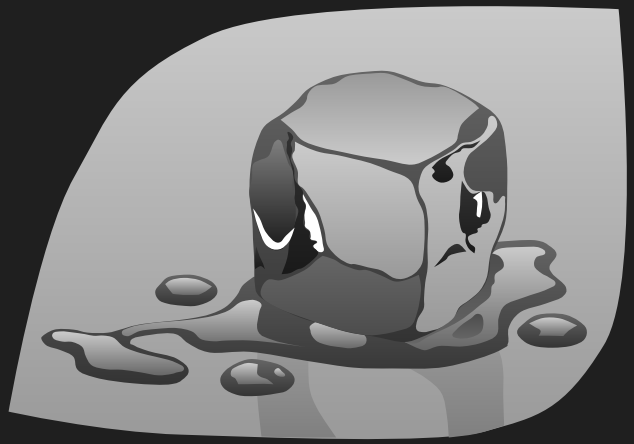
Wall-Crossing Phenomena
Past & Future

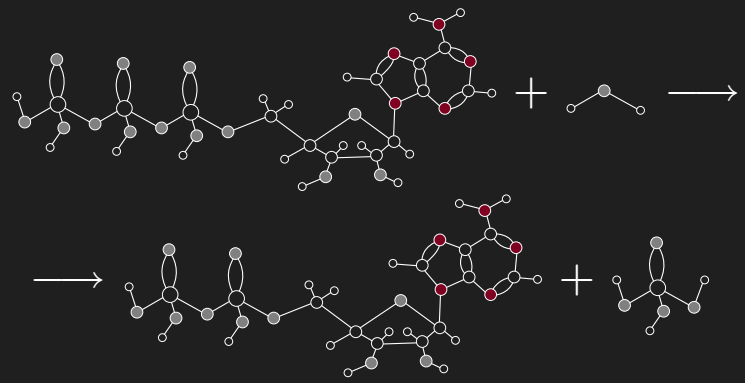
Dmitry Galakhov

Institute for Information Transmission Problems
Moscow Institute for Physics and Technology
NRC Kurchatov Institute (ITEP)

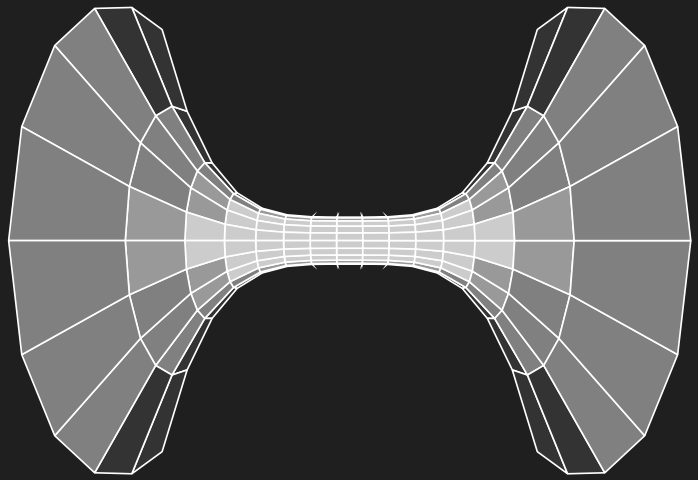
Problems of the Modern Mathematical Physics
JINR, Dubna, February 17th, 2026

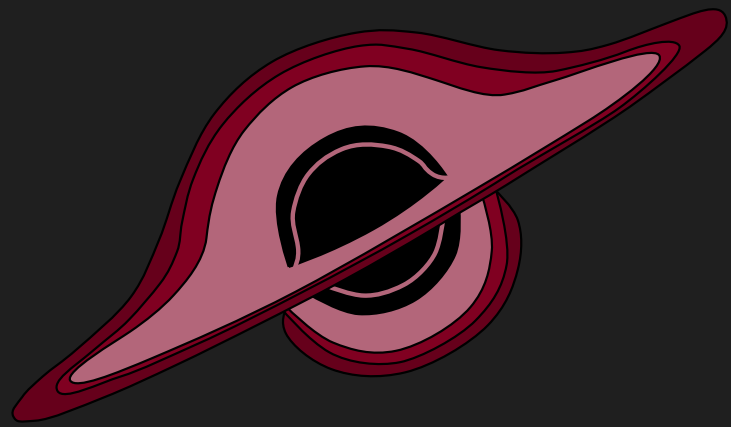
Wall-crossing in everyday life





Wall-crossing phenomena encompass a class of phenomena in **supersymmetric** QFTs where spectra of **BPS** states **jump abruptly** across **walls** of marginal stability on parameter (**moduli**) spaces. They are reminiscent of **critical phenomena**.





Simple wall-crossing

[Denef'02, Manschot-Pioline-Sen'10'11]

Supersymmetric quantum mechanics: a light electron coupled to a heavy monopole

$$L = \frac{m\dot{\vec{x}}^2}{2} + im\bar{\lambda}\dot{\lambda} + D^2 + \left(\frac{\kappa}{2r} + \zeta\right) D - \kappa \vec{A}^D(x)\cdot\dot{\vec{x}} - \frac{\kappa\vec{x}}{|\vec{x}|^3}\cdot\bar{\lambda}\vec{\sigma}\lambda$$

Magnetic flux

FI term

Dirac vector-potential

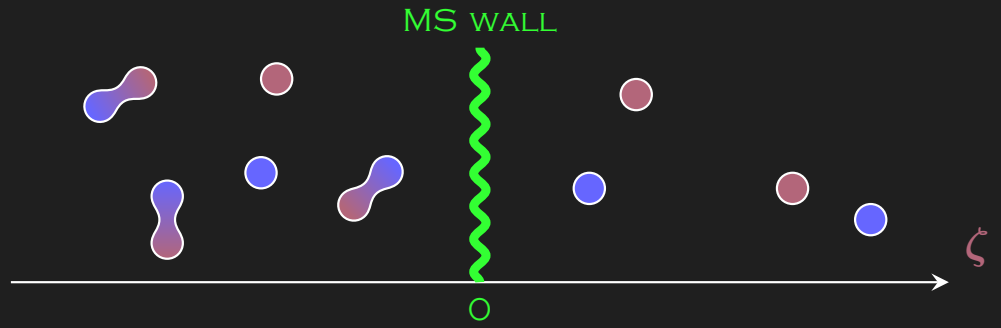
Superalgebra reads: $\left\{Q_\alpha, Q_\beta\right\} = 0,\ \left\{\bar{Q}^\alpha, \bar{Q}^\beta\right\} = 0,\ \left\{Q_\alpha, \bar{Q}^\beta\right\} = \delta_\alpha^\beta H$

Hamiltonian reads: $H = -\frac{1}{2m}\vec{D}^2 + \frac{1}{2m}\left(\zeta + \frac{\kappa}{2|\vec{x}|}\right)^2 + \frac{\kappa\vec{x}}{|\vec{x}|^3}\cdot\bar{\lambda}\vec{\sigma}\lambda$

(Supersymmetric) BPS states = ground sates,

$$\Psi = r^{\frac{\kappa}{2}-1} e^{\zeta r} e^{i\ell\varphi} (1+\cos\theta)^{\frac{\ell-1}{2}} (1-\cos\theta)^{\frac{\kappa}{2}-\frac{\ell+1}{2}} \begin{pmatrix} 1-\cos\theta \\ -e^{i\varphi}\sin\theta \end{pmatrix},$$

with Landau level ℓ, $0 \le \ell \le \kappa - 1$



Kronecker quiver

κ arrows



$\kappa = 1$: simple molecules

$\kappa = 2$: ∞ many molecules

$\kappa \ge 3$: wild spectra

Cohomological Hall Algebras

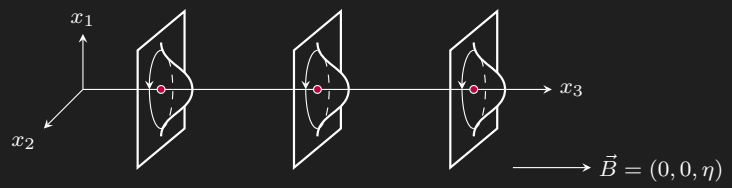
Nodes $\longrightarrow$ Vector $(A_\mu, \lambda_\alpha, D)$, Arrows $\longrightarrow$ chiral Higgs (H, ψ_α, F)
Coulomb branch localization: $\langle A_\mu \rangle \neq 0$, $\langle H \rangle = 0$
SQM: dimensional reduction $(A_0, A_1, A_2, A_3) \longrightarrow (A_0, X_1, X_2, X_3)$,
+potential $Tr\,[X_i, X_j]^2 \longrightarrow X_i = \mathrm{diag}(x_i^{(1)}, \ldots, x_i^{(N)})$
Gauge theory $U(N) \longrightarrow N$-particle BPS molecule
$x_{i=1,2,3}^{(r)}$ are 3d coordinates of the r^{th} particle

Wall-crossing $\rightsquigarrow$ "chemical reactions"

[Harvey-Moore'95,96, Kontsevich-Soibelman'10, G.'18]

BPS algebra:

$$\cdot \star \cdot : \ \mathscr{H}_{\mathrm{BPS}} \otimes \mathscr{H}_{\mathrm{BPS}} \longrightarrow \mathscr{H}_{\mathrm{BPS}}, \quad S(2 \to 1) = \frac{\langle \mathcal{F} | \Psi_1 \star \Psi_2 \rangle}{s - m_{\mathcal{F}}^2} + \ldots$$



LL w.f.: $\psi_\ell = e^{-\eta |z|^2} z^\ell$, $z = x_1 + \mathrm{i} x_2$

For SQM BPS algebra $\longrightarrow$ **CoHA** of Kontsevich-Soibelman ($\star \to \sqcup$):

$$(\psi_1 \sqcup \psi_2)\left(\{z_{i,a}\} \cup \{z'_{i,a}\}\right) = \underset{\text{shuffles}}{\mathrm{Sym}} \left\{ \psi_1\left(\{z_{i,a}\}\right) \psi_2\left(\{z'_{i,a}\}\right) \frac{\prod\limits_{a \in \mathfrak{Q}_0} \prod\limits_{b \in \mathfrak{Q}_0} \prod\limits_{i=1}^{d_a} \prod\limits_{j=1}^{d'_b} \left(z'_{j,b} - z_{i,a}\right)^{\#\{a \to b\}}}{\prod\limits_{c \in \mathfrak{Q}_0} \prod\limits_{i=1}^{d_c} \prod\limits_{j=1}^{d'_c} \left(z'_{j,c} - z_{i,c}\right)} \right\}$$

Wall-crossing formulae

Non-commutative characters for shuffle product:

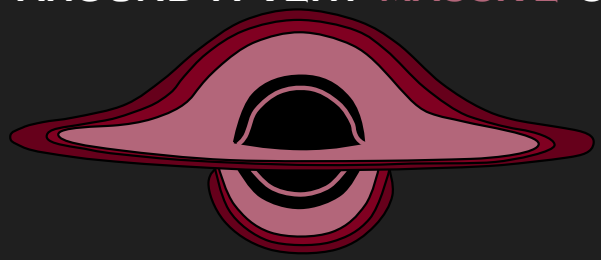
$$X_{(e,m)}X_{(e',m')} = q^{em'-e'm}X_{(e+e',m+m')}\,,$$
$$\Phi(X) := \mathrm{Tr}_{\mathrm{Fock}}\,(-1)^F q^{-2J} X_{\mathcal{Q}}$$

For a single hypermultiplet particle:

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{q^{-k^2}}{\prod_{j=1}^{k}(1-q^{-2j})} X^k = \left(-q^{-1}X|q^{-2}\right)_\infty$$

BPS molecular haloes around a very massive core

$$\text{Eff. FI } \zeta^{\mathrm{eff}} = \frac{\sum_{v\in\mathfrak{Q}_0} d_v\zeta_v}{\sum_{v\in\mathfrak{Q}_0} d_v}$$



$\zeta_1^{\mathrm{eff}} < \zeta_2^{\mathrm{eff}} < \zeta_3^{\mathrm{eff}}$

$$\mathrm{Fock}_{(1,0)} \sqcup \mathrm{Fock}_{(0,1)} \cong \mathrm{Fock}_{(0,1)} \sqcup \mathrm{Fock}_{(1,1)} \sqcup \mathrm{Fock}_{(1,0)} \;\Leftrightarrow\; \Phi(X_{(1,0)})\Phi(X_{(0,1)}) = \Phi(X_{(0,1)})\Phi(X_{(1,1)})\Phi(X_{(1,0)})$$

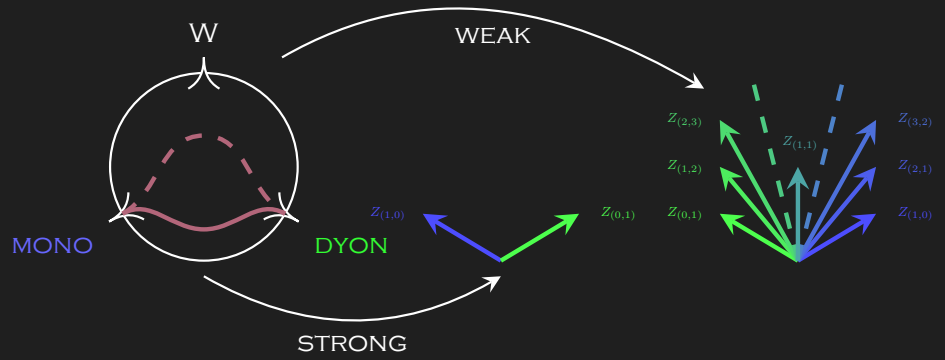
Wall-crossing formulae: $\overrightarrow{\prod_{\vec{\gamma}}} \Phi^{\Omega_{\vec{\gamma}}(q)}(X_{\vec{\gamma}}) = \mathrm{const}$

Wall-crossing in class S theories

class S (S=six) • 4d $\mathcal{N}=2$ theory obtained by a compactification of 6d $(0,2)$ on UV curve $\mathcal{C}$, dyons $\zeta = \arg\left(Z_\gamma \bar{Z}\right)$

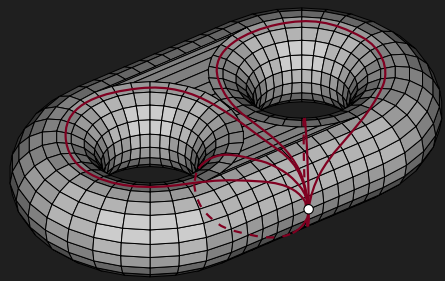
Pure $SU(2)$ SYM ($\kappa = 2$):

$$\lambda_{SW}^2 = \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}\right) dz^2$$



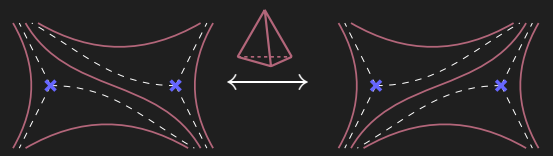
[Gaiotto-Moore-Neitzke'09'10'11'12]

N-stack of M5's (UV)	$\mathcal{C}$	$\times$	$\varnothing$	$\times$	$\mathbb{R}^{1,3}$
M5-brane (IR)	Σ_u	$\times$	$\varnothing$	$\times$	$\mathbb{R}^{1,3}$
M2 (BPS particle)	γ(cycle)	$\times$	$1d$	$\times$	$\mathbb{R}$(w.-l.)



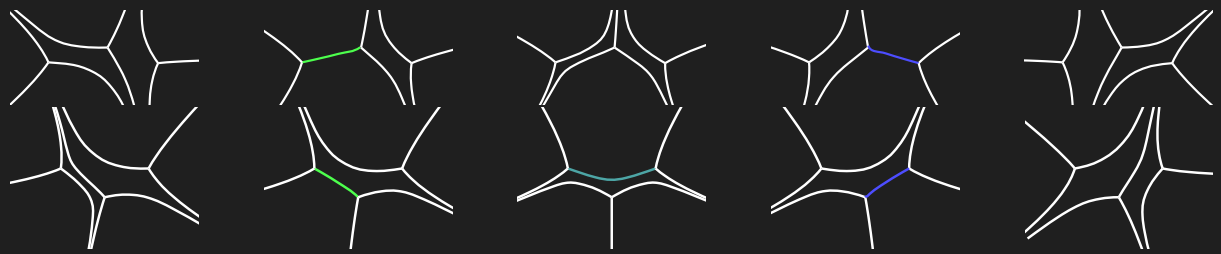
γ: $\partial_t \lambda_{SW} \in e^{i\varphi}\mathbb{R}_+$, $\varphi = \arg Z$

AD_3 thy ($\kappa = 1$) $\lambda_{SW}^2 = \left(z^3 - 3z + u\right) dz^2$



$u = 0.3 + 1.0\mathrm{i}$:

$u = -3.0\mathrm{i}$:



Wall-crossing and AGT correspondence

$4d$ class S $\xrightarrow{\text{AGT}}$ $2d$ CFT

$3d$ domian wall $\xrightarrow{\text{AGT}}$ $3d$ top. theory = $3d$ Chern-Simons

$$\mathcal{A}_\mu = e^a_\mu (\sigma_a)^c_b + \mathrm{i}\omega^c_{\mu b},\ S_{\text{grav}} = \int \mathrm{Tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) + \text{c.c.} = \int d^3x\,\sqrt{g}\,(R+1)$$

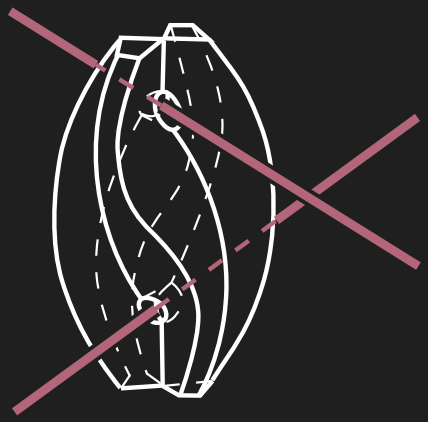


Fig. from 1612.02928

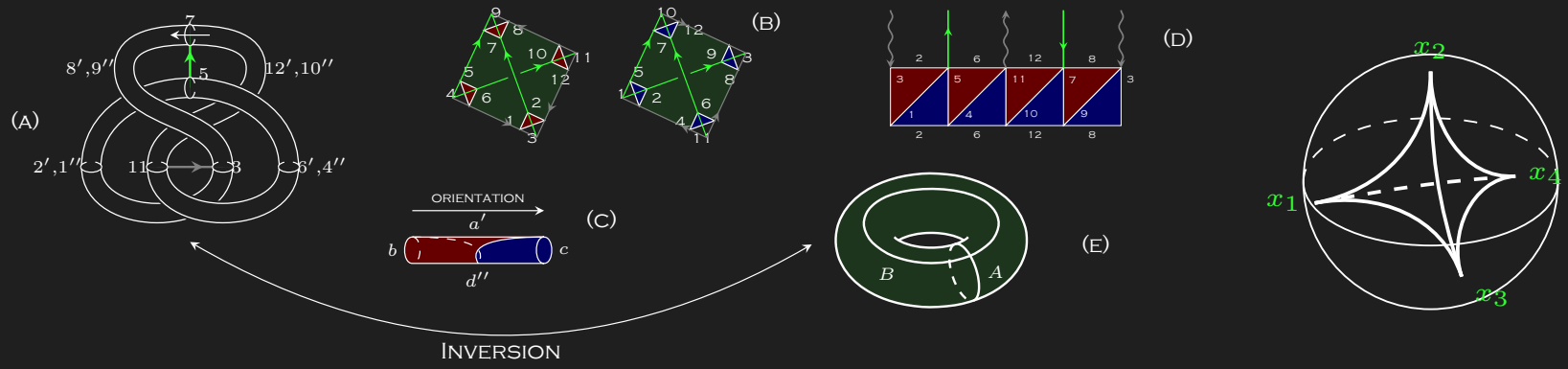
[Hikami-Inoue '14, G.-Mironov-Morozov '15]

$$\mathrm{Tr}\,\mathrm{Pexp}\oint_A \mathcal{A} = X + X^{-1},\quad \mathrm{Tr}\,\mathrm{Pexp}\oint_B \mathcal{A} = Y + Y^{-1}$$

$$XY = e^{\frac{1}{\hbar}} YX,\quad \Phi_\hbar(x) \sim e^{\frac{1}{\hbar}\mathrm{Li}_2(x)} \sim e^{\frac{1}{\hbar}S_{\text{grav}}}$$

$$\langle x_1, x_2|R|y_1, y_2\rangle = \frac{\Phi_\hbar(x_1-x_2)\Phi_\hbar(y_2-y_1)e^{\cdots}}{\Phi_\hbar\left(x_1-y_1+\frac{\mathrm{i}}{2}\left(\hbar+\hbar^{-1}\right)\right)\Phi_\hbar\left(y_2-x_2+\frac{\mathrm{i}}{2}\left(\hbar+\hbar^{-1}\right)\right)}$$

[Dimofte-Gaiotto-Gukov '11, Lee-Yamazaki '13, Kucharski-Reineke-Stosic- Sulkowski '17,...]



Calabi-Yau crystals

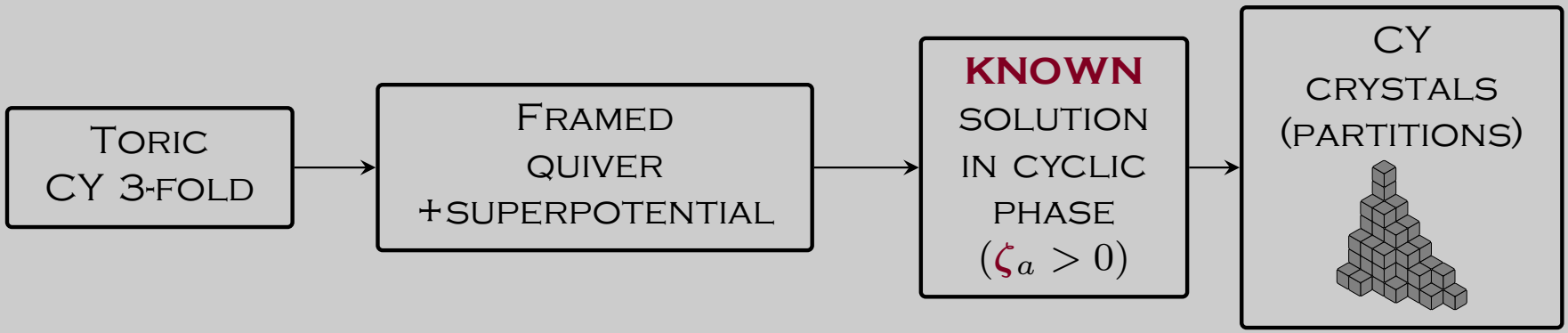
Quiver theory Higgs branch equivariant (flavor action) vacua...
ADHM-like equations:

$$\sum_{\alpha:\bullet\to a} H_\alpha H_\alpha^\dagger - \sum_{\beta:a\to\bullet} H_\beta^\dagger H_\beta = \zeta_a \mathbb{1}, \quad \forall a \in \mathfrak{Q}_0\,,$$

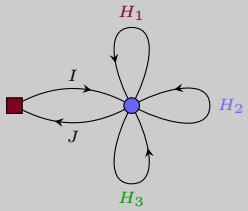
$$\frac{\partial}{\partial H_\alpha} W = 0, \quad \forall \alpha \in \mathfrak{Q}_1\,,$$

$$\Phi_b H_\gamma - H_\gamma \Phi_a = \epsilon_\gamma H_\gamma, \quad \forall (\gamma : a \to b) \in \mathfrak{Q}_1$$

H_α – Higgs fields associated to quiver arrows
Φ_a – scalars in the vector multiplets associated to quiver nodes
ϵ_γ – flavor charges of Higgs fields
[Ooguri-Yamazaki'09, Okounkov-Reshetikhin-Vafa'06, Li-Yamazaki'20, ...]



Calabi-Yau crystals II

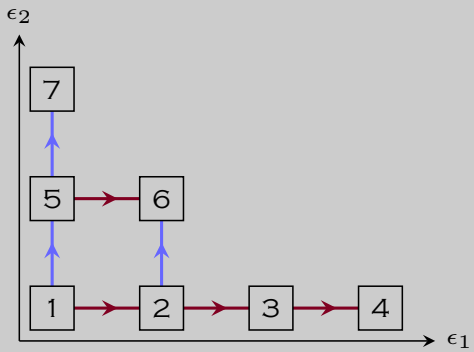


$$, \; W = \mathrm{Tr}\, H_3 \left([H_1, H_2] + IJ \right)$$

$$[H_1, H_1^\dagger] + [H_2, H_2^\dagger] + II^\dagger - J^\dagger J = \zeta$$

$$[H_1, H_2] + IJ = 0$$

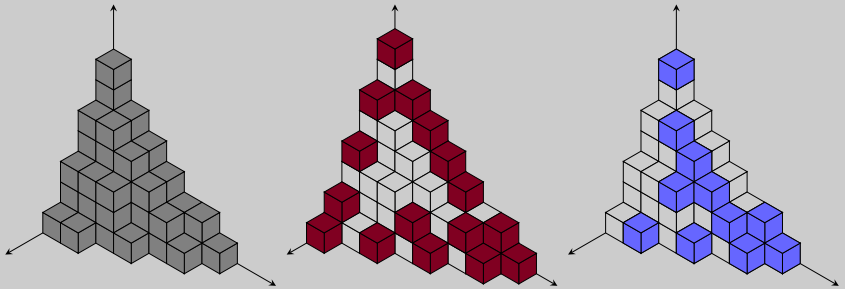
$$[\Phi, H_1] = \epsilon_1 H_1, \quad [\Phi, H_2] = \epsilon_2 H_2, \quad \Phi I = 0, \quad J\Phi = -(\epsilon_1 + \epsilon_2) J$$

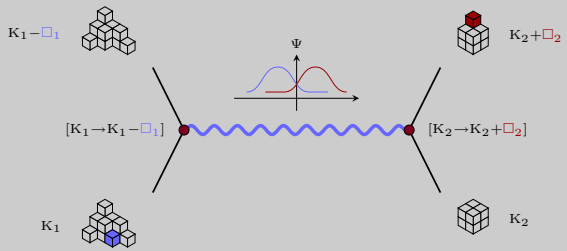


$$, \quad I = \begin{pmatrix} \sqrt{7\zeta} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi = \mathrm{diag}\,(0, \epsilon_1, 2\epsilon_1, 3\epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2, 2\epsilon_2)$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{\frac{6\zeta}{7}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2\zeta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\zeta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{4\zeta}{7}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{\frac{2\zeta}{7}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3\zeta}{7}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\zeta} & 0 & 0 \end{pmatrix}$$

BPS ALGEBRA

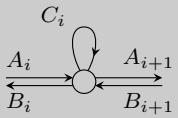
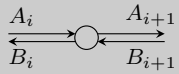


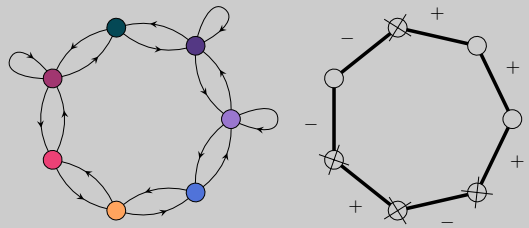


$$e_k|\mathrm{K}\rangle = \sum_{\square\in \mathrm{Add}(\mathrm{K})} E[\mathrm{K},\square]\omega_\square^k|\mathrm{K}+\square\rangle \qquad f_k|\mathrm{K}\rangle = \sum_{\square\in \mathrm{Rem}(\mathrm{K})} F[\mathrm{K},\square]\omega_\square^k|\mathrm{K}-\square\rangle$$

WHERE $\omega_\square = \epsilon_1 x_\square + \epsilon_2 y_\square + \epsilon_3 z_\square$

BPS ALGEBRA = QUIVER YANGIAN $Y(\mathfrak{Q})$

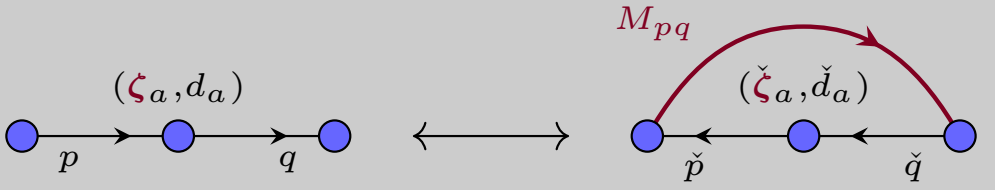
SPIN ARRANGEMENT	$\sigma_i\sigma_{i+1}=1$	$\sigma_i\sigma_{i+1}=-1$
$\mathbb{Z}_2$-PARITY	EVEN	ODD
DYNKIN NODE		
QUIVER NODE		
SUPERPOTENTIAL δW_i	$\sigma_i \operatorname{Tr} C_i\,(B_{i+1}A_{i+1} - A_iB_i)$	$-\sigma_i \operatorname{Tr} B_{i+1}A_{i+1}A_iB_i$



CoHA (COULOMB) $\approx$ **BOREL$_+$** SUBALGEBRA OF **QUIVER YANGIAN** (HIGGS)

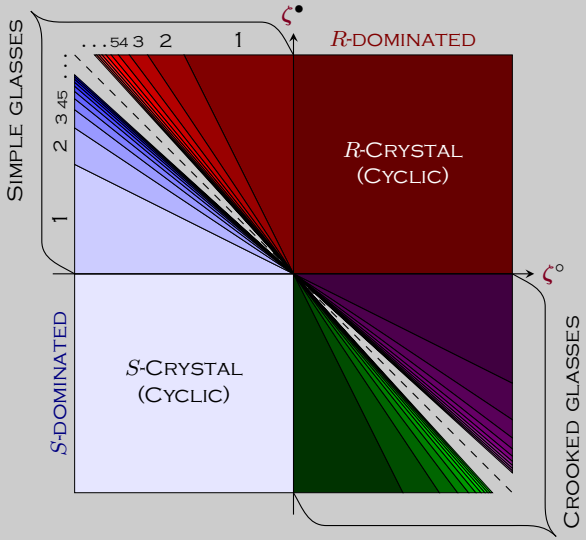
Seiberg duality, a.k.a. mutation

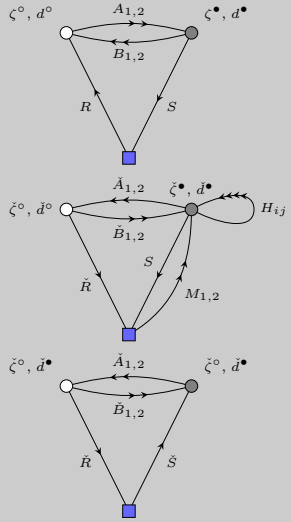
[Kolyaskin-Litvinov-Zhukov'22, G.-Morozov-Tselousov '24]



$$\check{\zeta}_a = -\zeta_a$$
$$\check{d}_a = \max_{b\in\{\text{neighbors}\}} d_b - d_a$$
$$W(qp) \to \check{W} = W(M_{pq}) + \mathrm{Tr} M_{pq}\check{p}\check{q}$$

Conifold $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_{1|1})$





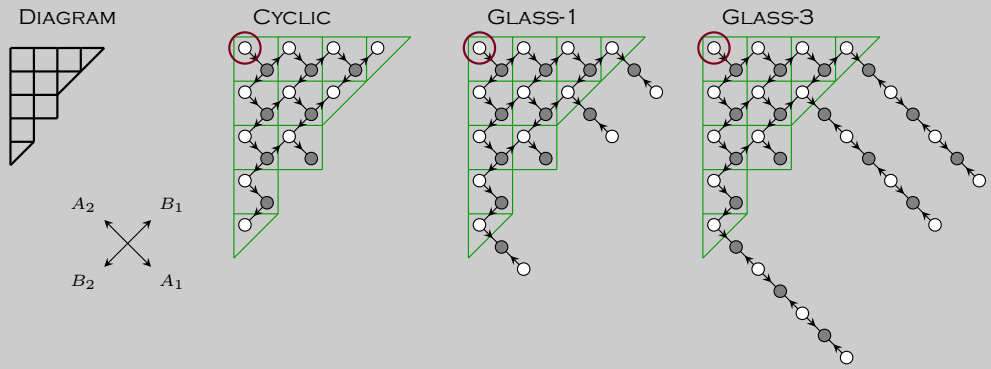
$$W = \mathrm{Tr}\left(A_1B_1A_2B_2 - A_1B_2A_2B_1 + A_2RS\right),$$

Fields	A_1	A_2	B_1	B_2	R	S
Weights	ϵ_1	$-\epsilon_1$	ϵ_2	$-\epsilon_2$	0	ϵ_1

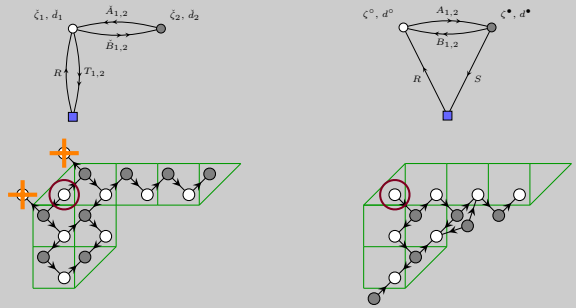
$$, \check{W} = \mathrm{Tr}\left[H_{11}H_{22} - H_{12}H_{21} + M_2S + \sum_{i,j=1}^{2} H_{ij}\check{B}_j\check{A}_i + M_2S + \sum_{i=1}^{2} M_i\check{R}\check{A}_i\right].$$

$$\check{W} = \mathrm{Tr}\left(\check{B}_1\check{A}_2\check{B}_2\check{A}_1 - \check{B}_1\check{A}_1\check{B}_2\check{A}_2 + \check{A}_1\check{S}\check{R}\right),$$

Fields	$\check{A}_1$	$\check{A}_2$	$\check{B}_1$	$\check{B}_2$	$\check{R}$	$\check{S}$
Weights	$-\epsilon_1$	ϵ_1	$-\epsilon_2$	ϵ_2	0	ϵ_1

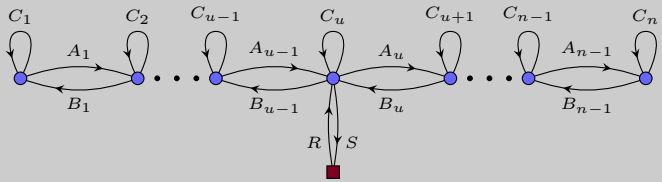


Crooked glasses:



Weyl mutations

[Bykov-Zinn-Justin'19, G.-Morozov-Gavshin '26]



$$, W = \mathrm{Tr}\left[A_1C_1B_1 + \sum_{a=2}^{n-1}(A_aC_aB_a - B_{a-1}C_aA_{a-1}) - B_{n-1}C_nA_{n-1} + SC_u^hR\right]$$

Rules for s_a:

- $\check{\zeta}_a = -\zeta_a, \quad \check{\zeta}_{a\pm1} = \zeta_{a\pm1} + \zeta_a, \quad \check{\zeta}_b = \zeta_b,$ for $|a-b| > 1$.
- $\check{d}_a = h\delta_{a,u} + d_{a+1} + d_{a-1} - d_a, \quad \check{d}_b = d_b,$ for $b \neq a$.
- $\check{R} = R,\ \check{S} = S,$ if $u \neq a; \quad \check{A}_b = A_b,\ \check{B}_b = B_b,$ for $b \neq a,\ b \neq a-1$.
- Maps α and β constructed as:

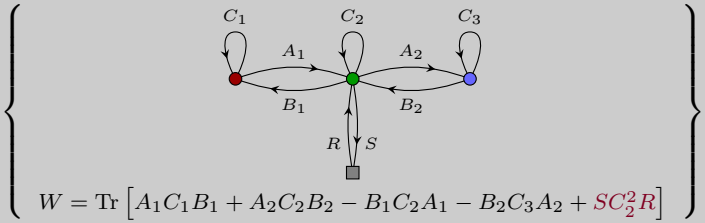
$\alpha = \begin{pmatrix} \check{B}_{a-1} & \check{A}_a \end{pmatrix}^T, \quad \beta = \begin{pmatrix} -A_{a-1} & B_a \end{pmatrix},$ if $u \neq a$;
$\alpha = \begin{pmatrix} \check{S} & \check{S}\check{C}_u & \dots & \check{S}\check{C}_u^{h-1} & \check{B}_{u-1} & \check{A}_u \end{pmatrix}^T,$
$\beta = \begin{pmatrix} C_u^{h-1}R & C_u^{h-2}R & \dots & R & -A_{u-1} & B_u \end{pmatrix},$ if $u = a$;

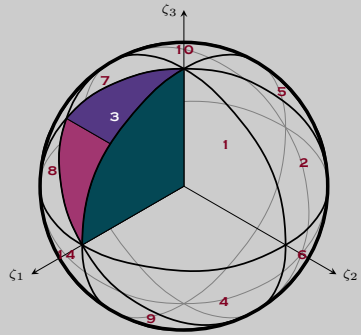
Define a **short exact sequence**:

$$0 \longrightarrow \check{V}_a \xrightarrow{\ \alpha\ } \mathbb{C}^{h\delta_{u,a}} \oplus V_{a-1} \oplus V_{a+1} \xrightarrow{\ \beta\ } V_a \longrightarrow 0 \quad .$$

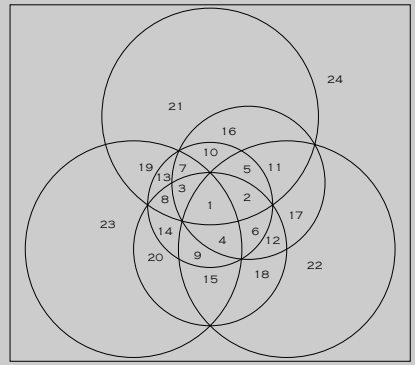
- $A_aA_{a-1} = \check{A}_a\check{A}_{a-1},\ B_{a-1}B_a = \check{B}_{a-1}\check{B}_a,\ B_{a-1}A_{a-1} = \check{B}_{a-1}\check{A}_{a-1},$ $A_aB_a = \check{A}_a\check{B}_a$ and $SA_{u-1} = \check{S}\check{A}_{u-1},\ SB_u = \check{S}\check{B}_u,\ A_uR = \check{A}_u\check{R},\ B_{u-1}R = \check{B}_{u-1}\check{R}_u,\ SR = \check{S}\check{R}$ if $a = u$

Wall-crossing in $\mathfrak{sl}_4$ quiver

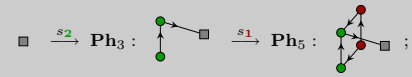


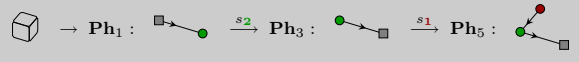


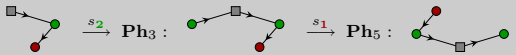
(A)

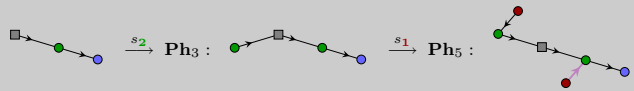


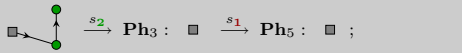
(B)

#1 : $\varnothing \to \mathbf{Ph}_1$:  ;

#2 : $\to \mathbf{Ph}_1$:  ;

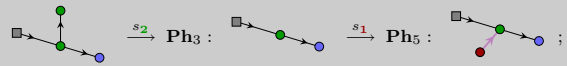
#3 :  $\to \mathbf{Ph}_1$:  ;

#4 :  $\to \mathbf{Ph}_1$:  ;

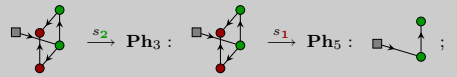
#5 :  $\to \mathbf{Ph}_1$:  ;

#6 :  $\to \mathbf{Ph}_1$:  ;

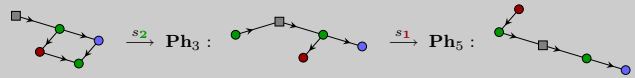
#7 :  $\to \mathbf{Ph}_1$:  ;

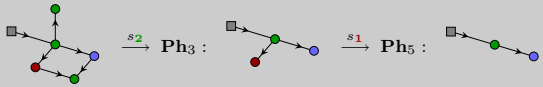
#8 :  $\to \mathbf{Ph}_1$:  ;

#9 :  $\to \mathbf{Ph}_1$:  ;

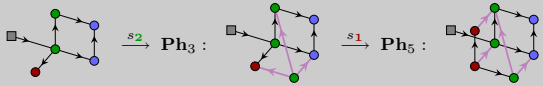
#10 :  $\to \mathbf{Ph}_1$:  ;

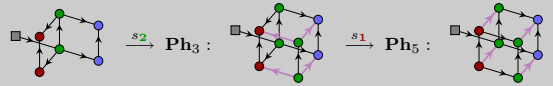
#11 :  $\to \mathbf{Ph}_1$:  ;

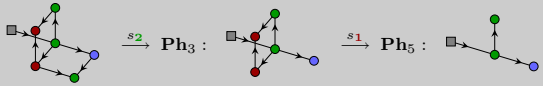
#12 :  $\to \mathbf{Ph}_1$:  ;

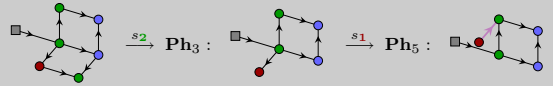
#13 :  $\to \mathbf{Ph}_1$:  ;

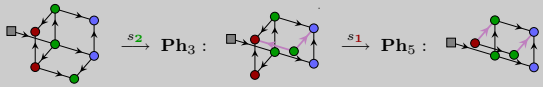
#14 :  $\to \mathbf{Ph}_1$:  ;

#15 :  $\to \mathbf{Ph}_1$:  ;

#16 :  $\to \mathbf{Ph}_1$:  ;

#17 :  $\to \mathbf{Ph}_1$:  ;

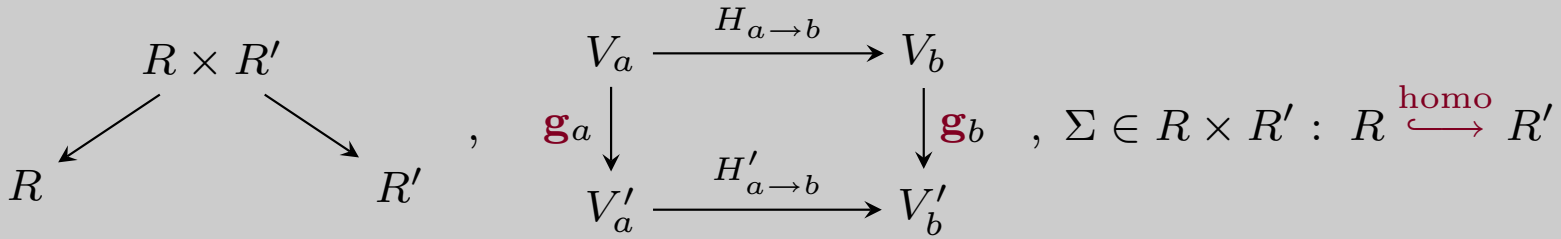
#18 :  $\to \mathbf{Ph}_1$:  ;

#19 :  $\to \mathbf{Ph}_1$:  ;

#20 :  $\to \mathbf{Ph}_1$: .

Mutated Yangians

Hecke modification ($\approx$ monopole operator, singular gauge transform $\mathbf{g}_a$):



Matrix elements:

$$\mathbf{E}[R,R'] = \int_\Sigma \mathbf{eu}(R) = \frac{\mathbf{eu}(R)}{\mathbf{eu}\left(\Sigma_{R,R'}\right)}, \; \mathbf{F}[R',R] = \int_\Sigma \mathbf{eu}(R') = \frac{\mathbf{eu}(R')}{\mathbf{eu}\left(\Sigma_{R,R'}\right)}$$

Consider a phase connected to the cyclic phase by a mutation $\boldsymbol{\mu}$, then $\boldsymbol{\mu}^{-1}(v)$ is a partition:

$$ {}^{\boldsymbol{\mu}}e_a^{(k)}|v\rangle = \sum_{\boxed{a}\in \mathrm{Add}\left(\boldsymbol{\mu}^{-1}(v)\right)} \mathbf{E}\left[v, \boldsymbol{\mu}\left(\boldsymbol{\mu}^{-1}(v)+\boxed{a}\right)\right] \phi^k_{\boxed{a}} \left|\boldsymbol{\mu}\left(\boldsymbol{\mu}^{-1}(v)+\boxed{a}\right)\right\rangle ,$$

$$ {}^{\boldsymbol{\mu}}f_a^{(k)}|v\rangle = \sum_{\boxed{a}\in \mathrm{Rem}\left(\boldsymbol{\mu}^{-1}(v)\right)} \mathbf{F}\left[v, \boldsymbol{\mu}\left(\boldsymbol{\mu}^{-1}(v)-\boxed{a}\right)\right] \phi^k_{\boxed{a}} \left|\boldsymbol{\mu}\left(\boldsymbol{\mu}^{-1}(v)-\boxed{a}\right)\right\rangle .$$

$$\boxed{{}^{\boldsymbol{\mu}}Y(\mathfrak{Q}) \cong Y(\mathfrak{Q})}$$

Open questions:

- Quiver Yangian mutations on the Coulomb branch
- All R-matrices = wall-crossing?
- Weyl mutations in physical terms
- Intrinsic definition for quiver Yangian outside the cyclic chamber
- M-theory spectra for CY3
- Higher dimensional defects



Thank you for your attetnion!