$ or $D_- \mathcal{Y} = 0$ [Buscher Lindström Roček 1988]

General Lagrangian $\quad L = \int d^4\theta\, K(\mathcal{Z}, \bar{\mathcal{Z}}, \mathcal{S}, \bar{\mathcal{S}}, \mathcal{X}, \bar{\mathcal{X}}, \mathcal{Y}, \bar{\mathcal{Y}})$

Kähler geometry

* Only chiral & anti-chiral superfields $Z_A, \bar{Z}_A$

$$g_{A\bar{B}} = \frac{\partial^2 K}{\partial Z^A \partial \bar{Z}^B} \qquad K(Z_A, \bar{Z}_A) = \text{Kähler potential}$$

$$\mathcal{L} = g_{A\bar{B}}(Z, \bar{Z})\, \partial_\alpha Z^A \partial^\alpha \bar{Z}^B + i\, \bar{\psi}^B \not{\partial} \psi^A g_{A\bar{B}} + \dots$$

"Invariant" definitions

complex structure

$$(g, J, \omega = g \cdot J)$$

$$\omega := \frac{i}{2} g_{A\bar{B}}\, dZ^A \wedge d\bar{Z}^B \,, \quad d\omega = 0 \quad \Leftrightarrow \quad \nabla J = 0$$

Levi-Civita

Baby generalized Kähler geometry

* Bi-Hermitian geometry: J_+, J_- g Hermitian w.r.t. both

* Non-topological B-field $\int B_{\alpha\beta} dX^\alpha \wedge dX^\beta$

$$T := dB \rightsquigarrow \text{Torsion}$$

$$(\nabla \pm T) J_\pm = 0$$

[Gates, Hull Roček 1984]

* $[J_+, J_-] = 0 \Rightarrow$ chiral Z_A & tw.chiral S_B fields

Ex. $SU(2) \times U(1)$ WZNW model [Roček, Schoutens Sevrin 1991]

Full-fledged generalized Kähler geometry

* $[J_+, J_-] \neq 0$ [Severin, Troost 1996]

  ⇒ Add semi-chiral fields [Hitchin 2003]

* Math definitions [Gualtieri 2004, 2010]

* Equivalence with physics SUSY models [Lindström, Roček, v. Unge, Zabzine 2005]

Examples from T-duality

* Sigma model on $(S^1)_R \sim (S^1)_{R^{-1}}$ [Cremmer Scherk 1975]

* Extends to U(1)-inv. backgrounds [Buscher 1587-88]

$$(g, B) \rightsquigarrow (g', B')$$

$$L = L(\partial_\alpha \phi, \dots)$$

① Gauge $S = \int d^2x \, L(\partial_\alpha \phi - A_\alpha, \dots) + \int (A - d\psi) \wedge \chi$

② Set $\phi = 0$

③ Integrate out A_α $\rightsquigarrow$ T-dual geometry

χ: Lagrange multiplier 1-form

SUSY T-duality

[Roček, Verlinde 1992]

Assume a holomorphic isometry V leaving K invariant

$$\mathcal{L}_V K = 0$$

General model: $Z \mapsto Z + i\varepsilon$, $\varepsilon \in \mathbb{R}/\mathbb{Z}$

$$K = K(Z + \bar{Z}, \text{all other coords.})$$

① Gauging: $Z \mapsto Z + \Phi$ (Φ: chiral), $V \mapsto V - \Phi - \bar{\Phi}$ (V: real)

$$K_g = K[Z + \bar{Z} + V, \ldots]$$

② Add Lagrange multiplier $V\cdot(\Sigma+\bar{\Sigma})$ ← twisted chiral

③ Set $\Sigma=\bar{\Sigma}=0$

④ Integrate out V: $\frac{\partial K}{\partial V}(V,\dots)=\Sigma+\bar{\Sigma}$

$\Rightarrow V=V[\Sigma+\bar{\Sigma},\dots]$

$$L_g=\int d^4\theta\,\Big(K_g[V,\dots]-V\cdot(\Sigma+\bar{\Sigma})\Big)\Big|_{V=V[\Sigma+\bar{\Sigma},\dots]}$$

"T-Dual" potential depending on chiral & tw. chiral fields = Legendre transform of K

Interim Summary

* T-DUALITY FOR KÄHLER / GEN. KÄHLER MANIFOLDS

⇓

GAUGE ISOMETRIES IN SUPERSPACE

* WE HAVE SEEN HOW TO DO IT

FOR INVARIANT K

Toric manifolds

Practical definition

* (M^{2n}, J) = complex manifold, $\dim_{\mathbb{C}} M^{2n} = n$

* n commuting holomorphic Killing vector fields

$\mathcal{L}_{V_A} J = 0$ $\mathcal{L}_{V_A} g = 0$ V_A $A = 1 \dots n$ $[V_A, V_B] = 0$

Example: $\mathbb{CP}^{n-1}$ Counter-example: $Gr(k,n)$

$k \neq 1, n-1$

"Invariant" case

* Let M^{2n} - toric, all isometries preserving K — chiral

$$\Rightarrow K = K(Z_1+\bar{Z}_1,\ Z_2+\bar{Z}_2,\dots)$$

$$Z_k \mapsto Z_k + i\varepsilon_k \qquad k=1\dots n$$

* T-dualizing all variables $\mapsto$ $K^\vee = K^\vee(S_1+\bar{S}_1, S_2+\bar{S}_2,\dots)$ — tw. chiral

The dual geometry is Kähler again $(B=0)$ $\theta^- \leftrightarrow \bar{\theta}^-$

* But recall $\mathbb{CP}^{n-1}_{\eta}$ $\longleftrightarrow$ (T-duality) $(\mathbb{CP}^{n-1}_{\eta})^\vee$

gen. Kähler — ? — Kähler

Non-invariant case

* Inv. of metric does not imply inv. of K

$$\mathcal{L}_v K = f(z) + \bar{f}(\bar{z}) \sim \text{up to Kähler transf.}$$

Ex. Torus T^2 $\quad z \mapsto z + i\varepsilon_1 + \tau\varepsilon_2$

$\exists\ K_{1,2}$ inv. under ε_1 or ε_2 but NOT under both

$$K_1 = \frac{1}{2}(z+\bar{z})^2$$

* An obstruction to gauging [Hull, Karlhede, Lindström, Roček 1986]

Ex. $\mathbb{C} \times \mathbb{C}$ $\quad z_1, z_2$

General flat metric $\quad ds^2 = \sum_{i,j=1}^{2} a_{ij}\, dz_i \cdot d\bar{z}_j$ $\quad$ ← 4 parameters

Isometries: $\quad z_j \mapsto z_j + i\,\varepsilon_j \qquad \varepsilon_j \in \mathbb{R}$

$\nexists$ invariant Kähler potential

Invariants: $\quad t_1 = z_1 + \bar{z}_1\,, \quad t_2 = z_2 + \bar{z}_2$

Most general potential $\quad K = \sum_{i \le j} d_{ij}\, t_i\, t_j$ $\quad$ ← 3 parameters

$\mathbb{C} \times \mathbb{C}$ continued

$$ds^2 = \sum_{i,j=1}^{2} a_{ij}\, dz_i \cdot d\bar{z}_j \qquad a_{ij} = \begin{pmatrix} a_1 & \alpha - i\beta \\ \alpha + i\beta & a_2 \end{pmatrix}$$

$$K = \frac{a_1}{2}(z_1+\bar{z}_1)^2 + \frac{a_2}{2}(z_2+\bar{z}_2)^2 + \alpha(z_1+\bar{z}_1)(z_2+\bar{z}_2) +$$

$$+ \boxed{i\beta(z_1+\bar{z}_1)(z_2-\bar{z}_2)} \quad \leftarrow \text{non-invariant piece}$$

$\leftarrow$ tw. chiral

* Replace by $i\beta(z_1+\bar{z}_1)(z_2-\bar{z}_2-\zeta_2+\bar{\zeta}_2)$

* This is a gen. Kähler transform (i.e. total derivative)

But now inv. under $z_2 \mapsto z_2 + i\varepsilon_2$, $\zeta_2 \mapsto \zeta_2 + i\varepsilon_2$

Gauging

* Only chiral fields: replace $Z+\bar{Z} \mapsto Z+\bar{Z}+V$

  Gauge transfos: $Z \mapsto Z+\Phi$ (Φ chiral)

  $V \mapsto V-\Phi-\bar{\Phi}$

* If symmetry acts on both chiral & tw. chiral fields

  $\Rightarrow$ V is not enough

  cf. [Lindström, Roček, Ryb, von Unge, Zabzine 2007]

* One could introduce V' for tw. chirals: $S+\bar{S} \mapsto S+\bar{S}+V'$

  But this corresponds to 2 independent symmetries

Our proposal

[DB, Kutsubin Kuzovchikov 2025]

* (V, X) — V: real gauge superfield; X: semi-chiral; $\overline{D}_+ X = 0$

Gauge transfos: $V \mapsto V - \Phi - \overline{\Phi}$ (Φ: chiral)

$X \mapsto X + \Phi - \Lambda$ (Λ: tw. chiral)

Introduce $V' = V + X + \overline{X} \Rightarrow V' \mapsto V' - \Lambda - \overline{\Lambda}$

* Gauge field strength $F = D_-(V + X)$

$F = \overline{F} = 0 \leftrightarrow (V, X)$ pure gauge

* FI term: $\xi \int d^4\theta\, V \simeq \xi \int d^4\theta\, V'$

T-duality

$\mathbb{C}\times\mathbb{C}$ model $\xrightarrow{\text{compactify}}$ $\mathbb{C}^*\times\mathbb{C}^*$ model

$z_1 \sim z_1 + 2\pi i$, $z_2 \sim z_2 + 2\pi i$

$$K = \frac{a_1}{2}(z_1+\bar{z}_1)^2 + \frac{a_2}{2}(z_2+\bar{z}_2)^2 + a(z_1+\bar{z}_1)(z_2+\bar{z}_2) + i\beta(z_1+\bar{z}_1)(z_2-\bar{z}_2-\zeta_2+\bar{\zeta}_2)$$

① Gauge:

$$K_g = \frac{a_1}{2}(z_1+\bar{z}_1+V_1)^2 + \frac{a_2}{2}(z_2+\bar{z}_2+V_2)^2 + a(z_1+\bar{z}_1+V_1)(z_2+\bar{z}_2+V_2) + i\beta(z_1+\bar{z}_1+V_1)(z_2-\bar{z}_2-\zeta_2+\bar{\zeta}_2+X-\bar{X})$$

② Add Lagrange multipliers $Y\cdot(V_2+X) + T\cdot V_1 + \text{c.c.}$

③ Set $z_1 = z_2 = \zeta_2 = 0$

Y: semi-chiral $D_-Y = 0$

T: tw.chiral

④ Integrate out V_1, V_2 $\qquad x = X + \bar{X}\,,\; y = Y + \bar{Y}$

$$K^V = -\frac{1}{2\Delta}\left(a_1 y^2 - 2d\,xy + a_2 x^2\right) + \frac{i}{\beta}\left(XY - \bar{X}\bar{Y}\right)$$

$\Delta = a_1 a_2 - d^2$

* $\beta \to 0$ limit "steepest descent"

X: $\bar{D}_+ Y = 0$ $\qquad$ $D_- Y = 0$

recall $\Rightarrow$ X, Y

Y: $D_- X = 0$ $\qquad$ $\bar{D}_+ X = 0$ $\qquad$ tw. chiral

* Inversion of radii: set $d = 0$ $\Rightarrow$ $K^V = -\frac{1}{2a_2} y^2 - \frac{1}{2a_1} x^2 + \dots$

$\Rightarrow$ T-duality !

Summary & Outlook

* A new method for gauging isometries
  in gen. Kähler geometry / $\mathcal{N}=(2,2)$ sigma models
* Overcomes previously known obstructions to gauging
* Any obstructions remaining?
* Gen. Kähler quotient construction?
* Applications to integrable sigma models

TO BE CONTINUED...

FOR APPLICATIONS TO

THE $\mathbb{C}P^{n-1}_{?}$ MODEL

SEE SAVVA KUTSUBIN'S

TALK LATER TODAY!

THANK YOU!

