The deformed $\mathbb{CP}^{n-1}_{\eta}$ model

based on work with
Dmitri Bykov and Andrew Kuzovchikov
2512.24726v1

Kutsubin Savva

MIPT
The Landau Phystech School of Physics and Research

February 16, 2026



$\mathcal{N} = (2,2)$ Lagrangian

Having developed the necessary techniques, we are now ready to tackle the so-called η-deformed $\mathbb{CP}^{n-1}_{\eta}$ geometry. In [D. Bykov, D. Lüst 2021] it was shown that the *T-dual* $\left(\mathbb{CP}^{n-1}_{\eta}\right)^{\vee}$, is an ordinary Kähler geometry:

$$\mathcal{K}^{\vee} = \frac{1}{2}\sum_{j=2}^{n-1} i\left(\mathbf{Z}_j\bar{\mathbf{Z}}_{j-1} - \bar{\mathbf{Z}}_j\mathbf{Z}_{j-1}\right) + \sum_{j=1}^{n}\mathcal{P}\left(\mathbf{t}_j - \mathbf{t}_{j-1}\right),$$

$$\text{where} \quad \mathcal{P}(t) = i\left(\mathrm{Li}_2\left(e^{it}\right) + \frac{t\,(2\pi - t)}{4}\right) \quad \text{and} \quad \mathbf{t}_j = \mathbf{Z}_j + \bar{\mathbf{Z}}_j$$

Here we have made a rescaling $\mathbf{Z}_i \mapsto \frac{\mathbf{Z}_i}{2\eta}$ and dropped an overall factor of $\frac{1}{2\eta}$.
Isometries have the form:

$$\mathbf{Z}_j \mapsto \mathbf{Z}_j + i\,\epsilon_j$$

In order to implement the method proposed in the talk of Dmitri Bykov we write an equivalent potential (n - odd):

$$\frac{i}{2}\sum_{j=1}^{\frac{n-3}{2}}\left(\mathbf{Z}_{2j-1}+\bar{\mathbf{Z}}_{2j-1}-\mathbf{Z}_{2j+1}-\bar{\mathbf{Z}}_{2j+1}\right)\left(\mathbf{Z}_{2j}-\bar{\mathbf{Z}}_{2j}\right)+$$
$$+\frac{i}{2}\left(\mathbf{Z}_{n-2}+\bar{\mathbf{Z}}_{n-2}\right)\left(\mathbf{Z}_{n-1}-\bar{\mathbf{Z}}_{n-1}\right)$$

For each j we introduce an unconstrained real gauge superfield $\mathbf{V}_j$, an auxiliary t.c. field $\mathbf{S}_j$, and a s.c. gauge field $\mathbf{X}_j$ satisfying $\bar{\mathcal{D}}_+\mathbf{X}_j=0$:

$$\mathbf{Z}_j+\bar{\mathbf{Z}}_j\mapsto\mathbf{Z}_j+\bar{\mathbf{Z}}_j+\mathbf{V}_j$$
$$\mathbf{Z}_j-\bar{\mathbf{Z}}_j\mapsto\mathbf{Z}_j-\mathbf{S}_j-\mathbf{X}_j-\bar{\mathbf{Z}}_j+\bar{\mathbf{S}}_j+\bar{\mathbf{X}}_j$$

Setting $\mathbf{Z}_j = \mathbf{S}_j = 0$ by a gauge transformation and adding Lagrange multipliers which are similar to those in $\mathbb{C} \times \mathbb{C}$ case:

$$+ \sum_{j=1}^{\frac{n-1}{2}} \left(\mathbf{Y}_{2j} \left(\mathbf{V}_{2j} + \mathbf{X}_{2j} \right) + \bar{\mathbf{Y}}_{2j} \left(\mathbf{V}_{2j} + \bar{\mathbf{X}}_{2j} \right) \right) + \sum_{j=1}^{\frac{n-1}{2}} \mathbf{V}_{2j-1} \left(\mathbf{\Sigma}_{2j-1} + \bar{\mathbf{\Sigma}}_{2j-1} \right)$$

where $\mathbf{V}_0 \equiv \mathbf{t}_0 \, , \mathbf{V}_n \equiv \mathbf{t}_n$, after some computations we arrive to:

$$\mathcal{K} = \sum_{j=1}^{n-1} \mathbf{V}_j \, \mathbf{w}_j + \sum_{j=1}^{n} \mathcal{P} \left(\mathbf{V}_j - \mathbf{V}_{j-1} \right) + 2i \sum_{j=1}^{\frac{n-1}{2}} \left(\mathbf{Y}_{2j} \mathbf{X}_{2j} - \bar{\mathbf{Y}}_{2j} \bar{\mathbf{X}}_{2j} \right)$$

where $\mathbf{Y}_{2j-1} = \mathbf{X}_{2j} - \mathbf{X}_{2j-2}, \ \mathbf{X}_0 \equiv \mathbf{X}_n \equiv 0$; $\mathbf{w}_j = \mathbf{Y}_j + \bar{\mathbf{Y}}_j$.

Therefore system is described in terms of semi-chiral fields in the odd n case.

Legendre transform

It remains to extremize the potential w.r.t. gauge fields. It's convenient to rewrite $\mathcal{K}$ in the homogeneous coordinates:

$$\mathcal{K} = \sum_{j=1}^{n} \mathbf{M}_j \tilde{\mathbf{w}}_j + \sum_{j=1}^{n} \mathcal{P}(\mathbf{M}_j) - \mathbf{L}\left(\sum_{k=1}^{n} \mathbf{M}_k - \rho\right) + 2i \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\mathbf{Y}_{2j}\mathbf{X}_{2j} - \bar{\mathbf{Y}}_{2j}\bar{\mathbf{X}}_{2j}\right)$$

where $\tilde{\mathbf{w}}_k = \sum_{j=k}^{n-1} \mathbf{w}_j \equiv -\log(\mathbf{U}_k \bar{\mathbf{U}}_k)$. Then e.o.m are

$$2\sin\frac{\mathbf{M}_k}{2} = e^{\mathbf{L}} |\mathbf{U}_k|^2 ; \; k = \overline{1,n} ; \; \sum_{k=1}^{n} \mathbf{M}_k = \rho$$

In the limit of vanishing deformation one obtains:

$$\mathcal{K} \sim -\eta \log\left(\sum_{j=1}^{n} |\mathbf{U}_j|^2\right) + \dots ,$$

$\mathcal{N} = (1,1)$ reduced $\mathbb{CP}^2_\eta$ case

In this case, where the generalized Kähler potential is:

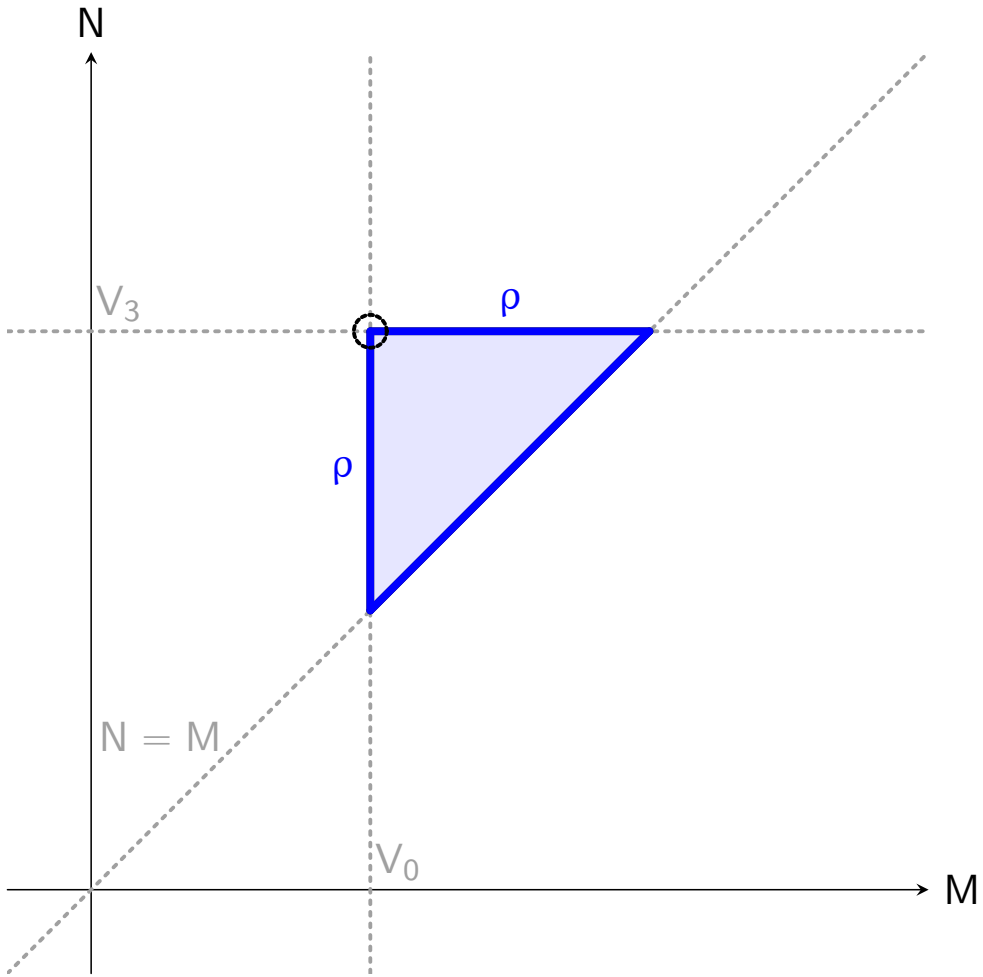
$$\mathcal{K} = \underbrace{\mathbf{V}_1\mathbf{x} + \mathbf{V}_2\mathbf{y} + \mathcal{P}(\mathbf{V}_3 - \mathbf{V}_2) + \mathcal{P}(\mathbf{V}_2 - \mathbf{V}_1) + \mathcal{P}(\mathbf{V}_1 - \mathbf{V}_0)}_{\equiv \mathrm{F}(\mathbf{x},\mathbf{y})} + \\ + 2i(\mathbf{X}\mathbf{Y} - \bar{\mathbf{X}}\bar{\mathbf{Y}})$$

Just like in the usual Kähler setup (cf.[V.Guillemin 2024]), it is often more illuminating to express the metric in terms of the symplectic potential

$$\mathrm{G}(\mathrm{M},\mathrm{N}) := \mathrm{M}\mathrm{x} + \mathrm{N}\mathrm{y} - \mathrm{F}(\mathrm{x},\mathrm{y})\,,\ \mathrm{x} = x\big|,\ \mathrm{y} = y\big|$$
$$\mathrm{G}(\mathrm{M},\mathrm{N}) = -\mathcal{P}(\mathrm{V}_3 - \mathrm{N}) - \mathcal{P}(\mathrm{N} - \mathrm{M}) \ - \mathcal{P}(\mathrm{M} - \mathrm{V}_0)\,,$$

where one defines the new coordinates $\mathrm{M} := \frac{\partial \mathrm{F}}{\partial \mathrm{x}}, \mathrm{N} := \frac{\partial \mathrm{F}}{\partial \mathrm{y}}$, which play the role of generalized moment maps for the two $U(1)$ actions.

Moment polytope of $\mathbb{CP}^2_\eta$



Useful limit is where the deformation vanishes, i.e. $\rho = V_3 - V_0 \to 0$. This is the limit where the triangle in Fig. 7 formally shrinks to zero:

$$\frac{1}{\rho}\mathcal{K}_{(1,1)} = \frac{N - V_0}{4\,(N-M)\,(M-V_0)}\mathscr{D}_+ M \mathscr{D}_- M + \frac{V_3 - M}{4\,(N-M)\,(V_3-N)}\mathscr{D}_+ N \mathscr{D}_- N -$$
$$- \frac{1}{2\,(N-M)}\mathscr{D}_+ M \mathscr{D}_- N + \frac{(M-V_0)\,(V_3-N)}{V_3-V_0}\left(\mathscr{D}_+\Omega\mathscr{D}_-\Phi + \mathscr{D}_+\Phi\mathscr{D}_-\Omega\right) +$$
$$+ \frac{(V_3-M)\,(M-V_0)}{V_3-V_0}\mathscr{D}_+\Phi\mathscr{D}_-\Phi + \frac{(V_3-N)\,(N-V_0)}{V_3-V_0}\mathscr{D}_+\Omega\mathscr{D}_-\Omega + \dots$$

This can be written in terms of the symplectic potential of $\mathbb{CP}^2$:

$$\frac{1}{\rho}\mathcal{K}_{(1,1)} = \frac{1}{4}\frac{\partial^2 G}{\partial\mu_i\partial\mu_j}\mathscr{D}_+\mu_i\mathscr{D}_-\mu_j + \left(\frac{\partial^2 G}{\partial\mu^2}\right)^{-1}_{ij}\mathscr{D}_+\phi_i\mathscr{D}_-\phi_j + \dots$$

where $\mu_1 = M,\ \mu_2 = N,\ \phi_1 = \Phi,\ \phi_2 = \Omega$.

Finally we recover symplectic potential of $\mathbb{CP}^2$:

$$\mathrm{G} = (\mathrm{M}-\mathrm{V}_0)\log(\mathrm{M}-\mathrm{V}_0) + (\mathrm{V}_3-\mathrm{N})\log(\mathrm{V}_3-\mathrm{N}) + \\ + (\mathrm{N}-\mathrm{M})\log(\mathrm{N}-\mathrm{M})$$

$$\left(\frac{\partial^2 G}{\partial\mu_i\partial\mu_j}\right)_{i,j\in\{1,2\}} = \frac{1}{\mathrm{N}-\mathrm{M}}\begin{pmatrix} \frac{\mathrm{N}-\mathrm{V}_0}{\mathrm{M}-\mathrm{V}_0} & -1 \\ -1 & \frac{\mathrm{V}_3-\mathrm{M}}{\mathrm{V}_3-\mathrm{N}} \end{pmatrix}$$

Bibliography

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[Abr] M. Abreu. "Kähler geometry of toric manifolds in symplectic coordinates"