Combinatorial Formulas in Electrical Network Theory and the Geometry of Lagrangian Grassmannian

Anton Kazakov

YarSU, HSE

Brief History of Combinatorial Formulas

Combinatorial (topological) formulas express effective resistances, electrical impedances, entries of the response matrices, and the scattering matrices purely in terms of an electrical network topology, via partition functions of spanning forests, groves, paths, and related combinatorial substructures.

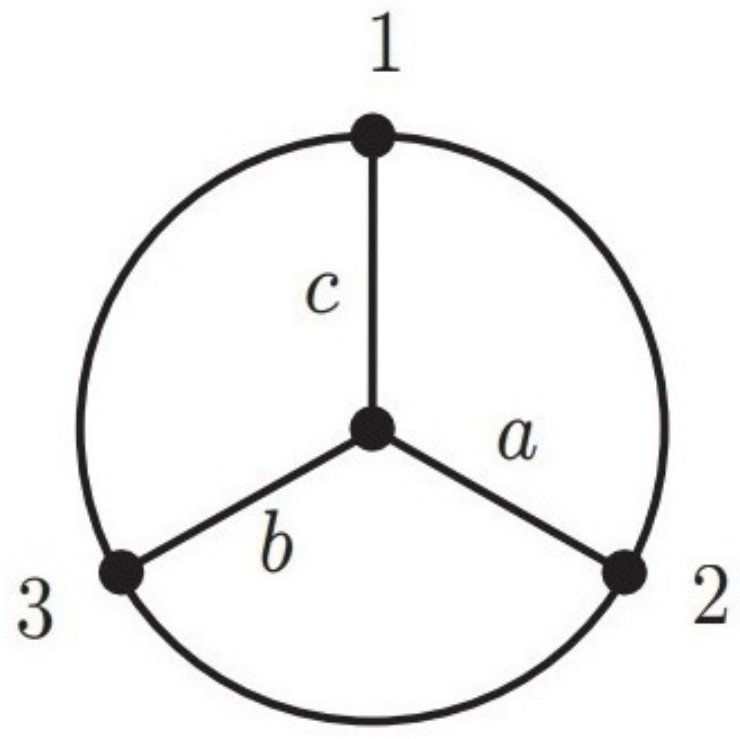
- G. Kirchhoff, «On the solution of the equations obtained from the investigation of the linear distribution of galvanic currents»
- L. Robichaud, M. Boisvert, J. Robert, «Graphes de fluence: applications à l'électrotechnique et à l'électronique calculateurs analogiques et digitaux»
- S. P. Chan, «Introductory topological analysis of electrical networks»

Brief History: Kirchhoff Formula

Effective resistance between nodes i and j of an electrical network on a graph G can be calculated using the following formula:

$$R_{ij} = \frac{T_{ij}(G)}{T(G)},$$

where $T(G)$ is the sum of weights of all spanning trees of the graph G, and $T_{ij}(G)$ is the sum of weights of all 2-spanning groves.



$$\mathrm{T(G)} = abc$$

$$\mathrm{T_{12}(G)} = ab + bc$$

Brief History of Combinatorial Formulas

Characterization problems: which matrices can occur as effective resistance matrices (or as resistance-distance matrices)?

Inverse (synthesis) problems: from a given response matrix (Dirichlet–to–Neumann map), recover conductances of an electrical network and/or its topology.

- R. Bott, R. J. Duffini, «Impedance Synthesis without Use of Transformers»
- S. Seshu, M. B. Reed, «Linear graphs and electrical networks»
- L. Borcea, V. Druskin, F.G. Vasquez, «Electrical impedance tomography with resistor networks»
- E.B. Curtis, J.A. Morrow, «Inverse problems for electrical networks»

Electrical networks

Curtis, Morrow, «Inverse problems for electrical networks»

Definition

An **electrical network** $e(\Gamma, \omega)$ is an embedded into a disk connected weighted planar graph $\Gamma(V, E)$, that satisfies the following conditions:

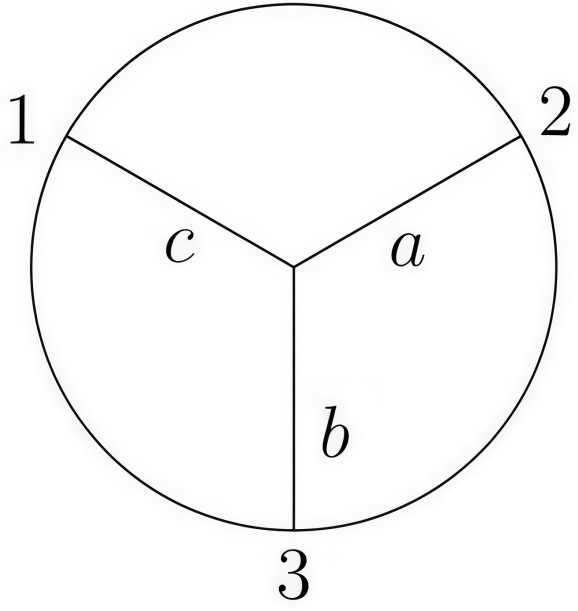
- All nodes are divided into the set of inner nodes and the set of boundary nodes V_B and each boundary node lies on the boundary circle;
- Boundary nodes are enumerated clockwise from 1 to $|V_B| = n$. Inner nodes are enumerated arbitrarily from $n + 1$;
- Every edge is equipped with a positive weight ω_{ij}, which denotes the conductivity of this edge.

Electrical networks

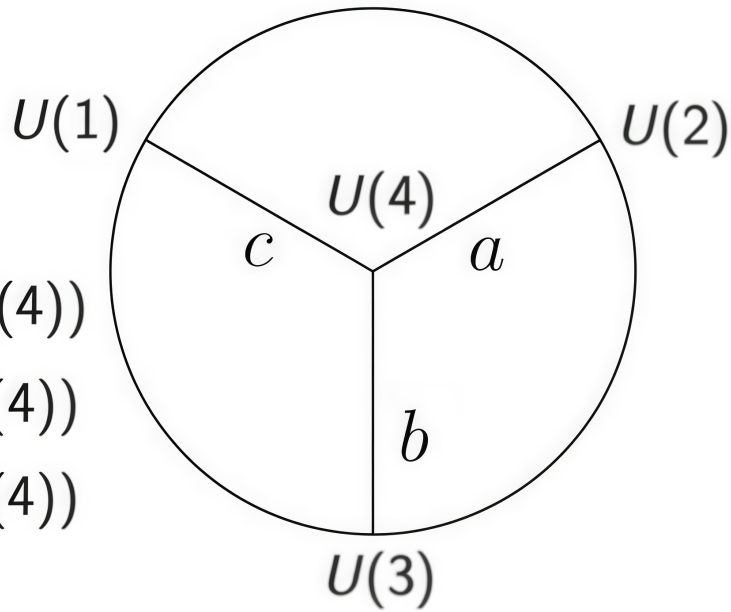
Let us apply voltages $\mathbf{U} : V_B \to \mathbb{R}$ to its boundary nodes V_B, these boundary voltages give rise to the unique harmonic extension $U : V \to \mathbb{R}$ on all vertices, which might be restored by the Kirchhoff and Ohm laws:

$$\sum_{j \in V} \omega_{ij} \big(U(i) - U(j)\big) = 0, \ \forall i \in V_I.$$

$$c(U(4) - U(1)) + a(U(4) - U(2)) + b(U(4) - U(3)) = 0$$



$$I_1 = c(U(1) - U(4))$$
$$I_2 = a(U(2) - U(4))$$
$$I_3 = b(U(3) - U(4))$$



Response matrix and the black box problem

Definition

Voltages $\mathbf{U}$ induce boundary currents $\mathbf{I} = \{I_1, \ldots, I_n\}$ flowing through boundary nodes:

$$I_k := \sum_{j \in V} \omega_{ij}\big(U(k) - U(j)\big), \;\; k \in \{1, \ldots, n\}.$$

Theorem

There is a matrix $(x_{ij}) = M(e) \in Mat_{n\times n}(\mathbb{R})$, which relates $\boldsymbol{U}$ and $\boldsymbol{I}$ to each other as follows:

$$M(e)\boldsymbol{U} = \boldsymbol{I}.$$

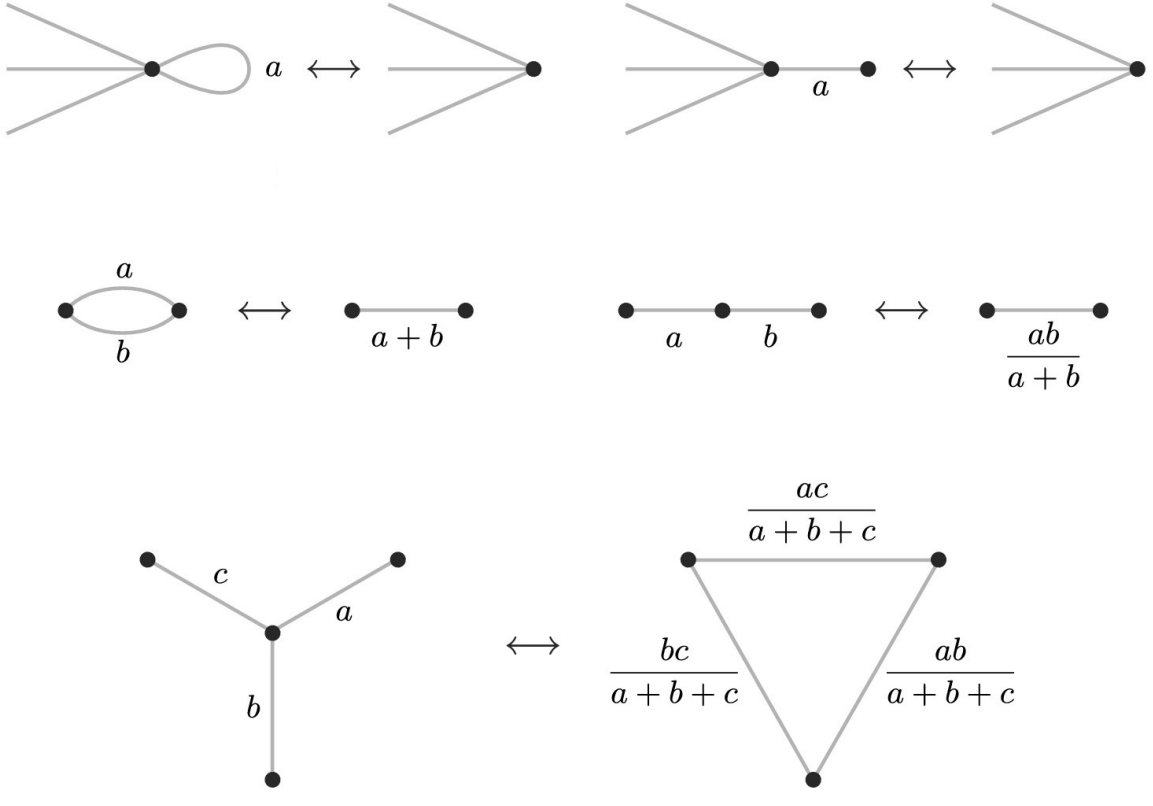
*This matrix is called the **response matrix** of the network e.*

Electrical transformations

Theorem

Two electrical networks share the same response matrix if and only if they can be converted to each other only the electrical transformations.

We will consider electrical networks up to the electrical transformations.



Projective Coordinates: Groves

Definition

A grove F on Γ is a spanning forest, such that each connected component $F_i \subset F$ contains at least one boundary node.
The boundary partition $\sigma(F)$ is the set partition of the set $\{1, 2, \ldots, n\}$ that specifies which boundary nodes lie in the same connected component of F.

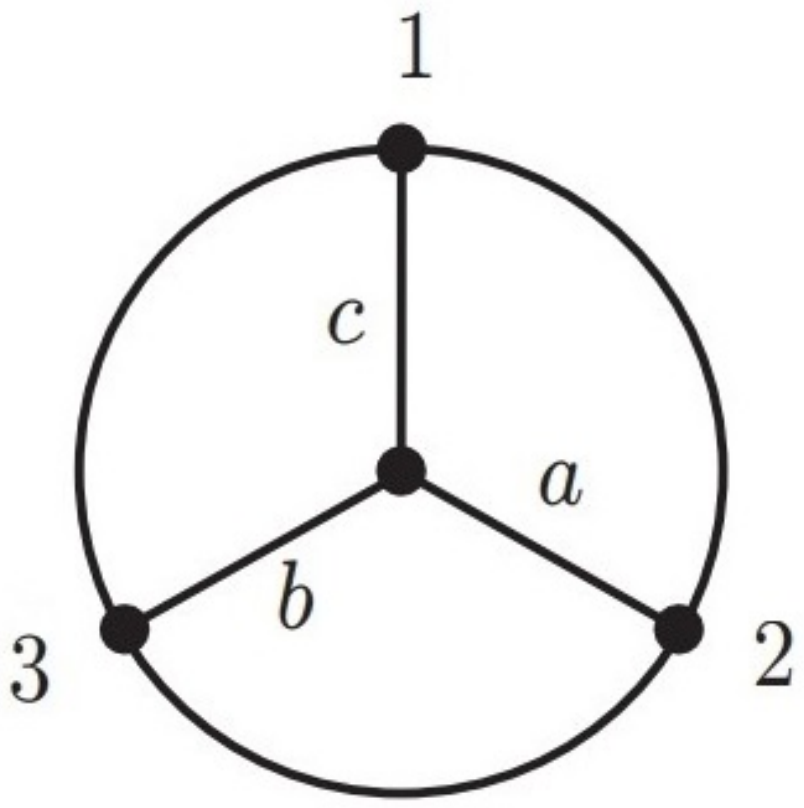
Definition

Each partition σ defines the grove measurement:

$$L_\sigma := \sum_{F|\sigma(F)=\sigma} wt(F),$$

where the summation is over all groves with boundary partition σ, and $wt(F)$ is the product of the weights of the edges in F.

Groves: Example



$$L_{1|2|3} = a + b + c$$

$$L_{123} = abc$$

$$L_{12|3} = ac$$

$$L_{23|1} = ab$$

$$L_{13|2} = bc$$

Plücker Coordinates via Groves

T. Lam, «Electroid varieties and a compactification of the space of electrical networks»

Theorem

Consider an electrical network $e(G, \omega)$ with n boundary vertices. For every index set $I \subset \{1, \ldots, 2n\}$ of size $|I| = n - 1$ define

$$\Delta_I = \sum_{\sigma} a_{\sigma,I} \, L_\sigma, \quad a_{\sigma,I} \in \{0, 1\},$$

where $a_{\sigma,I}$ are specified by the generalized Temperley trick. The set of Δ_I satisfies the Plücker relations with nonnegative coordinates and therefore determines a point of the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(n-1, 2n)$.

From electrical networks to the non-negative Grassmannians

Bychkov, Kazakov, Talalaev, «Electrical networks, Lagrangian Grassmannians and symplectic groups»

Theorem

Consider an electrical network $e \in E_n$. Using its response matrix $M_n(e) = (x_{ij})$ we define a point of the Grassmannian $\mathrm{Gr}(n-1, 2n)$ associated to e as the row space of the matrix:

$$\Omega_n(e) = \begin{pmatrix} x_{11} & 1 & -x_{12} & 0 & x_{13} & 0 & \cdots & (-1)^n \\ -x_{21} & 1 & x_{22} & 1 & -x_{23} & 0 & \cdots & 0 \\ x_{31} & 0 & -x_{32} & 1 & x_{33} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

Then $\Omega_n(e)$ defines a non-negative point in $\mathrm{Gr}(n-1, 2n)$ i.e. the dimension of the row space of $\Omega_n(e)$ is equal to $n-1$ and each $n-1 \times n-1$ minor (each Plücker coordinate) of the matrix $\Omega_n(e)$ is non-negative.

From electrical networks to the non-negative Grassmannians

Bychkov, Gorbounov, Guterman, Kazakov, «Symplectic geometry of electrical networks».

Theorem

Consider an electrical network $e(\Gamma, \omega)$. Using its resistance matrix $R = (R_{ij})$ let us define a point of the Grassmannian $\mathrm{Gr}(n-1, 2n)$ associated to e as the row space of the matrix:

$$\Omega_{n,R}(e) = \begin{pmatrix} 1 & m_{11} & 1 & -m_{12} & 0 & m_{13} & 0 & \dots \\ 0 & -m_{21} & 1 & m_{22} & 1 & -m_{23} & 0 & \dots \\ 0 & m_{31} & 0 & -m_{32} & 1 & m_{33} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $m_{ij} = -\frac{1}{2}(R_{i,j} + R_{i+1,j+1} - R_{i,j+1} - R_{i+1,j})$. Then $\Omega_{n,R}(e)$ defines a non-negative point of Grassmannian $\mathrm{Gr}(n-1, 2n)$ i.e. dimension of the row space of $\Omega(e)$ is equal to $n-1$ and each $n-1 \times n-1$ minors (each Plücker coordinate) of the matrix $\Omega(e)$ is non-negative.

Examples

$$\Omega_4(e) = \begin{pmatrix} x_{11} & 1 & -x_{12} & 0 & x_{13} & 0 & -x_{14} & 1 \\ -x_{21} & 1 & x_{22} & 1 & -x_{23} & 0 & x_{24} & 0 \\ x_{31} & 0 & -x_{32} & 1 & x_{33} & 1 & -x_{34} & 0 \\ -x_{41} & 0 & x_{42} & 0 & -x_{43} & 1 & x_{44} & 1 \end{pmatrix}$$

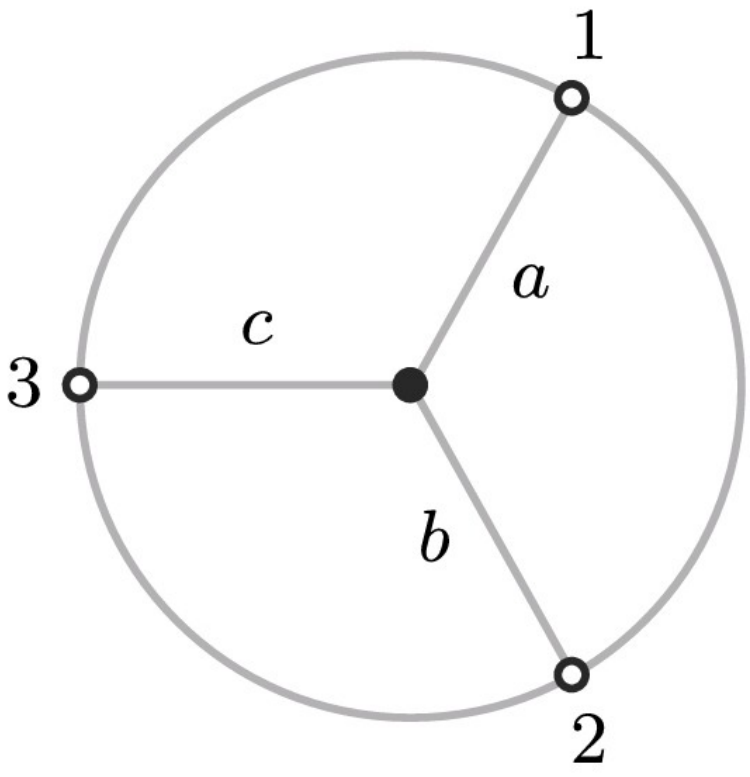
$$\Omega_{4,R}(e) = \begin{pmatrix} 1 & m_{11} & 1 & -m_{12} & 0 & m_{13} & 0 & -m_{14} \\ 0 & -m_{21} & 1 & m_{22} & 1 & -m_{23} & 0 & m_{24} \\ 0 & m_{31} & 0 & -m_{32} & 1 & m_{33} & 1 & -m_{34} \\ 1 & -m_{41} & 0 & m_{42} & 0 & -m_{43} & 1 & m_{44} \end{pmatrix}$$

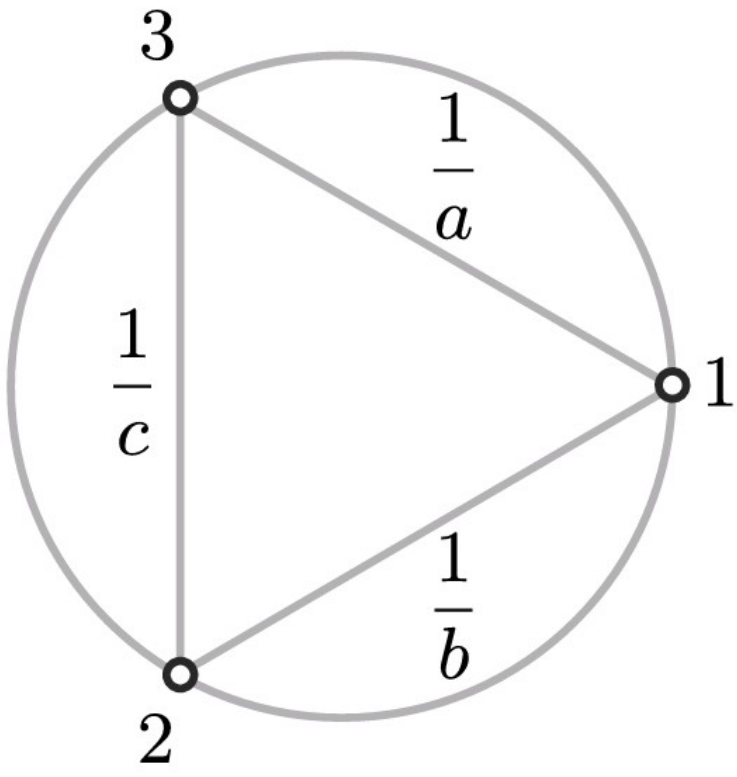
Some Combinatorial Formulas

- For each electrical network e,
$$x_{ij} = \frac{L_{ij|*|*|\ldots}}{L_{1|2|3|\ldots}};$$
- Let e be an electrical network and e^* its dual. Then
$$x^*_{ij} = -\tfrac{1}{2}\big(R_{ij} + R_{i+1,j+1} - R_{i,j+1} - R_{i+1,j}\big);$$
- Each ratio
$$\frac{L_\sigma}{L_{1|2|3|\ldots}}$$
is a homogeneous polynomial in the variables x_{ij} (for $i \neq j$) with integer coefficients.
- A set R_{ij} defines a Kalmanson metric on the boundary nodes; that is, for any four boundary i_1, i_2, i_3, i_4 taken in circular order one has
$$R_{i_1 i_3} + R_{i_2 i_4} \geq \max\{R_{i_1 i_2} + R_{i_3 i_4}; R_{i_2 i_3} + R_{i_1 i_4}\}.$$

Bychkov, Kazakov, Talalaev, «Modern theory of electrical networks: from matrix tree theorem to the theory of cluster varieties»

Dual Electrical Networks





From electrical networks to the non-negative Grassmannians

Bychkov, Kazakov, Talalaev, «Electrical networks, Lagrangian Grassmannians and symplectic groups»; Lam, «Electroid varieties and a compactification of the space of electrical networks»

Theorem

The embedding $E_n \hookrightarrow \mathrm{Gr}_{\geq 0}(n-1, 2n)$ *extends to an embedding* $\mathcal{L}\colon \overline{E}_n \hookrightarrow \mathrm{Gr}_{\geq 0}(n-1, 2n)$, *where* $\overline{E}_n$ *denotes the set of electrical networks modulo electrical transformations whose conductivities are allowed to be* 0 *or* ∞.

Denote by $\mathcal{X}_{\geq 0}$ the image of $\mathcal{L}$.

Theorem

Each point of $Y \in \mathcal{X}_{\geq 0}$ *can be rewritten in the form:*

$$Y = \sum_{\sigma} L_\sigma w_\sigma, \ L_\sigma \geq 0,$$

where $w_\sigma = \mathcal{L}(e_\sigma)$; e_σ *is an electrical network with all* 0 *or* ∞ *conductivities.*

From electrical networks to the Lagrangian Grassmannians

Denote by $\mathcal{X}$ the subset of $\mathrm{Gr}(n-1,2n)$ consisting of all points that can be written in the form

$$\sum_{\sigma} L_{\sigma} w_{\sigma}.$$

Theorem

The variety $\mathcal{X}$ is isomorphic to $\mathrm{LG}(n-1,2n-2)$.

Theorem

Consider a point in $\mathcal{X}$ then its Plücker coordinates have the form:

$$\Delta_I = \sum_{\sigma} a_{\sigma,I} L_{\sigma}, \;\; a_{\sigma,I} \in \{0,1\}.$$

From electrical networks to the Lagrangian Grassmannians

Gao, Lam, Xu, «Electrical networks and the grove algebra»

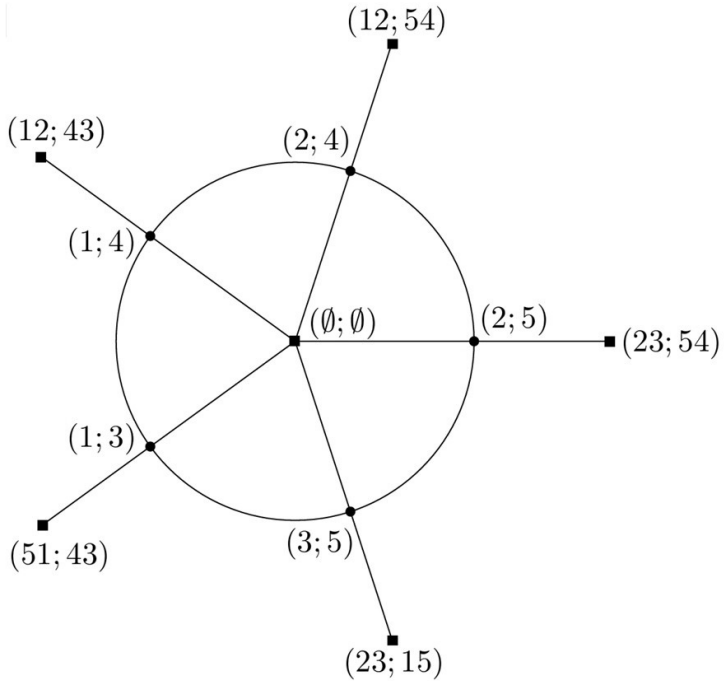
Theorem

The homogeneous coordinate ring of $\mathcal{X}$ is naturally isomorphic to the grove algebra $\mathcal{G} = \mathbb{C}[L_\sigma]/\mathcal{I}_n$ – the polynomial algebra over $\mathbb{C}$ in variables $\{L_\sigma |\ \sigma \in \mathcal{NC}_n\}$ modulo the ideal $\mathcal{I}_n$ generated by electrical Plücker relations:

$$\sum_{\sigma \in E(I), \kappa \in E(J)} L_\sigma L_\kappa = \sum_{I', J'} (-1)^{a(I', J')} \sum_{\sigma \in E(I'), \kappa \in E(J')} L_\sigma L_\kappa,$$

Cluster Structure on Grove Algebra

Bychkov, Guterman, Kazakov, «Electrical networks, grassmannians, and cluster algebras»



$$(\emptyset;\emptyset) \to 1|2|3|4|5$$
$$(1;4) \to 14|2|3|5 \quad (12;43) \to 14|23|5$$
$$(2;4) \to 24|1|3|5 \quad (12;54) \to 15|24|3$$
$$(2;5) \to 25|1|3|4 \quad (23;54) \to 25|34|1$$
$$(3;5) \to 35|1|2|4 \quad (23;15) \to 12|35|2$$
$$(1;3) \to 13|2|4|5 \quad (51;43) \to 54|13|2$$

Theorem

The localization of $G_{L_{1|2|\ldots|n}}$ has the cluster algebra structure.

Characterization

Bychkov, Kazakov, Talalaev, «Electrical networks, Lagrangian Grassmannians and symplectic groups»; Lam, «Electroid varieties and a compactification of the space of electrical networks»

Theorem

A matrix $M = (x_{ij}) \in Mat_{n\times n}$ is a response matrix of an electrical network with n boundary nodes if and only if:

- *M — symmetric;*
- $\sum_k x_{ik} = 0$;
- $\Omega_n(M) \in \mathrm{Gr}_{\geq 0}(n-1, 2n)$.

This result allows us to derive simple necessary-and-sufficient criteria for a matrix to be a network response matrix.

Bychkov, Guterman, Kazakov, «Electrical networks, grassmannians, and cluster algebras».

Inverse Problems

Gorbounov, Kazakov, «Electrical networks and data analysis in phylogenetics»

Theorem

Up to the star-triangle transformations, a topology of each minimal electrical network $e(G, w)$ with n boundary nodes can be uniquely recovered by a column «rank-patterns» of $\Omega_n(e)$ (or $\Omega_{n,R}(e)$).

Kazakov, «Inverse problems related to electrical networks and the geometry of non-negative Grassmannians»

Theorem

The generalized chamber ansatz applied to $\Omega_n(e)$ (or $\Omega_{n,R}(e)$) provides a non-recursive algorithm for recovering the conductivity function ω,