On Entanglement Entropy: another point of view

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PROBLEMS OF MODERN MATHEMATICAL PHYSICS

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Towards understanding Quantum Dynamics

- Understanding the role of Entanglement in Quantum systems

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  - Dynamics in deformed and undeformed holographic backgrounds
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- Our focus:
  - Entaglement in finitely many systems
  - Mathematical structures in such systems

Information

Definition: An information category is a collection $\mathcal{C}$ of random variables (on a common probability space) such that:

1. If $X \in \mathcal{C}$ and Y is a coarsening of X (i.e., $Y = f(X)$ for some function f), then $Y \in \mathcal{C}$.
2. If $X, Y \in \mathcal{C}$, then their join $X \vee Y$ (the random variable corresponding to the coarsest common refinement of their partitions) is in $\mathcal{C}$.
3. The trivial random variable $\mathbf{1}$ (constant everywhere) belongs to $\mathcal{C}$.

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$\Longrightarrow$ All information quantities are organized in a pair of categories:
• *Category I* (data on a system) : generates an algebraic structure like a monoid, or more generally a monad.
- $X \vee Y \rightarrow$ less fine partition that is finer than X & $Y \Longrightarrow$ monoid on the set $\Pi(\mathcal{C})$ of partitions of $\mathcal{C}$, with $\forall X,\ XX = X$.

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• *Category II* (on parameters: probability laws, coefficients of equations etc): generates a representation of the algebraic structure.

Formalization

Definition: Information structures are pairs (S, E), where: S is a poset; objects: observables; arrows: the refinement relations; E is a functor on S, that associated with each observable the set of possible values it can assume.

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• $S = \{e_1, e_2, \ldots, e_n\}$: finite choice system & $p_i = p(e_i)$: a probability distribution.

$\Longrightarrow$ The measure of uncertainty: $H(p_1, p_2, \ldots, p_n)$ of $\{p_1, p_2, \ldots, p_n\}$ is only defined by the conditions on H:

- (H1) $H(p_1, p_2, \cdots, p_n) = H(p_{\sigma(1)}, p_{\sigma(2)}, \cdots, p_{\sigma(n)})$ for any permutation σ of $1, 2, \cdots, n \Longrightarrow$ the measure does not depend on the numbering of the possible choices
- (H2) $H(p_1, p_2, \cdots, p_n)$: a continuous function in all variables $\Longrightarrow$ (requires δp_i provoke small $\delta H(p_i)$.

Formalization

- (H3) The equation (grouping axiom)

$$H(p_1, p_2, \cdots, p_n) = H(p_1 + p_2, p_3, \cdots, p_n) + \\ + (p_1 + p_2) H\big(p_1/(p_1 + p_2), p_2/(p_1 + p_2)\big)$$

  holds for all probability distributions $p_1, p_2, \cdots, p_n \Longrightarrow$ simple form of a chaining formula.
- (H4) For a uniform probability distribution $p_i = 1/n$ for $i = 1, \cdots, n$, $H(p_1, \cdots, p_n) = H(1/n, \cdots, 1/n)$ as a function of $n \geq 1$ is monotone increasing $\Longrightarrow$ with uniform distribution the uncertainty should increase with the number of possibilities

Theorem A. (H1), (H2), (H3) and (H4) are satisfied if, and only if,

$$H(p_1, p_2, \cdots, p_n) = -\sum_{i=1}^{n} p_i \log p_i.$$

Formalization

Theorem B Let $\alpha > 0$ and $u : [0,1] \to \mathbb{R}$ be a measurable function which satisfies (*Findamental equation of iformation theory*)

$$\forall (x,y) \in [0,1)^2 \quad \text{such that } x+y \in [0,1]$$
$$u(1-x) + (1-x)^\alpha u\left(\frac{y}{1-x}\right) = u(y) + (1-y)^\alpha u\left(\frac{1-x-y}{1-y}\right) \tag{1}$$

Then exists a real number λ such that $\forall x \in [0,1]$,

$$u(x) = \begin{cases} \lambda\Big(-x\log_2(x) - (1-x)\log_2(1-x)\Big) = \lambda S_1(x,1-x), & \text{if } \alpha = 1 \\ \lambda\left(\frac{1}{1-\alpha}(x^\alpha + (1-x)^\alpha - 1\right) = \lambda S_\alpha(x,1-x), & \text{if } \alpha \neq 1 \end{cases} \tag{2}$$

With the conventions $\log_2 0 := \lim_{x\to 0} x\log_2 x = 0$.

$\Longrightarrow$ *These theorems provide a direct link to the entropies.*

von Neumann

♦ The von Neumann entropy: a measure of how quantum information is stored in a quantum state.

- Pure state: a quantum system whose quantum state $|\Psi\rangle$ is known *exactly*

  $\Longrightarrow$ can be expressed as a superposition of the basis vectors of an orthonormal basis $B = \{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ as:

$$|\psi\rangle = \sum_{i=1}^{n} a_i |b_i\rangle, \qquad \sum_{i=1}^{n} a_i = 1$$

  $\Longrightarrow$ A pure state that is a linear combination of two or more component states is sometimes called a coherent superposition.

  $\Longrightarrow$ The evolution of a closed quantum system can be completely described as a *unitary transformation* of pure states in a Hilbert space.

von Neumann

- The density matrix of a pure state $|\Psi\rangle \rightarrow$ outer product:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha_1\alpha_1^* & \alpha_1\alpha_2^* & \cdots & \alpha_1\alpha_n^* \\ \alpha_2\alpha_1^* & \alpha_2\alpha_2^* & \cdots & \alpha_2\alpha_n^* \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n\alpha_1^* & \alpha_n\alpha_2^* & \cdots & \alpha_n\alpha_n^* \end{pmatrix}$$

  $\rho^\dagger = \rho$ & $\mathrm{tr}(\rho) = \sum_{i=1}^n \alpha_i\alpha_i^* = 1.$

- A *mixed state* is a statistical ensemble of different *pure states* $\{(|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_n\rangle, p_n)\}$ w/ $\sum_{i=1}^n p_i = 1.$
  – The density matrix of a mixed state:

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|.$$

  The density matrix of a mixed state: Hermitian operator,; trace one.

Entanglement Entropy: definitions

Von Neumann entanglement entropy

– The bipartite von Neumann entanglement entropy S; pure state:

$$\mathcal{S}(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] = -\operatorname{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B)$$

$\rho_A = \operatorname{Tr}_B(\rho_{AB})$ and $\rho_B = \operatorname{Tr}_A(\rho_{AB})$ - the reduced density matrices.

– Explicily,

$$|\Psi\rangle = \sum_{i=1}^{m} \alpha_i |u_i\rangle_A \otimes |v_i\rangle_B \qquad \Longrightarrow \qquad \mathcal{S}(\rho_A) = -\sum_i \alpha_i^2 \log(\alpha_i^2)$$

Renyi and von Neumann entanglement

The Rényi entanglement entropies $\mathcal{S}_\alpha$:

$$\mathcal{S}_n(\rho_A) = \frac{1}{1-n} \log \operatorname{tr}(\rho_A^n) = \mathcal{S}_\alpha(\rho_B),$$

n - folding number ($n = \alpha$ above).

A relation between Ranyi and on Neumann entanglement:

$$\mathcal{S}(\rho) = \lim_{\alpha \to 1^+} \mathcal{S}_\alpha \tag{3}$$

Holographic Entanglement Entropy

Holographic Entanglement: relates quantum entanglement in a CFT to geometric features in an anti-de Sitter (AdS) spacetime.

- *The Ryu–Takayanagi Formula*: for static state in a CFT_d dual to $\text{AdS}_{d+1} \to S_A$ of a boundary spatial region A:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

  γ_A - the minimal-area extremal surface (ES) in AdS; homologous to A (i.e., $\partial\gamma_A = \partial A$) $\Longrightarrow$ generalizes Bekenstein–Hawking BH entropy to general quantum states; identifies entanglement entropy with a geometric measure in the gravitational dual.

- *Extensions and Refinements*: HRT formula (Hubeny-Rangamani -Takayanagi): extends RT to dynamical spacetimes; Quantum corrections; Island formula: In evaporating BH contexts, the EE of radiation includes contributions from islands - regions inside the BH connected via quantum ES - resolving the information paradox.

Implications

- *Physical Implications*
  – Geometry from entanglement: HEE suggests spacetime connectivity emerges from entanglement structure ("ER = EPR").
  – Entanglement wedge reconstruction: The bulk region bounded by $A \cup \gamma_A$ (the entanglement wedge) is dual to the boundary region A; this defines the holographic subregion duality.
  – Quantum corrections encode bulk information: The quantum extremal surface formula incorporates bulk entropy, capturing fine-grained quantum information.
- *Applications*
  - Recovers known CFT results for 2D theories.
  - Satisfies entanglement inequalities (subadditivity, monogamy).
  - Provides a geometric interpretation of thermodynamic entropy in BH physics.
  - Underlies recent progress on black hole information paradox via the island rule.

Multipartite system: Notations

- A general Hilbert space of multipartite system:

$$\mathcal{H}_{I_1 \ldots I_n} = \bigotimes_{I_i = I_1}^{I_n} \mathcal{H}_{I_i} = \mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2} \otimes \ldots \otimes \mathcal{H}_{I_n} \,. \tag{4}$$

- The basis of $\mathcal{H}_{I_i}$ has an basis $|B_{I_i}\rangle = \{|e_{I_i=1}\rangle, \ldots, |e_{I_i=d_i}\rangle\}$

$$|\Psi\rangle_{I_1 \ldots I_n} = \sum_{I_1=1}^{d_1} \cdots \sum_{I_n=1}^{d_n} \psi_{I_1 I_2 \ldots I_n} |e_{I_1}\rangle \otimes |e_{I_2}\rangle \ldots |e_{I_n}\rangle \tag{5}$$

– *pure state*

$$|\psi_{I_1 \ldots I_n}\rangle = |\psi_{I_1}\rangle \otimes |\psi_{I_2}\rangle \otimes \cdots \otimes |\psi_{I_n}\rangle$$

– When $|\Psi\rangle_{I_1 \ldots I_n}$ cannot be represented as product of states as above it is called *entangled state*.

Density matrix

• A generic *density matrix* ρ: convex linear conbinations of *pure states* density matrices

$$\rho = \sum_{l-1}^{n} p_l |\psi_l\rangle\langle\psi_l|, \tag{6}$$

$\sum p_l = 1$. Note: if the state is entangled $\Longrightarrow \mathrm{tr}(\rho) = 1$ & $\mathrm{tr}(\rho^2) < 1$.
– For an arbitrary mixed state ρ can be diagonalized in an ortho-basis $\{|\alpha_l\rangle\}$ via Schmidt/Singular Value Decomposition (SVD)

$$\rho = \sum_{l=1}^{M} \lambda_l |\alpha_l\rangle\langle\alpha_l|.$$

– Projector onto the subsectors: it projects on the image of the density matrix $\mathrm{image}(\rho)$. For given ρ

$$\Pi_\rho = \sum_{l=1}^{M} |\alpha_l\rangle\langle\alpha_l|. \tag{7}$$

Partial trace

– A generic operator $\mathcal{O}_{I_1 \ldots I_n} : \mathcal{H}_{I_1 \ldots I_n} \longrightarrow \mathcal{H}_{I_1 \ldots I_n}$

$$\mathcal{O}_{I_1 \ldots I_n} = \sum_{I_1=1}^{d_1} \cdots \sum_{I_n=1}^{d_n} o_{e_{\{I\}}}^{f_{\{I\}}} |e_{I_1}\rangle\langle f_{I_1}| \otimes |e_{I_2}\rangle\langle f_{I_2}| \otimes \cdots \otimes |e_{I_n}\rangle\langle f_{I_m}|$$

Partial trace with respect to J_r-th component of $\mathcal{H}$

$$\mathrm{tr}_{J_r}(\mathcal{O}_{I_1 \ldots I_n}) = \sum_{J_r=1}^{d_r} \langle \alpha_{J_r} | \mathcal{O}_{I_1 \ldots I_n} | \beta_{J_r} \rangle$$

$$= \sum_{\{I_l\}=1}^{d_1,\ldots,d_n} o_{e_{\{I\}}}^{f_{\{I\}}} |e_{I_1}\rangle\langle f_{I_1}| \otimes |e_{I_2}\rangle\langle f_{I_2}| \otimes \cdots \sum_{J_r=1}^{d_r} (\langle \alpha_{J_r} || e_{J_r} \rangle)\langle f_{J_r} || \beta_{J_r} \rangle)$$

$$\otimes \cdots \otimes |e_{I_n}\rangle\langle f_{J_m}| \quad (8)$$

$\Longrightarrow$

$$\mathrm{tr}_{J_r}(\mathcal{O}_{\{I\}}) = \sum_{\{I_l\},\{J_l\}=1}^{d_1,\ldots,d_n} o_{e_{\{I\}}}^{f_{\{I\}}} |e_{I_1}\rangle\langle f_{J_1}| \otimes \ldots \cancel{|e_{J_r}\rangle\langle f_{J_r}|} \cdots \otimes |e_{I_n}\rangle\langle f_{J_n}| \quad (9)$$

Partial traces and associated subspaces

- generic operatos acts as $\mathcal{O}: \mathcal{H} \to \mathcal{H}, \qquad \mathcal{H} = \bigotimes_{I_i} \mathcal{H}_{I_i}$.

- $\mathcal{O}_{I_1 \ldots I_n}: \mathcal{H}_{I_1 \ldots I_n} \longrightarrow \mathcal{H}_{I_1 \ldots I_n} \Longrightarrow$ *not* related to ρ_i in general ;
$\Longrightarrow$ cannot associate image w/ relevant $p(x)$ to form info-structure
$\Longrightarrow$ cannot be used further $\Longrightarrow$ as it is $\to$ construct the right relations.

- The same applies for $\mathrm{tr}_{I_r} \mathcal{O}_{I_1 \ldots I_n}: \mathcal{H}_{I_1 \ldots I_n} \to \mathcal{H}_{\ldots \not{I_r} \ldots}$.

$*$ Consider $\rho_{\{I\} \setminus J_r}: \mathcal{H}_{\{I\}} \to \mathcal{H}_{\{I\} \setminus J_r}$
– For $K \subset J \subset I \quad \Longrightarrow \quad$ restriction map

$$\mathrm{tr}_J \left(\mathrm{tr}_K \rho_I \right) = \mathrm{tr}_{K \cup J} \rho_I = \rho_{I \setminus K \cup J} \quad \Longrightarrow \quad \mathrm{tr}_J \circ \mathrm{tr}_K = \mathrm{tr}_{K \cup J} \, .$$

– to make $\mathcal{O}$ acting *on* $\mathrm{image}(\rho_{\{I\} \setminus J_r}) \Longrightarrow$ use projectors Π_ρ eq. (7).

- *Importan*: after taking trace of ρ we are in its image $\to$ make sure that after the action on $\mathcal{O}$ we are still in that subspace(to preserve $\mathrm{tr}(\rho)) = 1$), i.e. one must use again Π_ρ.

Building spaces of operators.

Constructing ${}^{\rho}\mathcal{H}_k$:

$$ {}^{\rho}\mathcal{H}_k := \Pi_{\rho}\mathcal{H}_k \quad \& \quad \text{analogoulsly for } {}^{\rho_{\{I\}\setminus J_r}}\mathcal{H}_k. $$

- For operators define ${}^{\rho_*}\mathcal{O}_{\dots} := \Pi_{\rho_{\{*\}}}\mathcal{O}_{\dots}$.
- For the partial trace $\mathrm{tr}_*\,\mathcal{O} : \mathcal{H} \to \mathcal{H}_{\not *}, \quad \mathcal{H}_{\not *} \subset \mathcal{H} \Longrightarrow$

$$ {}^{\rho_*}\mathcal{O}_{\dots} := \mathcal{H} \longrightarrow {}^{\rho_*}\mathcal{H}, \quad {}^{\rho_*}\mathcal{H} \subset \mathcal{H} $$

Building C_k *spaces*

- The analog of "0-forms", C_0 - simply complex numbers,

$$ C_0(\rho_{1,\dots,n}) = \mathbb{C}, \tag{10} $$

- The analog of "top form", C_n space - just linear operators acting on the image of ρ

$$ C_n(\rho_{1,\dots,n}) = \mathrm{End}\left(\,\mathrm{image}(\rho_{1\dots,n})\right) \cong \mathbb{C}. \tag{11} $$

- Others, acting on images of ρ_{I_k}:

$$ C_k(\rho_{1,\dots,n}) = \bigtimes_{|I_k|=k} \mathrm{End}\left(\,\mathrm{image}(\rho_{I_k})\right) \tag{12} $$

Further refinement

- For an inclusion $V \subseteq U \rightarrow$ define the restriction:

$$r_V^U(\star) := \mathsf{Tr}_{U\setminus V}\big[\star(\mathbf{1}_V) \otimes \tau_{U\setminus V}\big] \implies r_V^U : \star(U) \rightarrow \star(V), \qquad (13)$$

– From $\tau_{U_1\setminus U_3} = \tau_{U_2\setminus U_3} \otimes \tau_{U_1\setminus U_2}$ and the cyclicity functoriality follows

$$r_{U_3}^{U_2} \circ r_{U_2}^{U_1} = r_{U_3}^{U_1}.$$

- For every $V \subset U$ define the linear extension ($\tau_{U\setminus V} := \bigotimes_{j\in U\setminus V_j} \tau_j$)

$$e_V^U : V(V) \rightarrow V(U), \quad e_V^U(X) := X \otimes \tau_{U\setminus V} \;\Big|\; \mathsf{Tr}(\tau_{U\setminus V}) = 1.$$

$$\implies \qquad r_V^U = \mathsf{Tr}_{U\setminus V} \text{ **satisfies** } r_V^U \circ e_V^U = \mathbf{1}_{V(V)}.$$

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$\implies$ *The correct action* of (reduced) operator on the *ordered* $\mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2} \otimes \ldots \otimes \mathcal{H}_{I_n}$:

a) re-orded the product pulling at the first place the space on which a reduced operator has to act;

b) apply the operator

c) re-order back to increasing order.

Hochschild (co)homology

Two ways of constructing Hochschild homology.
A) Let A be an associative algebra over a projective A-module & M be an A-bimodule. $\implies$ the Hochschild complex is the chain complex

$$C^n(A, M) := \hom(A^{\otimes n}, M) \quad \forall n; \quad C^0(A, M) = M$$

- For fixed $n \to$ define maps

$$f_{n,i}(ma_1 \otimes \cdots \otimes a_n) = \begin{cases} (ma_1) \otimes \cdots \otimes a_n & \text{if } i = 0 \\ (a_n m) \otimes \cdots \otimes a_{n-1} & \text{if } i = n \\ m \otimes a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n & \text{otherwise} \end{cases}$$

$\implies$ define the boundary operators $b_n : C_n \to C_{n-1}$ as

$$b_n = \sum_{i=0}^{n} (-1)^i f_{n,1} \tag{14}$$

Since $b_{n-1} \circ b_n = 0 \implies (C_n(A, M), b_n)$ forms a chain complex. *The Hochschild homology of is the homology of this complex.*

Hochschild (co)homology

B) Another perspective: $f_{n,i}$ make the family $C_n(A, M)$ a simplicial object in the category of R-modules. The map b_n in this case is called *face map*

& the *degeneracy maps* are

$$\delta_i(a_0 \otimes \cdots, \otimes a_n) = \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

The *Hochschild homology* can also be constructed as the homology of this simplicial module:

Let $Z_n(A, M) := \ker b_n$ and $B_n(A, M) := \text{range}(b_{n+1})$
$\Longrightarrow$ the Hochschild homology is defined as

$$HH_n(A, M) = \frac{Z_n(A, M)}{B_n(A, M)} \quad \text{for each integer } n \geq 0$$

.

Hochschild (co)homology

- Dualizing to Cochains

- For a bimodule M, cochains are dual to chains.: n-cochain is a linear map:

$$f : C_n(A) = A^{\otimes(n+1)} \to M.$$

Def. co-bdy operator $\delta^n : C^n(A, M) \to C^{n+1}(A, M)$ dual to b_n:

$$(\delta^n f)(a_1, \ldots, a_n) := f(b_{n+1}(a_1 \otimes \cdots \otimes a_{n+1})).$$

$$b_{n+1} : A^{\otimes(n+1)} \to A^{\otimes(n)} \quad \Longrightarrow \quad f \circ b_{n+1} : A^{\otimes(n+1)} \to M$$

$\Longrightarrow$ the coboundary is composition with the boundary:

$$\delta^n f = f \circ b_{n+1}.$$

- Cocycle Condition in bdy terms: $\delta^n f = 0 \quad \Longleftrightarrow \quad f \circ b_{n+1} = 0.$

Hodge-Hochschild Correspondence Table

Classical (Hodge)	**Noncommutative** (Hochschild + ρ)
Algebra of functions $C^\infty(M)$	Alg of observables A (e.g., matrices)
k-forms $\Omega^k(M)$	Hochschild k-cochains $C^k(A) = \mathrm{Hom}(A^{\otimes k}, A)$
Exterior derivative d	Hochschild coboundary b
de Rham cohomology $H^\bullet_{dR}(M)$	Hochschild cohomology $HH^\bullet(A)$
Riemannian metric g	ρ (or more generally, a weight/state)
Volume form $d\mathrm{vol}_g$	Modular operator Δ_ρ & J-antilinear map (Tomita theory)
Inner product on forms $\int \alpha \wedge \star\beta$	inner product $\langle \phi, \psi \rangle_\rho$ via GNS/cyclic completion
Hodge star $\star$	Modular operator Δ_ρ & J-antilinear map (Tomita theory)
Laplacian $\Delta = dd^* + d^*d$	Noncomm Laplacian $\Box_\rho = bb^*_\rho + b^*_\rho b$
Harmonic forms $\Delta\omega = 0$	Harmonic cochains $\Box_\rho \phi = 0$

Addapting Hochschild cohomology to Entanglement

In Hochschild cohomology

- a cochain $\phi : A^{\otimes k} \to A$ is just a multilinear functional
- Cyclic cochains:

$$\phi(a_1, \ldots, a_k, a_0) = (-1)^k \phi(a_0, a_1, \ldots, a_k)$$

  for all $a_0, a_1, \ldots, a_k \in A$.

- The completion

$$\phi_{\text{cyc}}(a_0, ..., a_k) = \frac{1}{k+1} \sum_{i=0}^{k} (-1)^{ki} \phi(a_i, ..., a_k, a_0, ..., a_{i-1})$$

$\Longrightarrow$ Cyclic cohomology $\to$ ideally designed to inorporate both, ρ and Tr

Density matrix as the bridge to cyclicity

The density matrix ρ provides a natural trace-class operator:

$$\tau_\rho(a) = \mathsf{Tr}(\rho a) \quad \Longrightarrow \quad \mathsf{Tr}(\rho a_0 a_1 \dots a_k) = \mathsf{Tr}(\rho a_k a_0 a_1 \dots a_{k-1})$$

- Mathematical structure: from Hochschild to Cyclic

$*$ The precise mathematical completion:
— Hochschild complex $C^k(A) = \mathsf{Hom}(A^{\otimes k}, A)$
— Cyclic subspace $C^k_\lambda(A) \subset C^k(A)$of λ-invariant cochains (λ = cyclic permutation)
$\rightarrow$ Cyclic cohomology $HC^k(A)$ from the subcomplex

$*$ With density matrix ρ: we get a map:

$$\Psi_\rho : C^k(A) \rightarrow C^k_\lambda(A)$$

$$\Psi_\rho(\phi)(a_0, \dots, a_k) = \mathsf{Tr}(\rho\phi(0_0, \dots, a_k)) + \text{cyclic perm w/ modular twist}$$

What we can do w/ Hohchschild cohomology

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- *Holography and Hochschild Duality*

$*$ In AdS/CFT: for bdry region A with reduced $\rho_A = \mathrm{Tr}_{\bar{A}}(\rho_{CFT})$
$\rightarrow$ modular Hamiltonian $K_A = -log\rho_A$ satisfies:

$$\langle K_A \rangle = S(\rho_A) = \frac{\mathrm{Area}(\gamma_A)}{4G_N} + \text{bulk entropy}$$

In Hochschild terms, K_A represents a cyclic 1-cocycle τ_A on boundary algebra $\mathcal{A}(\partial M)$:

$$\tau_A(a_0, a_1) = \mathrm{Tr}(\rho_A a_0 [K_A, a_1])$$

The holography: τ_A, which comes from partial trace on boundary, equals (via bulk-boundary correspondence) a cocycle on bulk algebra coming from geometric data (area operator).
$\Longrightarrow$ This suggests a Hochschild duality:

$$HH^1(\mathcal{A}_{\mathsf{CFT}}(A)) \cong HH^1(\mathcal{A}_{\mathsf{bulk}}(W[A]))$$

induced by partial trace + holographic dictionary.

. . . more . . .

• *Entropy Production as a Cyclic Cocycle*

Consider a nonequilibrium quantum system driven by time-dependent Hamiltonian $H(t)$ & initial state ρ_0.

– The entropy production over a cycle expressed as a cyclic 1-cocycle:

Let $U(t)$ be time evolution $\rightarrow$ define relative entropy cyclic cochain:

$$S(a,b) = \mathsf{Tr}(\rho_0 a^*(\log \rho_0 - \log \rho_t)b)$$

where $\rho_t = U(t)\rho_0 U(t)^\dagger$.

– Modular flow:

Under cyclic completion with modular flow of ρ_0:

$$S_{\mathsf{cyc}}(a,b) = \frac{1}{2}[S(a,b) - S(\sigma_t^{\rho_0}(b), a)] \quad | \quad \sigma_t^{\rho_0}(b) = \rho_0^{it}\, b\, \rho_0^{-it}$$

$\Longrightarrow$ **measures "time asymmetry = entropy production".**

$*$ For a periodically driven system (Floquet), the time-average of S_{cyc} computes the non-adiabatic entropy production rate .

. . . more . . .

- Studying modular flow: given a faithful normal state $\varphi_\rho(a) = \mathrm{Tr}(\rho a)$ $\rightarrow$ define the modular automorphism group:

$$\sigma_t^\rho(a) = \rho^{it} a \rho^{-it}$$

- State as Geometry: in QM, ρ completely specifies the "geometry" of the observable algebra in noncommutative sense. It provides both the "integration" and the "metric" data. Different states (e.g., thermal states at different temperatures) induce different "geometries" on the same algebra of observables.
- Modular Theory as Gravity: The modular automorphism group $\sigma_t^\rho(a) = \rho^{it} a \rho^{-it}$ associated with ρ is a purely noncommutative object; plays a role like the flow generated by a canonical vector field $\rightarrow$ deep connections between stat mechanics, QFT, and gravity .
- Entropy as Geometric Invariant: vN entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ emerges naturally, analogous to geometric invariants like the Einstein-Hilbert action. In fact, the Connes-Chern character of the cyclic cocycle defined by ρ encodes such information.

Summary

Conceptual Summary

The partial trace is not merely a mathematical tool - it embodies fundamental physical principles:

1. Locality and Causality : You can only access your local subsystem; partial trace tells you what you can know.
2. Information Distillation : From global state to local information, respecting quantum constraints.
3. Entanglement as Resource : The difference between ρ_{AB} and $\rho_A \otimes \rho_B$ (measured by mutual information) quantifies entanglement.
4. Emergence of Classicality : When systems become macroscopic, partial traces over environmental degrees of freedom lead to effective classical behavior (decoherence).
5. Holography : Partial traces in boundary theory encode bulk geometry - the ultimate reduction of degrees of freedom.

Conclusions

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Main result:

Presented construction shows that:

- Cyclic cohomology with coefficients in endomorphism algebras of reduced density matrices is equivalent to Hochschild cohomology of the algebra of locally consistent operators.
- We call this equivalence *Hochschild entanglement cohomology*.

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Presented construction shows that:

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Further

- Extension to QFT & gravity
- Refinements including relative entropy, modular theory etc
- Extension to QM & Quantum statistical mechanics
- Relations to integrability and chaos etc.

END

THANK YOU!