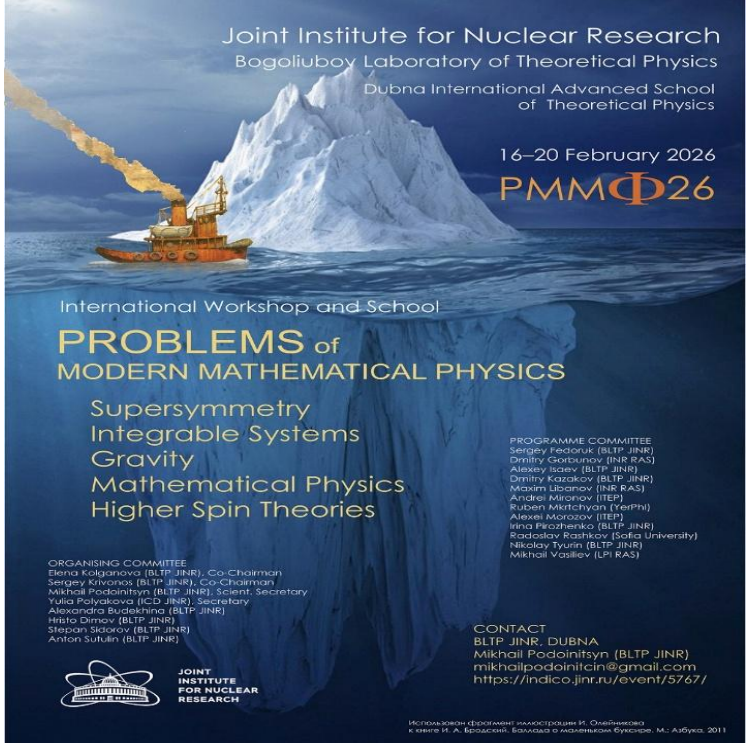


Schwinger-Keldysh-DeWitt technique and covariant pseudodifferential calculus

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Outline

Motivation: wave operator and its heat kernel in context of Lorentz and diffeomorphism invariance of the theory and their violation

$$\hat{H}(\nabla)\,\delta(x,y) = \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}, \qquad \hat{K}(\tau|\,x,y) = e^{-\tau\hat{H}(\nabla)}\,\delta(x,y)$$

Local Schwinger-DeWitt curvature expansion

Extension to higher derivative minimal operators and nonminimal causal operators

Perturbation theory and subtraction procedure

Covariant pseudodifferential calculus

Extension to Ho\v rava-Lifshitz Lorentz violating models

Nonlocal heat kernel and effective action in Lorentz-invariant state in diffeomorphism-invariant theory (A.O.B. and G.A.Vilkovisky, Nucl.Phys. B 333 (1990) 471-511)

$$\Gamma_{1-\mathrm{loop}} = \frac{1}{2}\,\mathrm{Tr}\,\ln H(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K(\tau)$$

$$H(\nabla) \sim -\Box + P, \quad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$$ ***minimal second-order operator***

Curvature expansion

$$\mathrm{Tr}\,K(\tau)=\frac{1}{(4\pi\tau)^\omega}\int dx\, g^{1/2}\,\mathrm{tr}\left\{\hat{1}+\tau\hat{P}+\tau^2\Big(R_{\mu\nu}F_1(-\tau\Box)R^{\mu\nu}\hat{1}+RF_2(-\tau\Box)R\hat{1}+\hat{P}F_3(-\tau\Box)R\right.$$
$$\left.+\hat{P}F_4(-\tau\Box)\hat{P}+\hat{\mathcal{R}}_{\mu\nu}F_5(-\tau\Box)\hat{\mathcal{R}}^{\mu\nu}\Big)+O[\Re^3]\right\}$$

$$f(\xi)=\int_0^1 d\alpha\, e^{-\alpha(1-\alpha)\xi}$$

Fundamental formfactor

Nonlocal formfactors

$$F_1(\xi)=\frac{f(\xi)-1+\frac{1}{6}\xi}{\xi^2},$$
$$F_2(\xi)=\frac{1}{8}\left[\frac{1}{36}f(\xi)+\frac{1}{3}\frac{f(\xi)-1}{\xi}-\frac{f(\xi)-1+\frac{1}{6}\xi}{\xi^2}\right],$$
$$F_3(\xi)=\frac{1}{12}f(\xi)+\frac{1}{2}\frac{f(\xi)-1}{\xi},\quad F_4(\xi)=\frac{1}{2}f(\xi),\quad F_5(\xi)=-\frac{1}{2}\frac{f(\xi)-1}{\xi}$$

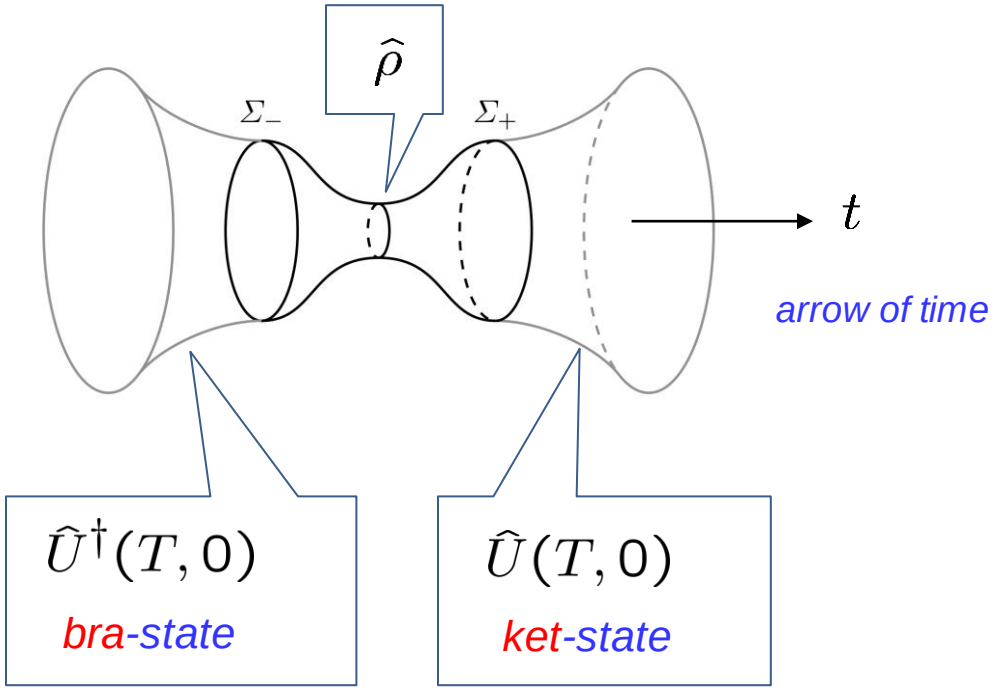
New challenges from cosmology and quantum gravity!

Cosmological context of Schwinger-Keldysh formalism

Closed cosmology with spatial slices $\Sigma \sim S^3$ *instant time,* $S^3 \times R^1$ *topology*

Lorentz symmetry violation of the quantum state --- initial density matrix evolving in time:

$$\hat{U}(T,0)\,\hat{\rho}\,\hat{U}^{\dagger}(T,0)$$



Unitary evolution in real Lorentzian time

$$\hat{U}(T,0) = \mathbb{T}\,\exp\left(-i\int_0^T dt\,\hat{H}(t)\right)$$

"Euclidean evolution" in imaginary time

$$\hat{\rho} = \frac{1}{Z}\,\mathbb{T}\exp\left(-\int_{\Sigma_-}^{\Sigma_+} d\tau\,\hat{H}(\tau)\right)$$

Generalization to Lorentz non-invariant and nonequilibrium state in diffeomorphism-invariant theory (tomorrow talk by Nikita Kolganov)

Another challenge --- diffeomorphism and Lorentz noninvariant theory:

Projectable Horava gravity *– single example of* ***local, unitary, resnormalizable asymptotically free quantum gravity*** *(A.B., D.Blas, M.Herrero-Valea, S.M.Sibiryakov and C.F.Steinwachs, Phys. Rev. D 93, 064022 (2016), arXiv:1512.02250)*

A.O. Barvinsky, A.V. Kurov, S. M. Sibiryakov, beta functions, UV asymptotic freedom, RG flows: Phys.Rev.D 105 (2022) 044009, arXiv:2110.14688; Phys.Rev.D 108 (2023) 12, L121503, arXiv:2310.07841; Phys. Rev. D 111, (2025) 024030, arXiv:2411.13574)

Lorentz symmetry violation $O(3,1) \to O(3)$*, anisotropic scaling invariance*

kinetic term -- unitarity

$$\int d^4x\,\sqrt{g}\,{}^{(4)}R \to \frac{1}{2G}\int dt\,d^3x\sqrt{\gamma}\Big(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}(\gamma)\Big), \qquad K_{ij}$$

extrinsic curvature

Potential term $\mathcal{V}(\gamma) \sim ({}^{(3)}\boldsymbol{R})^3 + (\boldsymbol{\nabla}{}^{(3)}\boldsymbol{R})^2$

spatial curvature

Nonminimal higher-derivative wave operator

$$H(\nabla) \sim \partial_t^2 + F(\boldsymbol{\nabla}), \qquad F(\boldsymbol{\nabla}) \sim \underbrace{\boldsymbol{\Delta}^3 + \boldsymbol{\nabla}^6 + ..., \quad \boldsymbol{\Delta} = g^{ij}\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j}$$

spatial covariant derivatives

Heat kernel and proper time method

Hessian of the theory $\hat{H}(\nabla)\,\delta(x,y) = \dfrac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}$

Proper time method $G \equiv -H^{-1}(\nabla) = \int_0^\infty d\tau\, K(\tau), \quad \frac{1}{2}\,\mathrm{Tr}\,\ln H(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K(\tau)$

Heat kernel and its functional trace $\hat{K}(\tau|\,x,y) = e^{-\tau\hat{H}(\nabla)}\,\delta(x,y), \qquad \mathrm{Tr}K(\tau) = \int dx\,\mathrm{tr}\,\hat{K}(\tau|\,x,x)$

Operator $\hat{H}(\nabla) = H^A_{\ B}(\nabla)$ **acting in the space of fields** $\varphi = \varphi^A(x)$

Generic matrix notation $\hat{P} = P^A_{\ B}$

Minimal second order operator in curved spacetime $\hat{H}(\nabla) = -\Box - \hat{P}, \quad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$

Covariant d'Alembertian

Potential term and curvatures of covariant derivatives $\Re = \hat{P},\ \hat{\mathcal{R}}_{\mu\nu},\ R_{\mu\nu\alpha\beta},$

Heat kernel (Schwinger-DeWitt) expansion for minimal second order operators

Synge world function

$$e^{-\tau\hat{F}(\nabla)}\delta(x,y)=\frac{\Delta^{1/2}(x,y)}{(4\pi\tau)^{d/2}}\,g^{1/2}(y)\,e^{-\frac{\sigma(x,y)}{2\tau}}\sum_{n=0}^{\infty}\tau^n\,\hat{a}_n(x,y)$$

Schwinger-DeWitt (Gilkey-Seely) coefficients and their coincidence limits

$$\hat{a}_0(x,x)=\hat{1},\quad \hat{a}_1(x,x)=\hat{P},$$

$$\hat{a}_2(x,x)=\frac{1}{180}\left(R^2_{\alpha\beta\gamma\delta}-R^2_{\mu\nu}+\Box R\right)\hat{1}+\frac{1}{12}\hat{\mathcal{R}}^2_{\mu\nu}+\frac{1}{2}\left(\hat{P}+\frac{\hat{1}}{6}R\right)^2+\frac{1}{6}\Box\left(\hat{P}+\frac{\hat{1}}{6}R\right),\ldots$$

$$\hat{a}_n(x,x)\propto\overbrace{\nabla\cdots\nabla}^{2p}\overbrace{\mathcal{R}\cdots\mathcal{R}}^{m},\quad m+p=n$$

One-loop divergences $$\Gamma^{\rm div}_{\rm one-loop}=-\frac{1}{32\pi^2\varepsilon}\int d^4x\,g^{1/2}{\rm tr}\,\hat{a}_2(x,x),\quad \varepsilon=2-\frac{d}{2}\to 0$$

Applicable only to minimal second order operators of the form $\hat{F}(\nabla)=-\Box+\hat{P}$ ***!***

Minimal higher-derivative operators

A.B., A.Kurov, W.Wachowski,,
arXiv:2408.10990

Generalized exponential functions

$$e^{-\frac{\sigma(x,x')}{2\tau}} \to \sum_{\alpha} \mathcal{E}_{N,\alpha}\Big(-\frac{\sigma(x,x')}{2\tau^{1/N}}\Big)$$

$$\mathcal{E}_{\nu,\alpha}(z) = \frac{1}{\nu}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\alpha+k}{\nu}\right)}{\Gamma(\alpha+k)}\frac{z^k}{k!}$$

$$\frac{d^n}{dz^n}\mathcal{E}_{\nu,\alpha}(z) = \mathcal{E}_{\nu,\alpha+n}(z)$$

Split form of the 2N-th order minimal operator

$$\hat{F}(\nabla) = (\Box + P)^N + \hat{W}(\nabla), \quad \hat{W}(\nabla) = O\Big[\nabla^{2N-1}\Big]$$

$$\hat{K}_F(\tau|x,x') = \frac{1}{(4\pi\tau^{1/N})^{d/2}}\Bigg\{\sum_{m=0}^{\infty}\tau^{\frac{m}{N}}\sum_{k=0}^{\infty}\mathcal{E}_{N,\frac{d}{2}+Nk-m}\Big(-\frac{\sigma}{2\tau^{1/N}}\Big)\,\hat{a}_{m,k}(F|x,x')$$
$$+\sum_{m=1}^{\infty}\tau^{-\frac{m}{N}}\sum_{k\geq\frac{m}{N-1}}^{\infty}\mathcal{E}_{N,\frac{d}{2}+Nk+m}\Big(-\frac{\sigma}{2\tau^{1/N}}\Big)\,\hat{a}_{-m,k}(F|x,x')\Bigg\}$$

Negative powers !

$$\hat{a}_{m,k}(F|x,x') = \sum_{l}\hat{U}_{m,k,l}(\nabla_x)\Big[\Delta^{1/2}(x,x')\,\hat{a}_l(-\Box + P|x,x')\Big]$$

special differential operators (fully algorithmic procedure)

SDW coefficients of $-\Box + P$

Three steps for the construction of $\left\{\hat{U}_k(\nabla)\right\}_n$

1) Multiple nested commutators:

$$\hat{V}_0(\nabla)=\hat{W},\quad \hat{V}_k(\nabla)=\Big[\dots\Big[\Big[\hat{W},\underbrace{\hat{H}^M\Big],\hat{H}^M}\Big],\dots,\hat{H}^M\Big],\quad k>0$$
$$k$$

2) Local differential operators:

$$\hat{U}_k(\nabla)=\hat{1},$$

$$\hat{U}_k(\nabla)=\sum_{\substack{n,k_1,\cdots k_n\\ n+|\mathbf{k}|=k}}\frac{(-1)^n\,\hat{V}_{k_1}(\nabla)\cdots\hat{V}_{k_n}(\nabla)}{k_1!\cdots k_n!\,(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_n+n)},\ k>0$$

3) n-th order ***SYNGIFICATION*** *of* $\hat{U}_k(\nabla),\quad \hat{U}_k(\nabla)\to\left\{\hat{U}_k(\nabla)\right\}_{\mathbf{n}}$:

Replacement of ***n*** *covariant derivatives in each of their monomials by Synge world function vectors* $\nabla_a\to-\nabla_a\sigma(x,x')/2$

$$\left\{\nabla_{a_1}\dots\nabla_{a_j}\right\}_n=\frac{1}{2^n n!}\frac{\partial^n}{\partial k^n}\left(\nabla_{a_1}-k\sigma_{a_1}\right)\dots\left(\nabla_{a_j}-k\sigma_{a_j}\right)\Big|_{k=0}$$

Nonminimal operators

$$\hat{H}(\nabla) = H^A_B(\nabla), \qquad \hat{1} \equiv \delta^A_B$$

$$\hat{H}(\nabla) = \hat{D}^{a_1 \ldots a_{2N}} \nabla_{a_1} \ldots \nabla_{a_{2N}} + O\Big(\nabla^{2N-1}\Big), \qquad \hat{D}^{a_1 \ldots a_{2N}} \not\propto \hat{1}$$

Examples:

Vector field in curved spacetime

$$H^a_b(\nabla) = -\Box \delta^a_b + \alpha \nabla^a \nabla_b + R^a_b, \quad \alpha \neq 1$$

Proca operator in curved spacetime

$$H^a_b(\nabla) = -\Box \delta^a_b + \nabla^a \nabla_b + R^a_b$$

4D Horava gravity (not causal, but spatial part $D^{kl}_{ij}(\nabla)$ is O(3)-invariant)

$$\hat{H}^{kl}_{ij}(\nabla) = -\delta^{kl}_{ij}\, \partial_\tau^2 + D^{kl}_{ij}(\nabla)$$

$$D^{kl}_{ij}(\nabla) = -\Big[A\, \delta_{ij}{}^{kl} + B\, g_{ij} g^{kl} \Big] \Delta^3 + C\, \delta^{(k}_{(i} \nabla_{j)} \nabla^{l)} \Delta^2$$
$$+ D\, g_{ij} \nabla^{(k} \nabla^{l)} \Delta^2 + E\, \nabla_{(i} \nabla_{j)} g^{kl} \Delta^2$$
$$+ G\, \nabla_{(i} \nabla_{j)} \nabla^{(k} \nabla^{l)} \Delta + \Big[\sim 200 \text{ lower derivative terms} \Big],$$

3D covariant Laplacian

A.O. Barvinsky, A.V. Kurov, S. M. Sibiryakov, beta functions, UV asymptotic freedom, RG flows: Phys.Rev.D 105 (2022) 044009, arXiv:2110.14688; Phys.Rev.D 108 (2023) 12, L121503, arXiv:2310.07841; Phys. Rev. D 111, (2025) 024030, arXiv:2411.13574)

Causal theories

Principal symbol

$$\hat{D}(\nabla) \to \hat{D}(ip) = (-1)^N \hat{D}^{a_1 \dots a_{2N}} p_{a_1} \dots p_{a_{2N}}$$

Causality condition

$$\det \hat{D}(ip) = C (p^2)^{NM}, \quad p^2 = g^{ab} p_a p_b$$

Wave equation characteristic surface is a light cone $p^2 = 0$

Eigenvalues of the principal symbol

$$\hat{D}(ip)\varphi_i = \lambda_i (p^2)^N \varphi_i$$

Heat kernel of the operator truncated to principal symbol

$$\hat{K}_H(\tau \,|\, x, y) \simeq \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \sum_i e^{-\tau \lambda_i (p^2)^N} \hat{\Pi}_i(n), \quad n_a = \frac{p_a}{\sqrt{p^2}}$$

Projectors onto lambda-eigenspaces

$$\hat{\Pi}_i \hat{\Pi}_k = \delta_{ik} \hat{\Pi}_i, \quad \sum_i \hat{\Pi}_i = \hat{1}$$

Projector structure for spin s theory

$$\hat{\Pi}_i = \Big[\Pi_i \Big]^A_B = \hat{\pi}_i^{a_1 \dots a_{2s}} \frac{p_{a_1} \dots p_{a_{2s}}}{(p^2)^s}, \quad A = a_1 a_2 \cdots a_s, \; B = b_1 b_2 \cdots b_s$$

Leading order approximation

Promotion to curved spacetime

$$\delta_{ab} \to g_{ab}(x), \quad p_a \to -i\nabla_a, \quad p^2 \to -\Box = -g^{ab}\nabla_a\nabla_b$$

From momentum space to coordinate space:

$$\hat{K}_H(\tau\,|\,x,y) \simeq \int \frac{d^dp}{(2\pi)^d}\, e^{ip(x-y)} \sum_i e^{-\tau\lambda_i (p^2)^N} \hat{\Pi}_i(n)$$

$$\Downarrow$$

$$\sum_i \hat{\pi}_i^{a_1\dots a_{2s}} \nabla_{a_1}\dots\nabla_{a_{2s}} \frac{e^{-\tau\lambda_i(-\Box)^N}}{(-\Box)^s}\,\delta(x,y)$$

Generalization to generic minimal operator $(-\Box)^N \Rightarrow F = \hat{F}(\nabla) = (-\Box + \hat{P})^N = (-\Box)^N + O[\mathfrak{R}]$

Quasi-projectors $\Pi_i = \hat{\pi}_i^{a_1\dots a_{2s}} \nabla_{a_1}\dots\nabla_{a_{2s}} \dfrac{\hat{1}}{[\,F(\nabla)\,]^{s/N}}, \quad \Pi_i\Pi_j = \delta_{ij} + O[\,\mathfrak{R}\,]$

curvature corrections

$$\Downarrow$$

$$\frac{\hat{1}}{F^{s/N}} = \frac{1}{\Gamma(s/N)} \int_0^\infty d\tau\, \tau^{s/N-1} e^{-\tau F}$$

source of IR divergences for Schwinger-DeWitt expansion of a heat kernel

Infrared problem in massless models

IR divergent at $m \geq d/2$

$$\frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K(\tau) = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\int_0^\infty d\tau\,\tau^{m-\frac{d}{2}-1}\int dx\,g^{1/2}(x)\,a_m(x,x)$$

Evisceration of DeWitt coefficients

$$a_m(x,x) = \sum \nabla^{2k}\,\mathfrak{R}^{m-k} = c_m\,\mathfrak{R}\,(-\Box)^m\,\mathfrak{R} + O[\mathfrak{R}]$$

$$\sum_m \int dx\,g^{1/2}(x)\,a_m(x,x)\tau^m = \int dx\,g^{1/2}(x)\left[\mathfrak{R}\underbrace{\sum_m c_m(-\tau\Box)^m}_{f(-\tau\Box)}\mathfrak{R} + \ldots\right] = \int dx\,g^{1/2}(x)\left[\mathfrak{R}\,f(-\tau\Box)\,\mathfrak{R} + \ldots\right]$$

$$\sum_m c_m \int_0^\infty d\tau\,\tau^{m-\frac{d}{2}-1}(-\Box)^m \quad \Rightarrow \quad \int_0^\infty \frac{d\tau}{\tau^{d/2+1}}\,f(-\tau\Box) < \infty$$

IR finite

We look for functorial properties of DeWitt expansion---keeping them intact.
Avoid proper time integration in the infinite proper time range.

Subtraction procedure

The goal --- to avoid IR problems of $\tau \to \infty$

source of IR divergences

$$\frac{\hat{1}}{F^s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} e^{-\tau F}$$

Choose any $\beta > 0$

$$K_H(\tau) \simeq e^{-\beta\tau F} + \sum_i \Pi_i \left(e^{-\tau\lambda_i F} - e^{-\beta\tau F} \right) = e^{-\beta\tau F} - \sum_i \Pi_i F \int_{\beta\tau}^{\lambda_i\tau} d\tau_1\, e^{-\tau_1 F}$$

Repeat n *times --- leading order approximation (**not** the flat-space one)*

$$K_H(\tau) = \sum_{k=0}^{n-1} \frac{\tau^k}{k!} (\beta F - H)^k\, e^{-\beta\tau F} + (-1)^n \int_{\beta\tau}^{\tau} d^n\tau \sum_i \Pi_i (F\lambda_i)^n\, e^{-\tau_n \lambda_i F}$$

local differential operator

$$\int_0^\tau d^n\tau\,(\ldots) \equiv \int_{\beta\tau}^{\tau} d\tau_1 \int_{\beta\tau}^{\tau_1} d\tau_2 \ldots \int_{\beta\tau}^{\tau_{n-1}} d\tau_n\,(\ldots)$$

$$\Pi_i F^n = \hat{\pi}_i^{a_1 \ldots a_{2s}} \nabla_{a_1} \ldots \nabla_{a_{2s}} \hat{F}^{n-s}(\nabla)$$

local for n>s/N *or* n=s/N *(assume N=1 below)*

Perturbation theory

$$\left(\frac{\partial}{\partial\tau}+H\right)\mathbb{K}_s(\tau)=-\mathbb{W}_s(\tau)=O[\,\mathfrak{R}\,]$$ *commutators* $\sim \mathfrak{R}, \nabla\mathfrak{R}$

$$\mathbb{K}(\tau)=\mathbb{K}_s(\tau)+\sum_{n=1}^{\infty}\mathbb{K}_s^{(n)}(\tau)$$

$$\mathbb{K}_s^{(n)}(\tau)=\int_0^{\tau} d^n\tau\,\mathbb{K}_s(\tau_n)\,\mathbb{W}_s(\tau-\tau_1)\mathbb{W}_s(\tau_1-\tau_2)\,\mathbb{W}_s(\tau_2-\tau_3)...\mathbb{W}_s(\tau_{n-1}-\tau_n)$$

Typical perturbation term multilinear in operator exponent $e^{-\tau F}$

$$\mathbb{K}_s^{(n)}(\tau)\sim\int d\tau_0\cdots d\tau_n\,K\,e^{-\tau_0 F}\,W\,e^{-\tau_1 F}\cdots W\,e^{-\tau_n F}$$

"vertices"

Convolution of kernels

$$\cdots\int d^dx'\,WK_F(\tau\,|\,x,x')\,WK_F(\tau'\,|\,x',x'')\cdots$$

INSTEAD: COVARIANT PSEUDODIFFERENTIAL CALCULUS

Commutation of all operator exponents to the right

$$\mathbb{K}_s^{(n)}(\tau) \sim \int d\tau_0 \cdots d\tau_n \, K \, e^{-\tau_0 F} \, W \, e^{-\tau_1 F} \cdots W \, e^{-\tau_n F}$$

$$\Big[e^{-\tau F}, W \Big] = \sum_{n=1}^{\infty} \frac{(-\tau)^n}{n!} \Big[F, W \Big]_n \, e^{-\tau F}$$

Multiple nested commutator

$$\Big[F, W \Big]_n \equiv \overbrace{\Big[F, [F, \cdots [F}^{n}, W] \cdots]] = O\Big[\nabla^n \mathfrak{R} + \mathfrak{R}^n \Big]$$

Linear map from minimal to nonminimal

$$e^{-\tau H(\nabla)} = \int d\tau' \, \underbrace{\mathfrak{B}(\tau, \tau', \mathfrak{R} \,|\, \nabla)}_{\text{differential operator}} \, e^{-\tau' F(\nabla)}$$

Substitute Schwinger-DeWitt expansion

Generic order

$$\mathbb{K}_s^{(n)}(\tau) = \int_0^\tau d^n\tau \sum_{i_0,i_1,\dots i_n} \int_{\beta\tau_n}^{\lambda_{i_0}\tau_n} d\tau_0' \int_{\beta\Delta\tau_1}^{\lambda_{i_1}\Delta\tau_1} d\tau_1' \cdots \int_{\beta\Delta\tau_n}^{\lambda_{in}\Delta\tau_n} d\tau_n' \sum_{k_1,\dots k_n=0}^{\infty} \frac{(-\tau_0')^{k_1}\cdots(-\tau_{n-1}')^{k_n}}{k_1!\cdots k_n!}$$

$$\times\ K_{i_0}(\tau_n,\tau_0')\ \mathbb{C}_{i_1\cdots i_n,k_1\cdots k_n}(\tau,\tau_1,\cdots\tau_n\,|\,\tau_1',\cdots\tau_n')\ e^{-\sum_{m=0}^{n}\tau_m' F},$$

Commutant---multiple nested $\left(\sum_{m=1}^{n} k_m\right)$-fold commutator

$$\mathbb{C}_{i_1\cdots i_n,k_1\cdots k_n}(\tau,\tau_1,\cdots\tau_n\,|\,\tau_1',\cdots\tau_n') = \underbrace{\Big[F,W_{i_1}(\Delta\tau_1,\tau_1')\cdots\Big[F,W_{i_{n-1}}(\Delta\tau_{n-1},\tau_{n-1}')}_{n-1}\Big[F,W_{i_n}(\Delta\tau_n,\tau_n')\Big]_{k_n}\Big]_{k_{n-1}}\cdots\Big]_{k_1}$$

$$\Delta\tau_m \equiv \tau_{m-1}-\tau_m, \quad m=1,\cdots n, \quad \tau_0=\tau.$$

Local differential operators

$$K_i(\tau,\tau') = \sum_{k=0}^{s}\frac{\tau^k}{k!}\Pi_i(\beta F-H)^k\,\delta(\tau'-\beta\tau) - \frac{(\tau'-\lambda_i\tau)^s}{s!}\Pi_i F^{s+1}$$

$$W_i(\tau,\tau') = \left\{\sum_{k=0}^{s}\frac{\tau^k}{k!}\Pi_i\Big[(\beta F-H)^k,\beta F\Big] - \frac{\tau^s}{s!}\Big((\beta\Pi_i F-H\Pi_i)F^s(\beta-\lambda_i)^s - \Pi_i(\beta F-H)^{s+1}\Big)\right\}\delta(\tau'-\beta\tau)$$
$$+\frac{(\tau'-\lambda_i\tau)^{s-1}}{(s-1)!}(H\Pi_i-\lambda_i\Pi_i F)F^s$$

Example: nonmimimal vector field operator

$$H^a_b(\nabla) = -\Box\delta^a_b + \alpha\nabla^a\nabla_b + R^a_b, \qquad \alpha \neq 1$$

Projectors and eigenvalues

$$\Pi_1 = \delta^a_b - \nabla^a\frac{1}{\Box}\nabla_b, \;\; \lambda_1 = 1, \qquad \Pi_2 = \nabla^a\frac{1}{\Box}\nabla_b, \;\; \lambda_2 = 1-\alpha$$

Leading order is exact, choice of $\beta = 1$ *and the minimal operator* $F^a_b(\nabla) = -\Box\delta^a_b + R^a_b$

$$\Big[K_H\Big]^a_{b'}(\tau\,|\,x,x') = \Big[K_F\Big]^a_{b'}(\tau\,|\,x,x') - \nabla^a\nabla_{b'}\int\limits_{\tau}^{(1-\alpha)\tau} d\tau_1\, K_{-\Box}(\tau\,|\,x,x')$$

minimal part — *nonminimal part*

$$\sum_{m=0}^{\infty}\frac{I\Big(\tau, m-\frac{d}{2}+1, \sigma(x,x')\Big)}{(4\pi)^{d/2}}\, a_m(-\Box\,|\,x,x')$$

SDW coefficients of the minimal box operator

$$I\Big(\tau,\gamma,\sigma\Big) = \Big(\frac{\sigma}{2}\Big)^{\gamma}\Big[\Gamma\Big(-\gamma, \frac{\sigma}{2\tau(1-\alpha)}\Big) - \Gamma\Big(-\gamma,\frac{\sigma}{2\tau}\Big)\Big] \to -\tau^{\gamma}\frac{1-(1-\alpha)^{\gamma}}{\gamma}, \quad \sigma\to 0$$

Incomplete gamma-functions — ***Regular coincidence limit***

Proca field operator with a degenerate principal symbol case $(\alpha = 1)$ ***---*** *heat kernel is singular for relevant and marginal terms:* $I\Big(\tau,\gamma,\sigma\Big)\Big|_{\alpha=1} \sim \sigma^{\gamma}\to\infty, \;\; \sigma\to 0, \;\; \gamma = m+1-\frac{d}{2} < 0$

Operator is not elliptic and its heat kernel is not a smooth bounded operator

Extension to projectable Ho\vrava-Lifshitz Lorentz violating models

Lorentz symmetry violation $O(3,1) \to O(3)$, *anisotropic scaling invariance*

Foliation preserving diffeomorphisms $x^i \mapsto \tilde{x}^i(x,t)$

ADM metric decomposition $ds^2 = dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\ , \quad i,j = 1,2,3$

Foliation preserving covariant time derivative $D_t,\ D_t g_{ij} \sim K_{ij}$ *--- extrinsic curvature*

Foliation preserving covariant wave operator

$$H(\nabla) \sim D_t^2 + F(\boldsymbol{\nabla}), \quad F(\boldsymbol{\nabla}) \sim \underbrace{\boldsymbol{\Delta}^3 + \boldsymbol{\nabla}^6 + ..., \quad \boldsymbol{\Delta} = g^{ij}\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j}_{\textbf{spatial covariant derivatives}}$$

$O(3)$-***invariant = 3D "causal" !***

$$\frac{e^{-\frac{(t-t')^2}{4\tau}}}{\sqrt{4\pi\tau}} a_0(t,t'),\ D_t a_0(t,t') = 0$$

one-dimensional heat kernel

$$e^{-\tau \hat{H}(\nabla)} = \hat{\mathbb{C}}(\nabla)\, e^{-\tau D_t^2}\, e^{-\tau \hat{F}(\boldsymbol{\nabla})}$$

Baker-Campbell-Hausdorff commutator factor

3D "causal" operator expansion

Conclusions

Systematic approach to heat kernel of higher-derivative and nonminimal causal operators:

principal symbol eigenvalues + auxiliary minimal operator F

LO heat kernel + subtraction scheme + perturbation theory + commutator algebra of operator exponents (covariant pseudodifferential calculus)

functorial use of Schwinger-DeWitt expansion coefficients

extension to Ho\vrava-Lifshitz gravity models