Baker-Akhiezer functions and Macdonald polynomials associated with different root systems

Mila Bishler

Lebedev Physical Institute

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Plan of the talk

1. Symmetric functions
2. Schur symmetric functions
3. Macdonald polynomials
4. BA function for root system A_n
5. Macdonald triad for root system A_n
6. Macdonald polynomials associated with root systems
7. BA function associated with root systems

Symmetric functions (SF)

$$f_\lambda(x_1, x_2, \ldots, x_n): \quad \sigma f_\lambda = f_\lambda. \tag{1}$$

$\sigma \in S_n$, S_n — symmetric group, λ — partition:

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n], \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0, \quad |\lambda| = \sum \lambda_i. \tag{2}$$

Power sum SF

$$p_k = x_1^k + \cdots + x_n^k, \quad p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i} \tag{3}$$

$$p_{[2,1]} = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) = x_1^3 + x_2 x_1^2 + x_3 x_1^2 + x_2^2 x_1 + x_3^2 x_1 + x_2^3 + x_3^3 + x_2 x_3^2 + x_2^2 x_3$$

Monomial SF

$$m_\lambda = \sum_{\sigma \in S_n} \prod_i \sigma x_i^{\lambda_i} \tag{4}$$

$$m_{[2,1]} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 \tag{5}$$

Schur symmetric functions

Schur symmetric functions can be defined with Cauchy's bialternant formula:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x^{n-j+\lambda_j}\right)}{\prod_{1\le i<j\le n}(x_i - x_j)} \tag{6}$$

1. **Orthogonality:** Schur functions are orthonormal

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}, \tag{7}$$

where the scalar product can be defined with power sum SF:

$$\langle p_\lambda(x), p_\mu(x) \rangle = \delta_{\lambda,\mu} z_\lambda, \tag{8}$$

z_λ depends on the partition λ.

2. **Triangular structure**

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu \tag{9}$$

where partitions $\mu < \lambda$ are determined with the partial ordering of partitions.

Partial ordering for root system A_n

For two Young diagrams λ and μ we say that $\lambda \geq \mu$ if $\forall j$

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \tag{10}$$

$$\begin{aligned}
|\lambda| &= 1: \quad [1] \\
|\lambda| &= 2: \quad [1,1] < [2] \\
|\lambda| &= 3: \quad [1,1,1] < [2,1] < [3] \\
|\lambda| &= 4: \quad [1,1,1,1] < [2,1,1] < [2,2] < [3,1] < [4] \\
|\lambda| &= 5: \quad [1,1,1,1,1] < [2,1,1,1] < [2,2,1] < [3,1,1] < [3,2] < [4,1] < [5]
\end{aligned} \tag{11}$$

In this case there is a natural grading depending on the weight $|\lambda|$ of partitions.

$c_{\lambda,\mu} = 0$ if we can't compare partitions λ and μ. For example we can't compare $[4,1,1]$ and $[3,3]$:

$$m = 6: \quad \cdots < [3,2,1] < \begin{matrix} [4,1,1] \\ [3,3] \end{matrix} < [4,2] < [5,1] < [6]. \tag{12}$$

Macdonald polynomials

Macdonald polynomials M_λ are orthogonal symmetric polynomials with two parameters q and t. They generalize Schur, Hall-Littlewood and Jack symmetric functions.

1. **Orthogonality**

$$\langle M_\lambda, M_\mu \rangle = 0 \quad \text{when} \quad \lambda \neq \mu, \tag{13}$$

where the scalar product is the following

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \tag{14}$$

2. **Triangular decomposition**

$$M_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu \tag{15}$$

3. **Eigenfunctions of Ruijsenaars Hamiltonians**

$$H_k = t^{\frac{k(k-1)}{2}} \sum_{1\leq i_1\leq\ldots\leq i_k\leq n} \prod_{a=1}^{k} \prod_{j\neq i_1,\ldots,i_k} \frac{tx_{i_a}-x_j}{x_{i_a}-x_j} q^{\sum_a x_{i_a}\partial_{i_a}} \tag{16}$$

From Macdonald polynomials to Baker-Akhiezer function

Macdonald polynomials associated with algebra sl_2 are the following:

$$M_{[r]}(x_1, x_2 \mid q, t) = x_1^r \sum_{\theta=0}^{r} \left(\frac{x_2}{x_1}\right)^{\theta} \prod_{i=0}^{\theta-1} \frac{(1-q^{i-r})}{(1-\frac{q}{t}q^{i-r})} \frac{(1-\frac{q}{t}q^{i-\theta})}{(1-q^{i-\theta})} = \\ x_1^r \sum_{\theta=0}^{r} \left(\frac{x_2}{x_1}\right)^{\theta} \prod_{i=0}^{\theta-1} C_\theta(r \mid q, t) \tag{17}$$

In we chose the parameter $t = q^{-m}$ and $2r > m$ this series can be expressed as

$$M_{[r]}(x_1, x_2 \mid q, t = q^{-m}) = x_1^r \sum_{\theta=0}^{m} \left(\frac{x_2}{x_1}\right)^{\theta} \prod_{i=0}^{\theta-1} C_\theta(r \mid q, q^{-m}) \\ + x_2^r \sum_{\theta=0}^{m} \left(\frac{x_1}{x_2}\right)^{\theta} \prod_{i=0}^{\theta-1} C_\theta(r \mid q, q^{-m}) \tag{18}$$

which is in fact a sum of two Baker-Akhiezer functions defined by O. Chalykh.

Definition of Baker-Akhiezer function for root system A_n

Baker-Akhiezer function $\Psi_m(\vec{z},\vec{\lambda})$ is a function of two sets of variables $\vec{\lambda}$ and $\vec{z}$ and $m \in \mathbb{Z}_+$ in the form:

$$\Psi_m(\vec{z},\vec{\lambda}) = q^{(\vec{\lambda}+m\vec{\rho})\vec{z}} \sum_{k_{ij}=0}^{m} \psi_{m,\vec{\lambda},k} q^{-\sum_{i>j} k_{ij}(z_i-z_j)} \tag{19}$$

satisfying the equations:

$$\Psi_m(z_k+j,\vec{\lambda}) = \Psi_m(z_l+j,\vec{\lambda}) \quad \text{for} \quad q^{z_k} = q^{z_l} \tag{20}$$

$\forall k, l$ and $1 \le j \le m$.
$\vec{\rho}$ is the Weyl vector, i.e. $\vec{\rho}\cdot\vec{z} = \frac{1}{2}\sum_{i=1}^{N}(N-2i+1)z_i$.

- BA function defined with these two conditions **exists** and is **unique** up to a normalization.
- BA function is a quasipolynomial in variables $q^{z_i} = x_i$. In fact, a polynomial up to a factor $q^{(\vec{\lambda},\vec{z})}$
- $\vec{\lambda}$ — is not a partition, but a set on arbitrary complex numbers

Properties of BA function

- Eigenfunction of the Ruijsenaars Hamiltonian:

$$\left(\sum_i \prod_{j\neq i} \frac{q^{z_j} - q^{-m-z_i}}{q^{z_j} - q^{z_i}} e^{\partial_{z_i}}\right) \Psi_m(\vec{z},\vec{\lambda}) = q^{-\frac{m}{2}} \left(\sum_i q^{\lambda_i}\right) \Psi_m(\vec{z},\vec{\lambda}) \tag{21}$$

- Properly normalized BA function $\overline{\Psi}_m$, where the coefficient at the leading term equal to

$$\psi_{m,\vec{\lambda},0} = \prod_{k>l} \prod_{j=1}^{m} \left(q^{\frac{\lambda_k-\lambda_l}{2}-j} - q^{-\frac{\lambda_k-\lambda_l}{2}}\right) \tag{22}$$

  is symmetric:

$$\overline{\Psi}_m(\vec{z},\vec{\lambda}) = \overline{\Psi}_m(\vec{\lambda},\vec{z}) \tag{23}$$

- It has the properties:

$$\overline{\Psi}_m(\omega\vec{z},\omega\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda}), \quad \omega \in W \quad \text{(equivariance)} \tag{24}$$

$$\overline{\Psi}_m(-\vec{z},-\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda}) \quad \text{(reflection)} \tag{25}$$

$$\overline{\Psi}_m(-\vec{z},\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda})\Big|_{q\to q^{-1}} \quad \text{(involution)} \tag{26}$$

Connection, similarities and differences with Macdonald polynomials

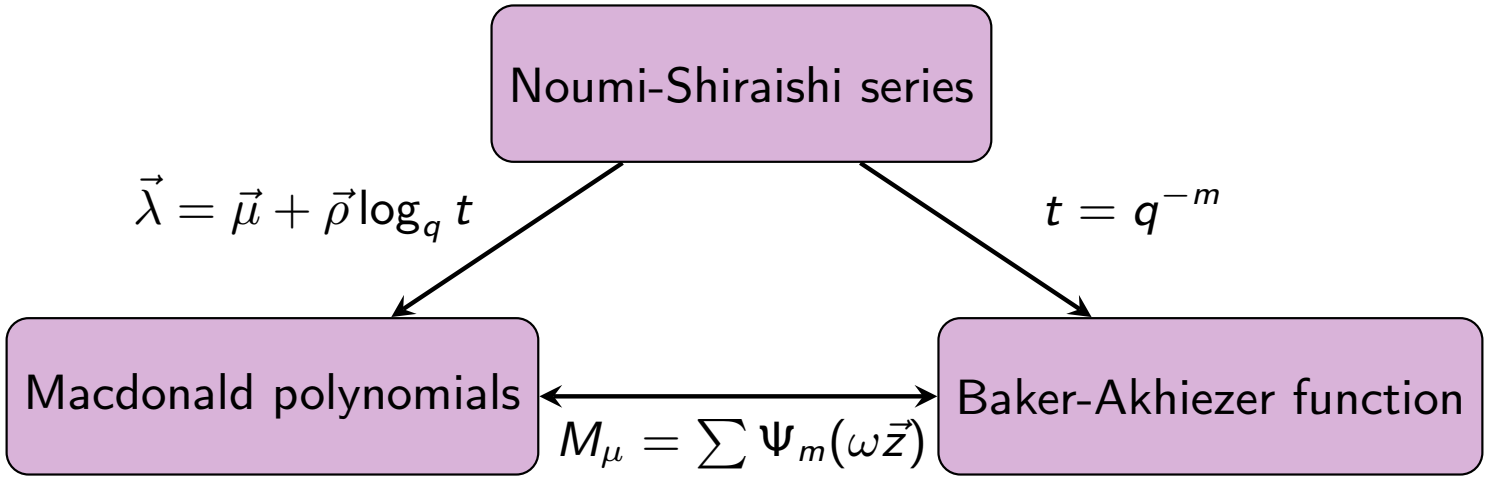
$$M_{\mu=\lambda+\rho}(\{x_i = q^{z_i}\} \,|\, q, t = q^{-m}) = \sum_{\omega \in W} \Psi_m(\omega \vec{z}, \vec{\lambda}) \tag{27}$$

Macdonald polynomials	BA functions
λ — partitions (dominant weights)	λ — a set of arbitrary complex numbers
defined with scalar product and triangular decomposition	defined with linear equations and general form
symmetric	non-symmetric
can be expressed via variables x_i as well as other SF $(p_k, m_\lambda, s_\lambda)$	can be expressed only via variables $q^{z_i} = x_i$

Macdonald triad for root system A_n

Macdonald polynomials $M_\mu(\{x_i\}\,|\,q,t)$ and BA function $\Psi_m(\vec{z},\vec{\lambda})$ are two elements of a more general system of three objects — Macdonald triad. The most general element of the triad in Noumi-Shiraishi power series

$$\mathfrak{P}_{q,t}(\vec{z},\vec{\lambda}) = q^{\vec{z}\cdot\vec{\lambda}}\cdot t^{-\vec{z}\cdot\vec{\rho}}\cdot\left(\sum_{k_{ij}}\psi(\vec{\lambda},k_{ij};q,t)\prod_{1\le i<j\le N}q^{k_{ij}(z_j-z_i)}\right)$$



Macdonald polynomials and BA functions are two polynomial reductions of NS series:

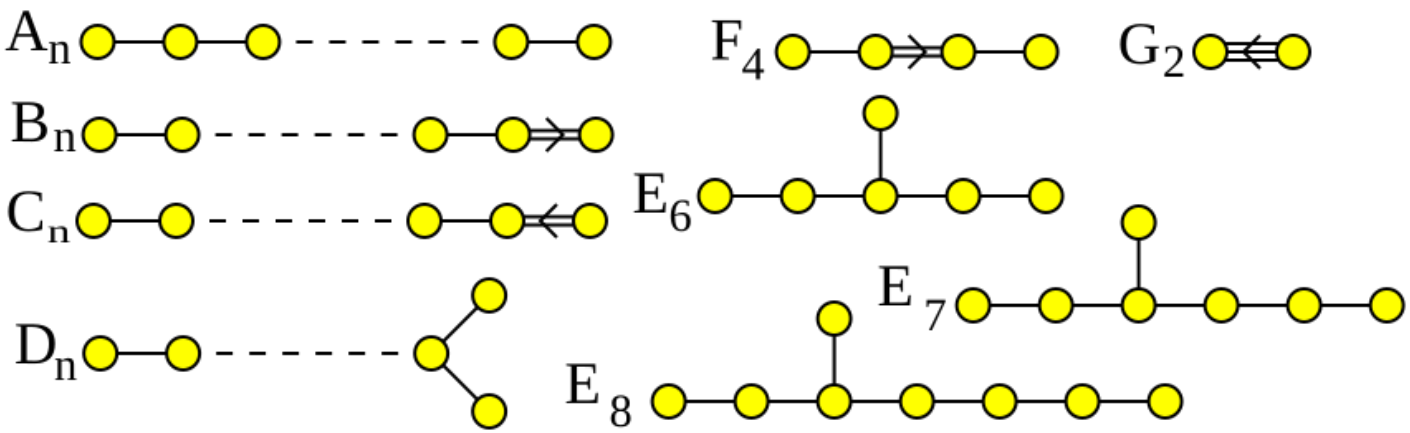
$$\mathfrak{P}_{q,t}(\vec{z},\vec{\mu}+\vec{\rho}\log_q t) = M_\mu(q^z\,|\,q,t) \tag{28}$$

$$\mathfrak{P}_{q,q^{-m}}(\vec{z},\vec{\lambda}) = \Psi_m(\vec{z},\vec{\lambda}) \tag{29}$$

We are interested in generalizing the Macdonald triad to other root systems.

Root systems

Dynkin diagrams:



Root system is defined with a set of simple roots α_i, $i = 1, \ldots, n$. Coroots:

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \tag{30}$$

Orthogonal to simple roots are dominant weights ω_i:

$$(\alpha_i^\vee, \omega_j) = \delta_{ij} \tag{31}$$

Combinations of dominant weights enumerate Schur and Macdonald polynomials:

$$\lambda = \sum_i \lambda_i \omega_i \tag{32}$$

Orthogonal Macdonald polynomials associated with root systems

For a reduced and irreducible root system R there exist a unique family of polynomials P_λ:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu, \tag{33}$$

which satisfy the orthogonality condition:

$$\langle P_\lambda, P_\mu \rangle = 0, \qquad \lambda \neq \mu. \tag{34}$$

The definition of a family of symmetric functions depending on a pair of root systems was given by I.G. Macdonald '87,'00.

Macdonald polynomials $P_\lambda^R(x \mid t_\alpha \mid q, t)$

- symmetric under the action of Weyl group W
- Depend on parameters q and t_α for each orbit of the Weyl group

$$t_\alpha = t_\beta \quad \text{for} \quad |\alpha| = |\beta| \tag{35}$$

- Enumerated with dominant weights λ

Symmetry

These functions are symmetric under the action of the Weyl group W of the corresponding root system. For classical Lie algebras:

algebra	root system	Weyl group
sl_n,	A_{n-1},	$W_A = S_n$,
so_{2n+1},	B_n,	$W_B = S_n \ltimes (\{\pm 1\})^n$,
sp_{2n},	C_n,	$W_C = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$,
so_{2n},	D_n,	$W_D = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

(36)

As a result one gets Laurent polynomials that depend on x_i as well as x_i^{-1} and are invariant under $x_i \leftrightarrow x_i^{-1}$ transformation.

Basis

Basis:

$$m_\lambda = \sum_{\omega \in W} q^{(\omega\lambda, z)}, \quad q^{z_i} = x_i \tag{37}$$

In the case of root systems C_n and B_n, D_n convenient basis consists of symplectic and orthogonal Schur functions correspondingly. They are Laurent polynomials that can be expressed with the following power sum functions symmetric under the action of Weyl group:

$$\begin{aligned} C_n, D_n: &\qquad p_k = \sum_i x_i^k + x_i^{-k}, \\ B_n: &\qquad p_k = 1 + \sum_i x_i^k + x_i^{-k} \end{aligned} \tag{38}$$

There is a correspondence between symplectic and orthogonal characters (Schur functions):

$$\chi_\lambda^{C_n} = (-1)^{|\lambda|} \chi_{\lambda^\vee}^{B_n/D_n}(-p_m) \tag{39}$$

Orthogonality and triangular structure

$$\langle f(x), g(x)\rangle = |W|^{-1}[f(x)g(x^{-1})\Delta]_0, \tag{40}$$

$$\Delta := \prod_{\alpha\in R} \frac{\left(t_{2\alpha}^{1/2} e^{\alpha}; q_\alpha\right)_\infty}{\left(t_\alpha t_{2\alpha}^{1/2} e^{\alpha}; q_\alpha\right)_\infty}, \tag{41}$$

the operator $[\dots]_0$ means picking up the constant term of polynomial, $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

Triangular structure of Macdonald polynomials

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu. \tag{42}$$

is built with the partial ordering of weights λ and μ:

$$\lambda \geq \mu \iff \lambda - \mu \in \mathbb{N}R_+. \tag{43}$$

Triangular structure for other classical root systems

In case of root system B_n there is weight $\omega_n = (1/2, 1/2, \ldots, 1/2)$ that divides all weights in two groups:

$$\begin{array}{ll} \text{integer:} & \varnothing < [1] < [1,1] < \begin{matrix}[2]\\ [1,1,1]\end{matrix} < [2,1] < \ldots \\ \text{half-integer} = \\ \text{integer} + \omega_n: & \varnothing + \omega_n < [1] + \omega_n < [1,1] + \omega_n < \begin{matrix}[2] + \omega_n\\ [1,1,1] + \omega_n\end{matrix} < \ldots \end{array} \tag{44}$$

In case of root system C_n dominant weights are divided into two groups: partitions of even and odd integers:

$$\begin{array}{ll} \text{even integers:} & \varnothing < [1,1] < \begin{matrix}[2]\\ [1,1,1,1]\end{matrix} < [2,1,1] < [2,2] < [3,1] < [4] < \ldots \\ \text{odd integers:} & [1] < [1,1,1] < [2,1] < [3] < \ldots \end{array} \tag{45}$$

In case of root system D_n there are four different groups of weights which are mutually incomparable:

- partitions of even integers
- partitions of odd integers
- (partitions of even integers)$+\omega_n$ + (partitions of odd integers)$+\omega_{n-1}$
- (partitions of even integers)$+\omega_{n-1}$ + (partitions of odd integers)$+\omega_n$

Eigenfunctions of Macdonald operator

Macdonald operator E_π depends on a minuscule fundamental weight π

$$E_\pi f = \sum_{\omega \in W} \omega \left(\Phi_\pi \cdot T_\pi f \right)$$

$$\Phi_\pi = \prod_{\substack{\alpha \in R_+ \\ (\pi, \alpha_*)=1}} \frac{1 - t_\alpha t_{2\alpha}^{1/2} q^{(\alpha, z)}}{1 - t_{2\alpha}^{1/2} q^{(\alpha, z)}}, \quad (T_\lambda f)(x) = f(x + \lambda) \tag{46}$$

E_π is self-adjoint, acts on the ring of symmetric functions A^W. Eigenvalues of E_π are all distinct for different λ.

Eigenvalue problem for Macdonald operator can be reformulated for $t_\alpha = q^{-m_\alpha}$ as a solution of a set of equations

$$f\left(z + \frac{1}{2} j\alpha\right) = f\left(z - \frac{1}{2} j\alpha\right) \quad \text{for} \quad q^{(\alpha^\vee, z)} = 1, \tag{47}$$

$\alpha \in R_+$, $j = 1, \ldots, m_\alpha$.

Construction of Baker-Akhiezer function

R — reduced and irreducible root system, $m = \{m_\alpha\}$ is a W-invariant set of multiplicities $m_\alpha \in \mathbb{Z}_+$. Let $\Psi_m(z, \lambda)$ be a function of two sets of variables in the form:

$$\Psi_m(z, \lambda) = q^{(\lambda, z)} \sum_{\nu \in \mathcal{N}} \psi_\nu \, q^{(\nu, z)}, \tag{48}$$

where the summation domain $\mathcal{N}$ is a polytope:

$$\mathcal{N} = \left\{ \nu = \rho_m^\vee - \sum_{\alpha \in R_+} l_\alpha \alpha^\vee \mid 0 \le l_\alpha \le m_\alpha \right\}, \tag{49}$$

satisfying the equations

$$\Psi_m\left(z + \frac{1}{2} j\alpha, \lambda\right) = \Psi_m\left(z - \frac{1}{2} j\alpha, \lambda\right) \quad \text{for} \quad q^{(\alpha^\vee, z)} = 1 \tag{50}$$

for $\alpha \in R_+$, $j = 1, \dots m_\alpha$, $\rho_m^\vee = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha^\vee$.

$\Psi_m(\lambda, z)$ is called a **Baker–Akhiezer (BA) function** associated with the data $\{R, m\}$

Properties

$$\Psi_m(z,\lambda) = q^{(\lambda,z)} \sum_{\nu\in\mathcal{N}} \psi_\nu \, q^{(\nu,z)}. \tag{51}$$

- BA function exists and is defined uniquely up to a normalization.
- BA function depends on two sets of arbitrary complex numbers: $\lambda = \{\lambda_i\}$ (not partitions) and $z = \{z_i\}$. m_α — non-negative integers
- Eigenfunction of the Macdonald operator E_π.
- Non-symmetric in z or λ
- Coincide with Macdonald polynomials at $t_\alpha = q^{-m_\alpha}$, $\mu = \lambda + \rho_m^\vee$ — partitions and $q^{z_i} = x_i$:

$$P_\mu(x = q^z \mid t_\alpha = q^{-m_\alpha} \mid q, t = q^{-m}) = \sum_{\omega\in W} \Psi_m(\omega z, \lambda) \tag{52}$$

Normalized BA function

When it's properly normalized

$$\psi_{\rho^\vee} = \prod_{\alpha\in R_+}\prod_{j=1}^{m_\alpha}[j-(\alpha,\lambda)], \quad [a]=q^a-1 \tag{53}$$

BA function is symmetric

$$\overline{\Psi}_m(z,\lambda) = \overline{\Psi}_m(\lambda,z) \tag{54}$$

It has the properties:

$$\overline{\Psi}_m(\omega\vec{z},\omega\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda}), \quad \omega\in W \quad \text{(equivariance)} \tag{55}$$

$$\overline{\Psi}_m(-\vec{z},-\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda}) \quad \text{(reflection)} \tag{56}$$

$$\overline{\Psi}_m(-\vec{z},\vec{\lambda}) = \overline{\Psi}_m(\vec{z},\vec{\lambda})\Big|_{q\to q^{-1}} \quad \text{(involution)} \tag{57}$$

Example: Root system C_2

There are four positive roots:

$$\begin{array}{cccc} \varepsilon_1-\varepsilon_2 & 2\varepsilon_2 & \varepsilon_1+\varepsilon_2 & 2\varepsilon_1 \\ m & m_l & m & m_l \end{array} \tag{58}$$

- $m=1,\ m_l=0$

$$\Psi_{1,0}=x_1^{\lambda_1}x_2^{\lambda_2}\left(x_1-\frac{1}{q}\frac{[\lambda_1-\lambda_2+1]}{[\lambda_1-\lambda_2-1]}x_2-\frac{1}{q}\frac{[\lambda_1+\lambda_2+1]}{[\lambda_1+\lambda_2-1]}x_2^{-1}+\right.$$
$$\left.+\frac{1}{q^2}\frac{[\lambda_1-\lambda_2+1]}{[\lambda_1-\lambda_2-1]}\frac{[\lambda_1+\lambda_2+1]}{[\lambda_1+\lambda_2-1]}x_1^{-1}\right) \tag{59}$$

- $m=0,\ m_l=1$

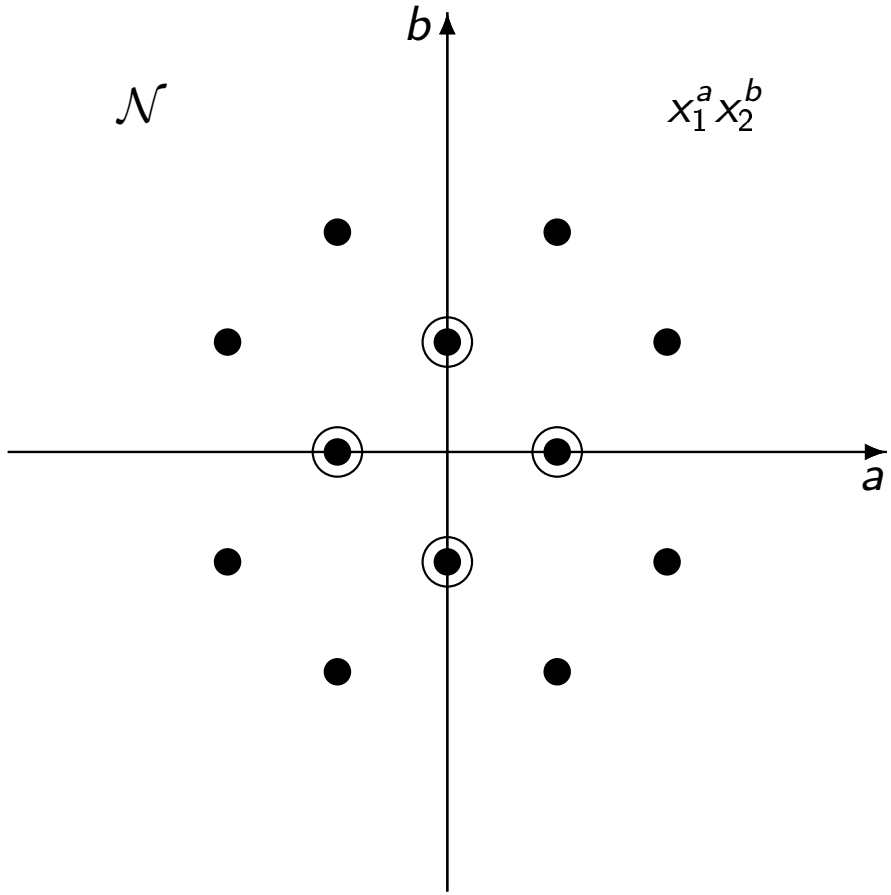
$$\Psi_{0,1}=x_1^{\lambda_1+1}x_2^{\lambda_2+1}-\frac{1}{q}\frac{[\lambda_1+1]}{[\lambda_1-1]}x_1^{\lambda_1-1}x_2^{\lambda_2+1}-\frac{1}{q}\frac{[\lambda_2+1]}{[\lambda_2-1]}x_1^{\lambda_1+1}x_2^{\lambda_2-1}+$$
$$+\frac{1}{q^2}\frac{[\lambda_1+1]}{[\lambda_1-1]}\frac{[\lambda_2+1]}{[\lambda_2-1]}x_1^{\lambda_1-1}x_2^{\lambda_2-1} \tag{60}$$

Symmetrization of monoms:

$$\mathfrak{m}_{[r_1,r_2]}=x_1^{r_1}x_2^{r_2}+x_1^{r_1}x_2^{-r_2}+x_1^{-r_1}x_2^{r_2}+x_1^{-r_1}x_2^{-r_2}+x_2^{r_1}x_1^{r_2}+x_2^{r_1}x_1^{-r_2}+x_2^{-r_1}x_1^{r_2}+x_2^{-r_1}x_1^{-r_2} \tag{61}$$

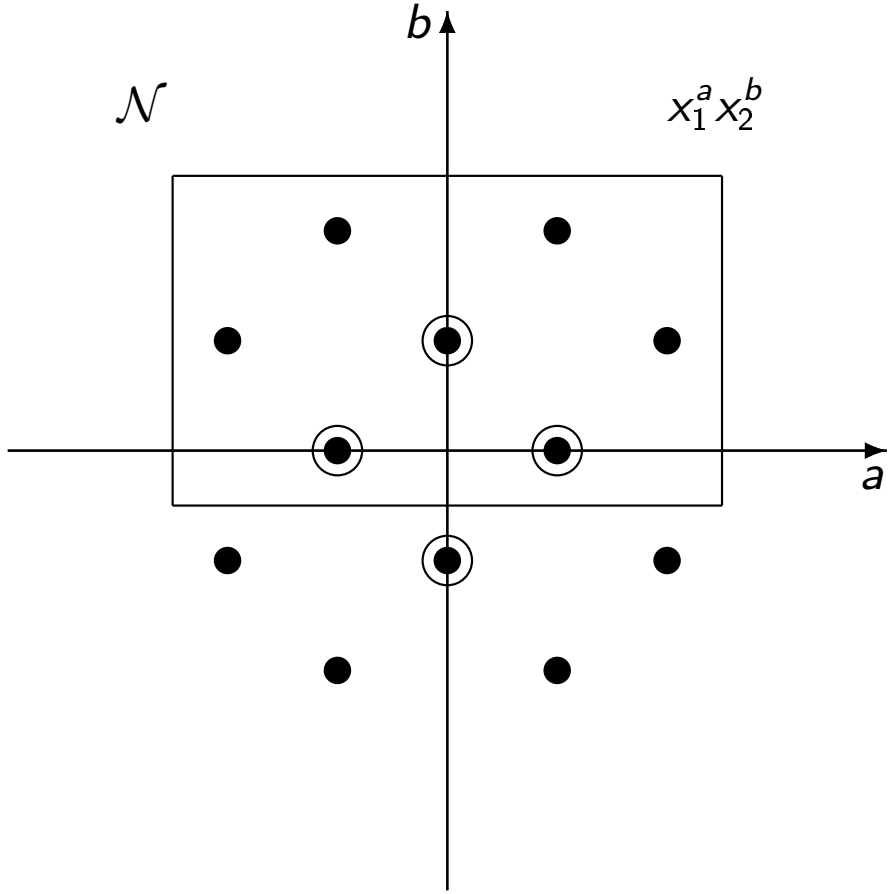
$$\Psi_m(z,\lambda) = q^{(\lambda,z)} \sum_{\nu\in\mathcal{N}} \psi_\nu\, q^{(\nu,z)}, \quad \mathcal{N} = \left\{ \nu = \rho_m^\vee - \sum_{\alpha\in R_+} l_\alpha \alpha^\vee \mid 0 \le l_\alpha \le m_\alpha \right\} \tag{62}$$

- $m = 1,\ m_l = 1$



Equations on BA function determine BA function up to a normalization coefficient $\psi_{\rho^\vee}$.
To obtain Macdonald polynomials one chooses $\psi_{\rho^\vee} = 1$.

- $m = 1,\ m_l = 1$



For $\lambda_1 = r - 2,\ \lambda_2 = -1$

$$\Psi_{1,1} = x_1^r + x_1^{r-2}\left(\frac{1}{q^2}\frac{[2r-4][r]}{[r-2]^2[r-3]}\right) + x_1^{r-4}\left(-\frac{1}{q^3}\frac{[r][r-1]}{[r-3][r-4]}\right) + \left(x_1^{r-1}x_2 + x_1^{r-1}x_2^{-1}\right)\left(-\frac{1}{q}\frac{[r]}{[r-2]}\right) + \left(x_1^{r-3}x_2 + x_1^{r-3}x_2^{-1}\right)\left(\frac{1}{q^2}\frac{[r][r-1]}{[r-2][r-3]}\right) \tag{63}$$

Thank you for your attention!