Entangling gates from cabling of knots

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Chern-Simons theory

Chern-Simons theory is a 3d topological gauge theory

$$S = \frac{k}{4\pi}\int \mathrm{Tr}_{\mathrm{adj}}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

We study Wilson-loop averages in such a theory:

$$\left\langle W_Y^K\right\rangle_{CS(N,q)} = \left\langle \mathrm{Tr}_Y Pexp\left(\oint\limits_K \mathcal{A}\right)\right\rangle_{CS(N,q)}$$

K - contour in 3d space - knots and links.

Knot polynomials

Wilson-loop averages in Chern-Simons theory are equal to knot polynomials.
Depending on the gauge group they are equal to

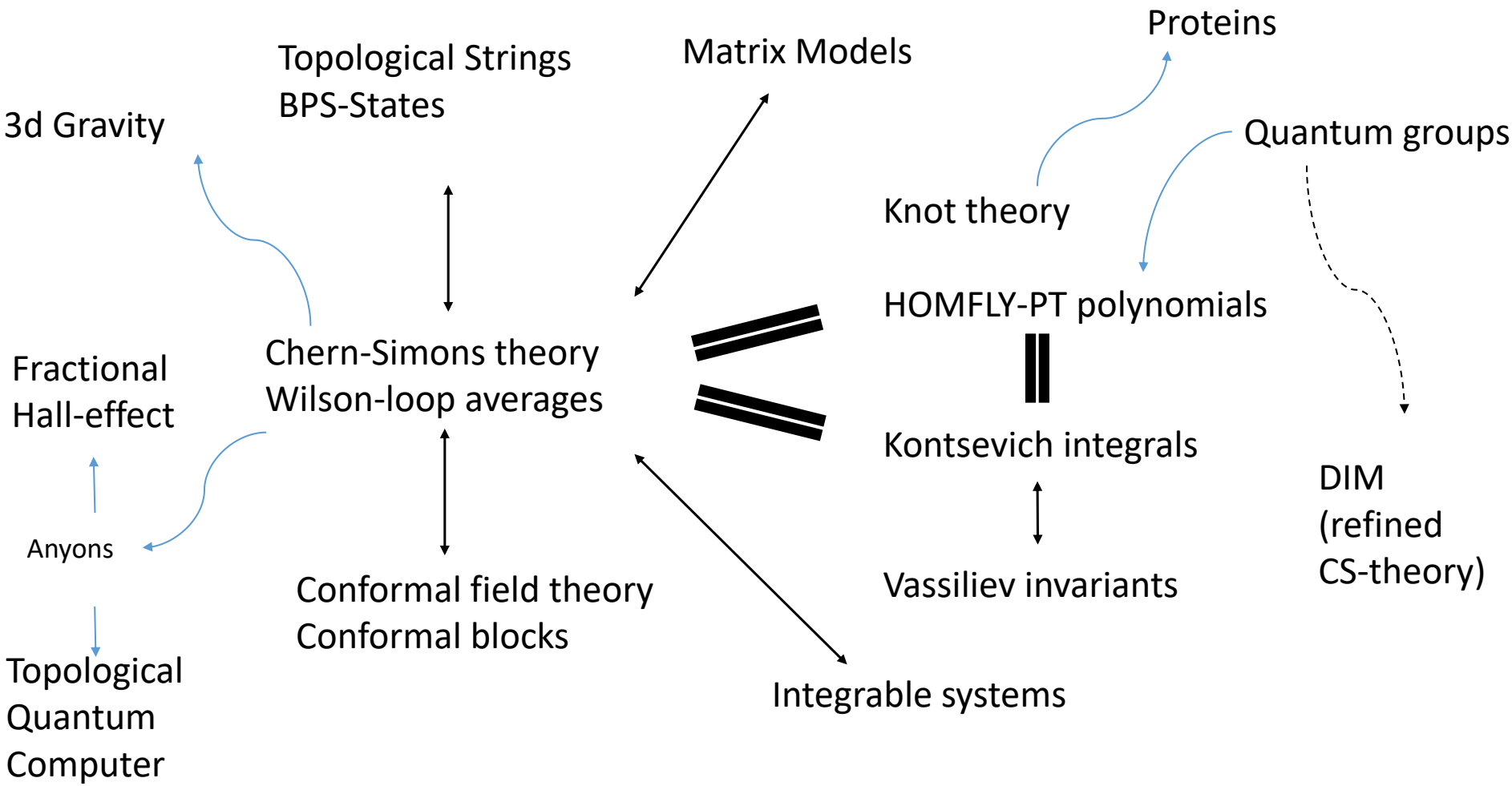
- SU(2) – Jones polynomials $\left\langle W_Y^K \right\rangle_{CS(2,q)} = J(q)$
- SU(N) – HOMFLY-PT polinomials $\left\langle W_Y^K \right\rangle_{CS(N,q)} = H(A,q)$
- SO(N) – Kauffmann polynomials

Variables in polynomials are functions of theory parameters:

$$q = exp(\frac{4\pi i}{k+N})$$

$$A = q^N$$

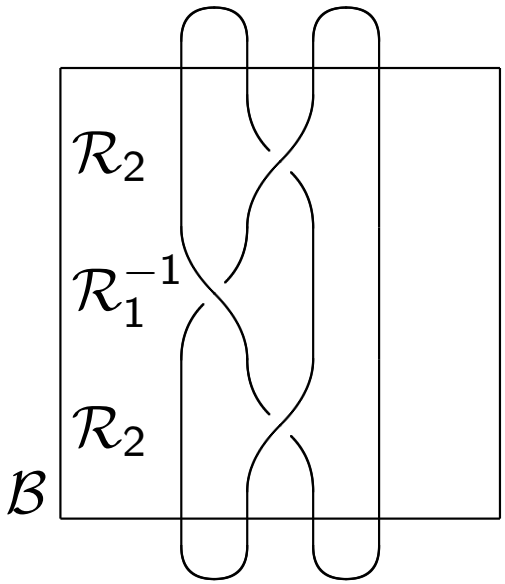
Chern-Simons theory and other theories



Reshetikhin-Turaev approach

Based on the field theory and topological properties, Wilson-loop averages correspond to the products of operators, placed on the crossings – quantum $\mathcal{R}$-matrices.

$$\mathcal{B} = \mathcal{R}_2 \mathcal{R}_1^{-1} \mathcal{R}_2$$

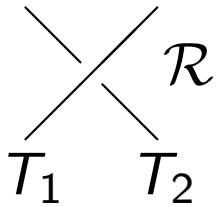


Quantum $\mathcal{R}$-matrix

Quantum $\mathcal{R}$-matrix is an operator from the quantum groups

$$\mathcal{R} = \mathcal{P} \otimes q^{\sum_{i,j} a^{-1}_{i,j} h_i \otimes h_j} \prod_{\beta \in \Phi^+}^{\rightarrow} exp_q \left((q - q^{-1}) E_\beta \otimes F_\beta \right),$$

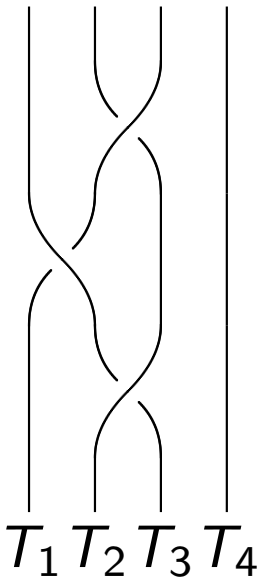
It acts on a product of two representations, placed on two strands.



Moreover it does not change the irreducible representation $Q \in T_1 \otimes T_2$

$\mathcal{R}$-matrices in a braid

If we study a braid, we should fix the irreducible representation in the whole braid and use the corresponding $\mathcal{R}$-matrices, acting on pairs of strands.



$$T_1 \otimes T_2 \otimes T_3 \otimes T_4 = \sum_i M_{Q_i} Q_i$$

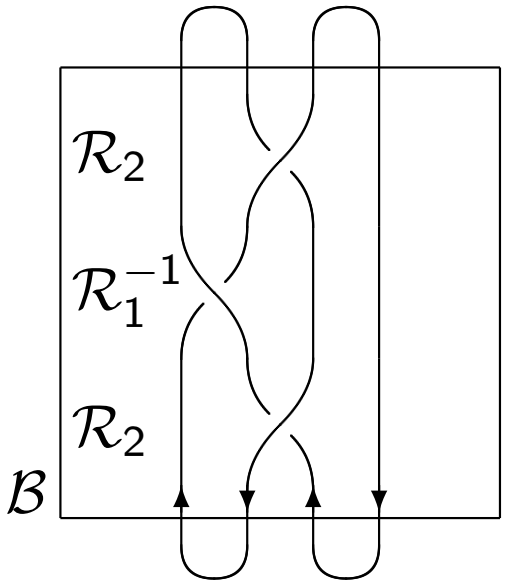
Example:

$$[1]^{\otimes 4} = [4] + 3 \cdot [3,1] + 2 \cdot [2,2] + 3 \cdot [2,1,1] + [1,1,1,1]$$

We can think of an $\mathcal{R}$-matrix as an operator, acting on the highest weights of the representations Q.

4-plat knot

"Hats" at the bottom and the top correspond to the projection on the trivial representation.



Representations expansion

Two strands connected with a "hat" should have a trivial representation in a product, therefore they are conjugate.

The product of representations, if we put fundamental representation on a Wilson loop has the following expansion:

$$[1] \otimes [\bar{1}] \otimes [1] \otimes [\bar{1}] = (\emptyset + \mathrm{Adj}) \otimes (\emptyset + \mathrm{Adj}) = 2 \cdot \emptyset + 3 \cdot \mathrm{Adj} + \ldots$$

At the beginning all representations are trivial, therefore total representation remains trivial.

4-plat operations

Answer for Wilson-loop average is given by

$$H = S_T \mathcal{B}_{\emptyset\emptyset}$$

$\mathcal{B}$ is constructed from the following matrices:

$$\mathcal{R}_1 = \begin{pmatrix} q & \\ & -q^{-1} \end{pmatrix}, \qquad \mathcal{R}_2 = \begin{pmatrix} -\frac{q^{-2}}{q+q^{-1}} & \frac{\sqrt{q^2+1+q^{-2}}}{q+q^{-1}} \\ \frac{\sqrt{q^2+1+q^{-2}}}{q+q^{-1}} & -\frac{q^2}{q+q^{-1}} \end{pmatrix}$$

$$q = \exp\left(\frac{4\pi i}{k+N}\right)$$

Anyons

Anyons are quasiparticles in two-dimensional space.

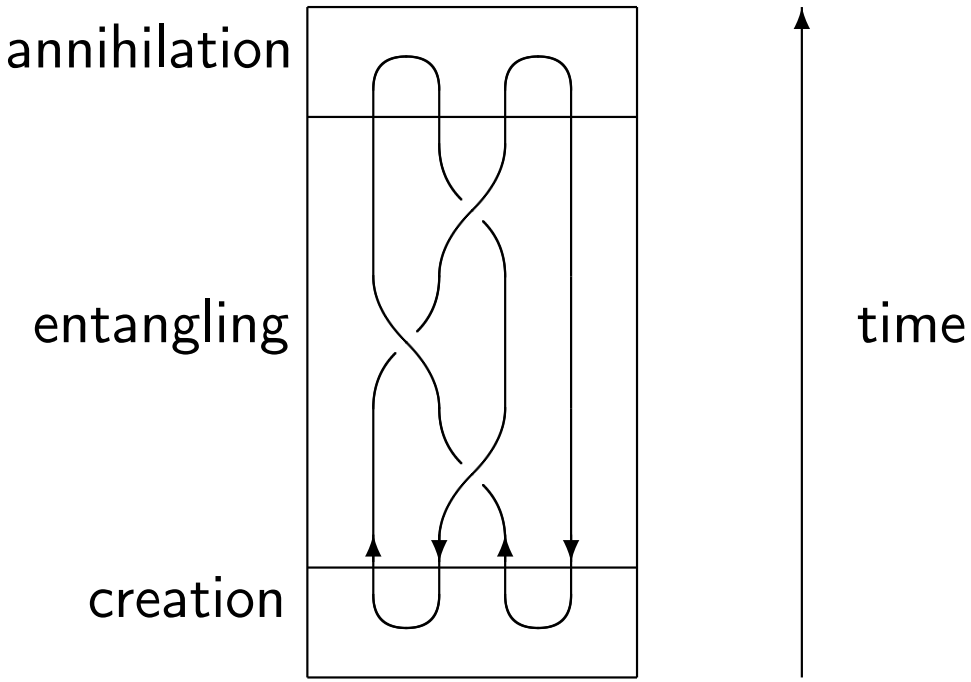
When we exchange them, anything can appear

- Bosons $\Psi(1,2) = \Psi(2,1)$
- Fermions $\Psi(1,2) = -\Psi(2,1)$
- Abelians Anyons $\Psi(1,2) = exp(i\alpha)\Psi(2,1)$
- Non-abelian Anyons $\Psi(1,2) = \hat{O}\Psi(2,1)$

Particles with effective Chern-Simons theory are anyons.

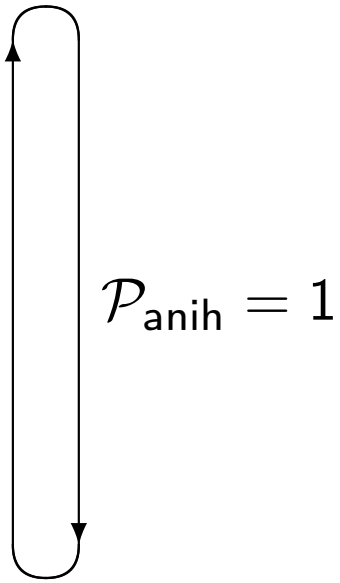
Anyonic evolution

"Hats" at the bottom correspond to the creation of pairs of anyons, while "hats" at the top correspond to their annihilation.

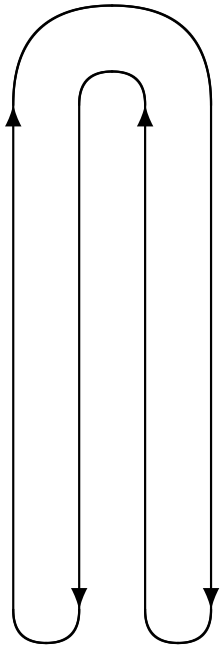


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Examples of probability of annihilation



$\mathcal{P}_{\text{anih}} = 1$

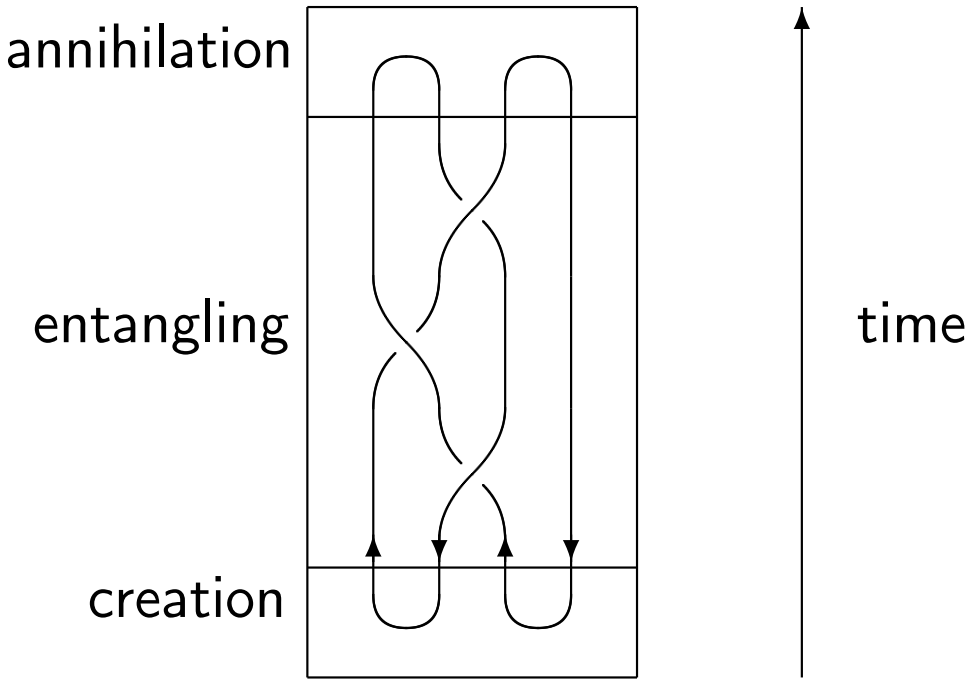


$\mathcal{P}_{\text{anih}} = \frac{1}{[N]_q^2} = \left(\frac{q-q^{-1}}{q^N-q^{-N}}\right)^2$

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Knot as an anyonic process

"Hats" at the bottom correspond to the creation of pairs of anyons, while "hats" at the top correspond to their annihilation.

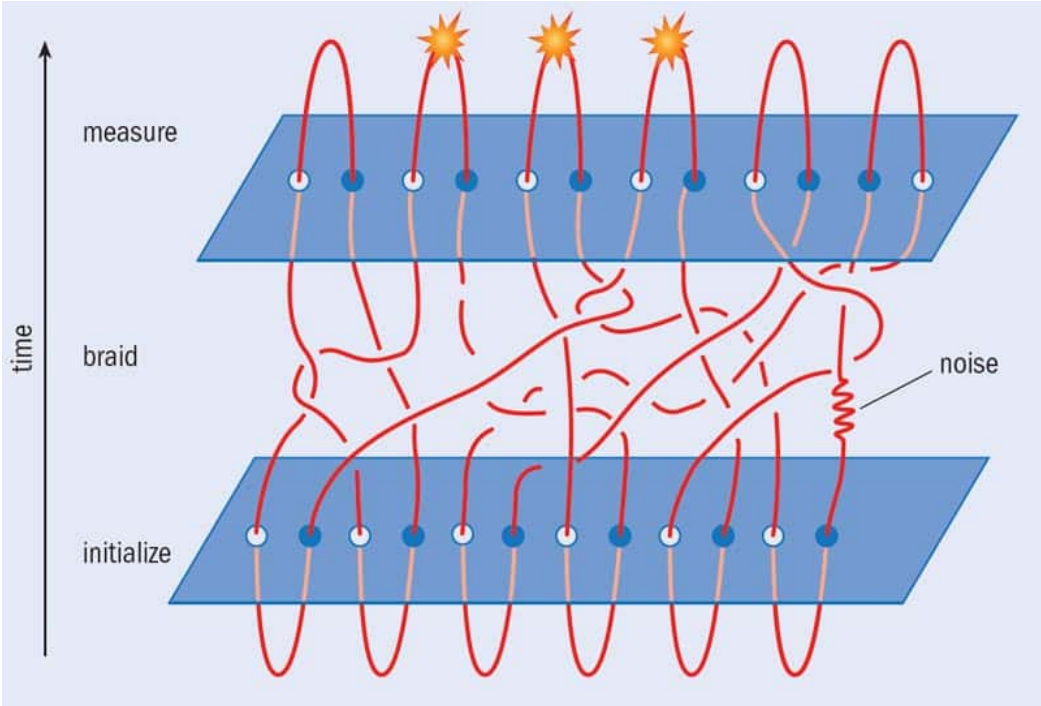


Knot polynomial is a probability amplitude of pairs of anyons annihilating after entangling.

Topological quantum computer

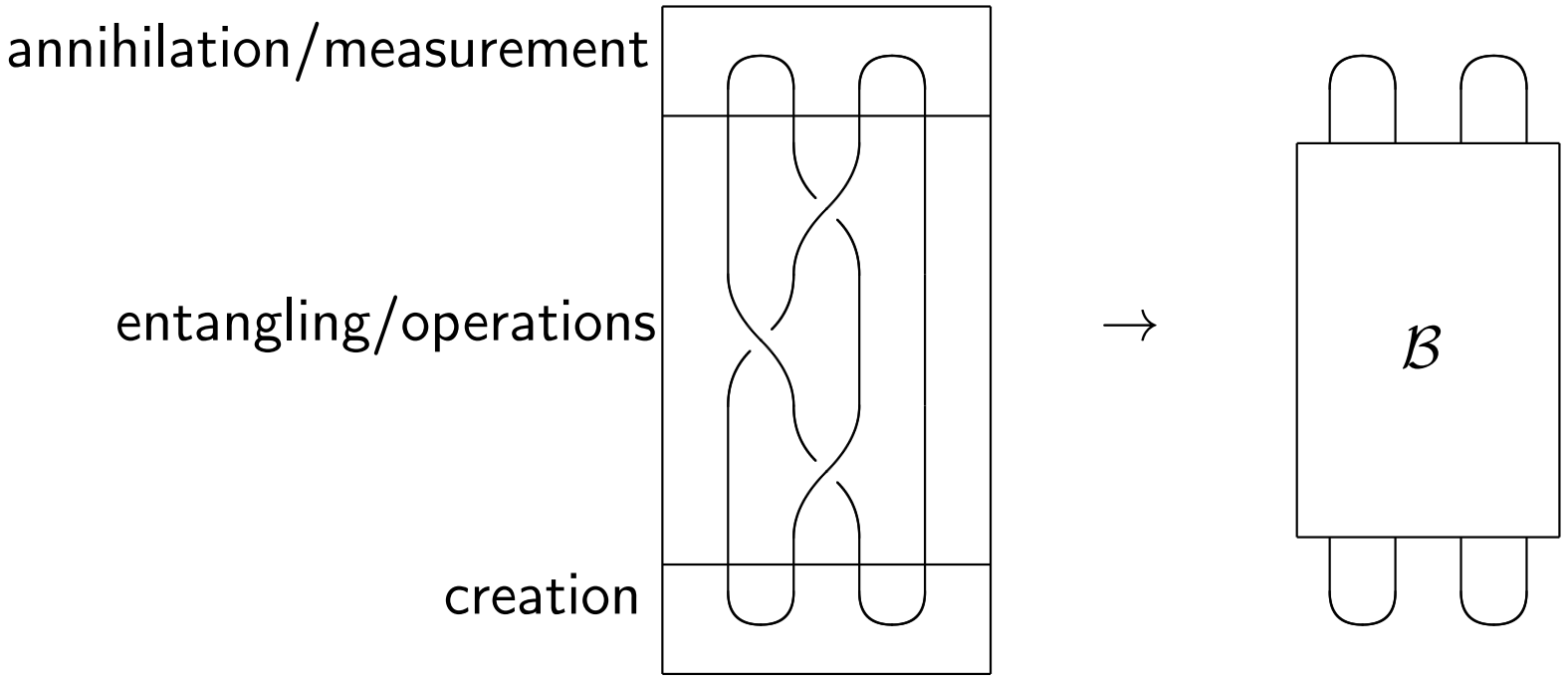
Using the basic idea of quantum computer, topological quantum computer, based on anyons was proposed.

It is supposed to be resistant to noises.



One-qubit operations

One qubit operations in topological quantum computer correspond to 4-plat knot.



One-qubit operations

Elementary operations are matrices 2-by-2

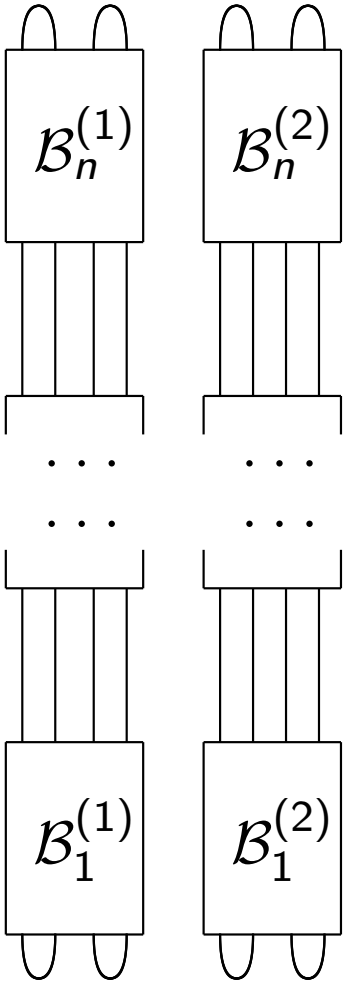
$$\mathcal{R}_1 = \begin{pmatrix} q & \\ & -q^{-1} \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} -\frac{q^{-2}}{q+q^{-1}} & \frac{\sqrt{q^2+1+q^{-2}}}{q+q^{-1}} \\ \frac{\sqrt{q^2+1+q^{-2}}}{q+q^{-1}} & -\frac{q^2}{q+q^{-1}} \end{pmatrix}$$

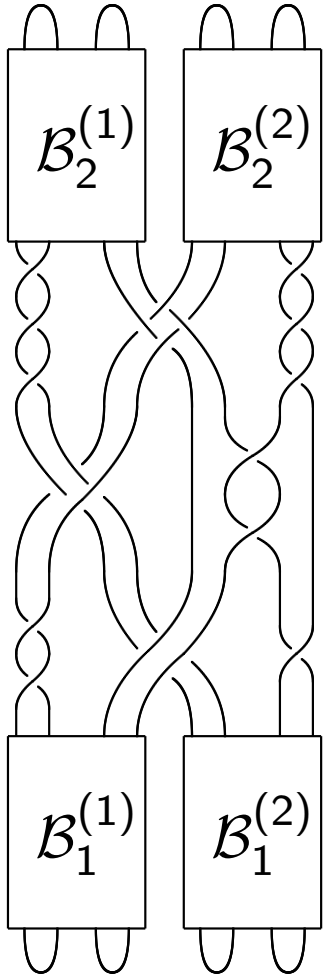
Dimension 2 is for the two trivial representations in the tensor product:

$$[1] \otimes [1] \otimes [\bar{1}] \otimes [\bar{1}] = \mathbf{2 \cdot \emptyset} + 3 \cdot \mathrm{Adj} + \ldots$$

Two-qubit operations

Straightforward idea is to write two qubit operation as a pair of 4-plat knots





Problem with computational space

If we take 4-strand braids which are unentangled with each other – everything is OK.
However, when we entangle them then there is a following product of representations:

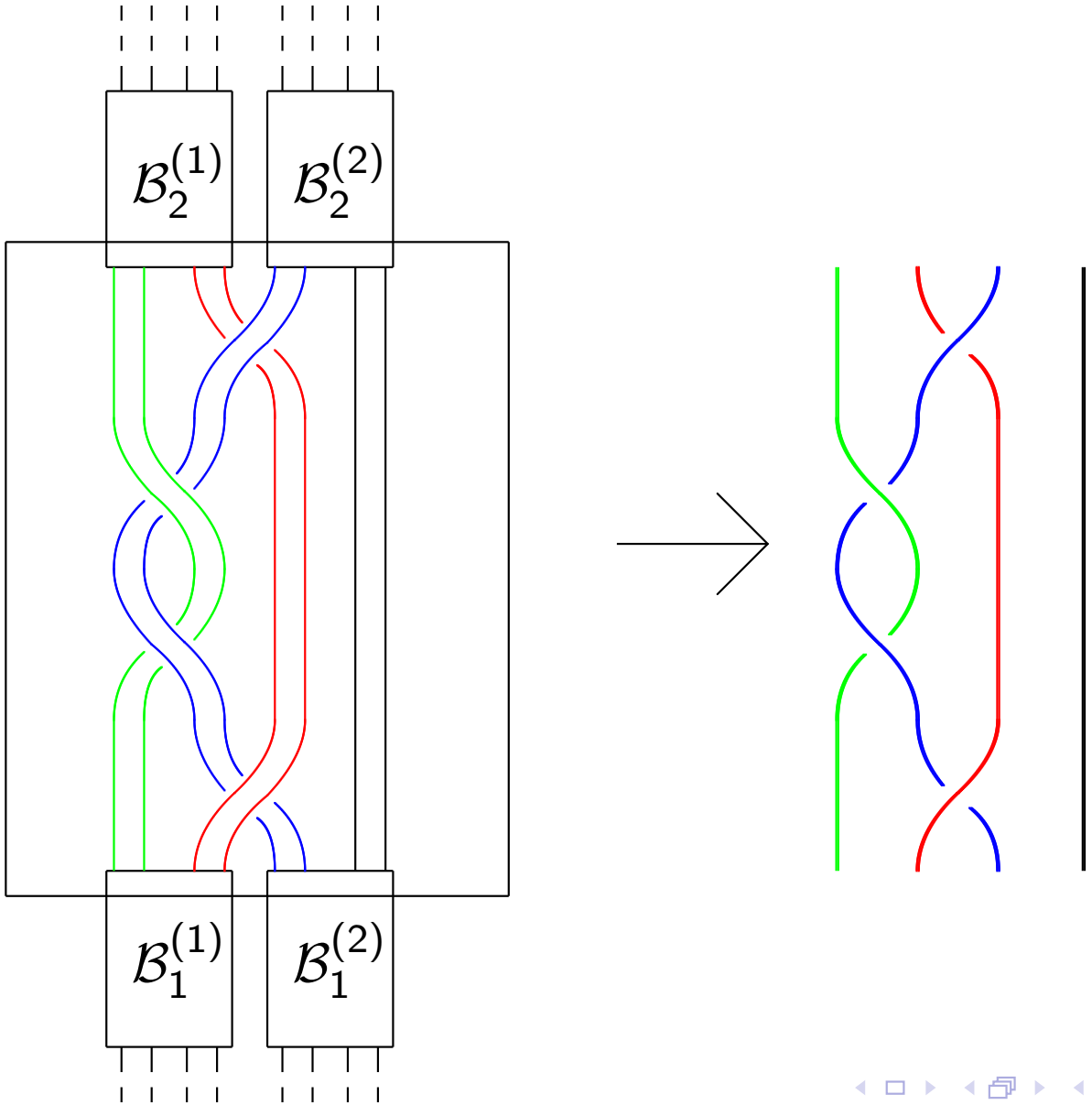
$$([1] \otimes [\bar{1}]) \otimes ([1] \otimes [\bar{1}]) \otimes ([1] \otimes [\bar{1}]) \otimes ([1] \otimes [\bar{1}]) = 24\ \emptyset + \dots$$

In the case of $U_q(SU(2))$:

$$([1] \otimes [1]) \otimes ([1] \otimes [1]) \otimes ([1] \otimes [1]) \otimes ([1] \otimes [1]) = 14\ \emptyset + \dots$$

which is more then 4 that we needed.

We can use and entangle cables instead of separate strands:



Cable consequence

In each qubit we have two cables, each one either in trivial or adjoint representation:

$$[1] \otimes [\bar{1}] = \emptyset + \text{adj}.$$

Therefore in total there are four cases

$$\begin{aligned} &I. && \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset, \\ &II. && \emptyset \otimes \emptyset \otimes \text{adj} \otimes \text{adj}, \\ &III. && \text{adj} \otimes \text{adj} \otimes \emptyset \otimes \emptyset, \\ &IV. && \text{adj} \otimes \text{adj} \otimes \text{adj} \otimes \text{adj}. \end{aligned}$$

Trivial representation $\emptyset$ interacts trivially with all other representations. Therefore, case *I*. will not be changed by any braiding. Cases *II*. and *III*. also do not change as long as cables do not change the position in the end.

Two qubit operation

The whole operation therefore is a 4-by-4 matrix

$$\mathcal{O} \approx \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{i\phi} \end{pmatrix},$$

where the last term comes form adjoint braid.

Adjoint braid operations

For adjoint representation there are following matrices:

$$R = \begin{pmatrix} 1 & & \\ & -q^2 & \\ & & q^6 \end{pmatrix}, \; S = \bar{S} = \begin{pmatrix} \frac{1}{[3]_q} & \frac{\sqrt{[3]_q}}{[3]_q} & \frac{\sqrt{[5]_q}}{[3]_q} \\ \frac{\sqrt{[3]_q}}{[3]_q} & \frac{[6]_q}{[3]_q[4]_q} & -\frac{[2]_q\sqrt{[3]_q[5]_q}}{[3]_q[4]_q} \\ -\frac{\sqrt{[5]_q}}{[3]_q} & \frac{[2]_q\sqrt{[3]_q[5]_q}}{[3]_q[4]_q} & -\frac{[2]_q}{[3]_q[4]_q} \end{pmatrix}$$

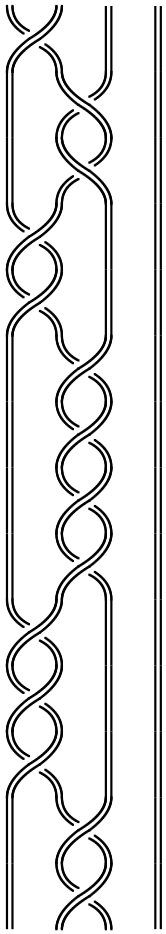
We need a braid where element [1,1] is as close to 1 in absolute value but with a nontrivial phase

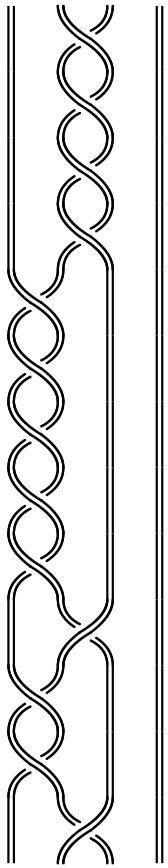
P is a probability of remaining in the computational space. $q = \exp\left(\frac{4\pi i}{k+2}\right)$

$k = 42$

$P = 0.99$

$\phi = 0.14\pi$



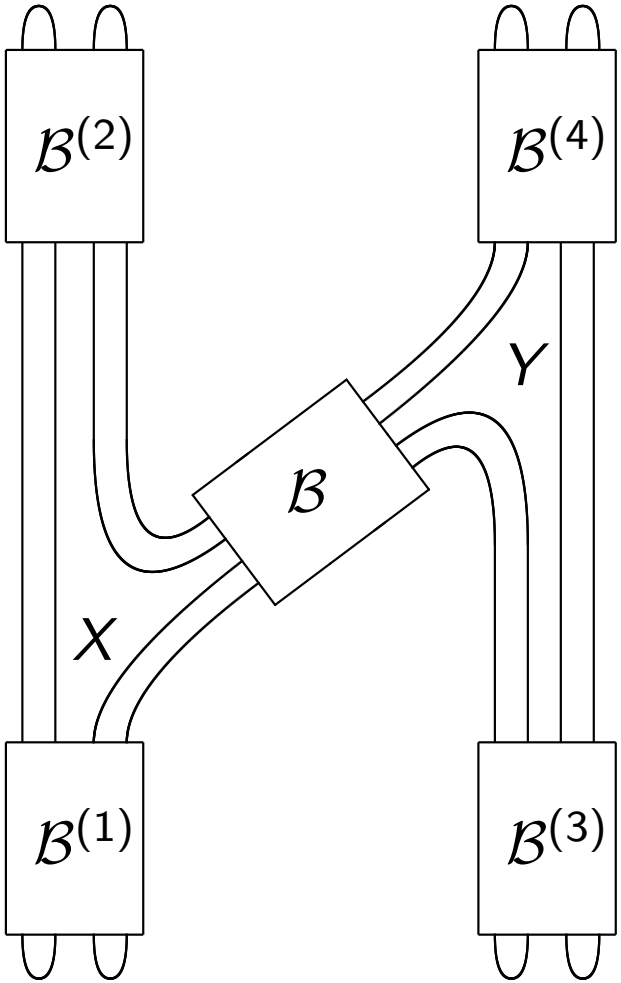


$k = 26$

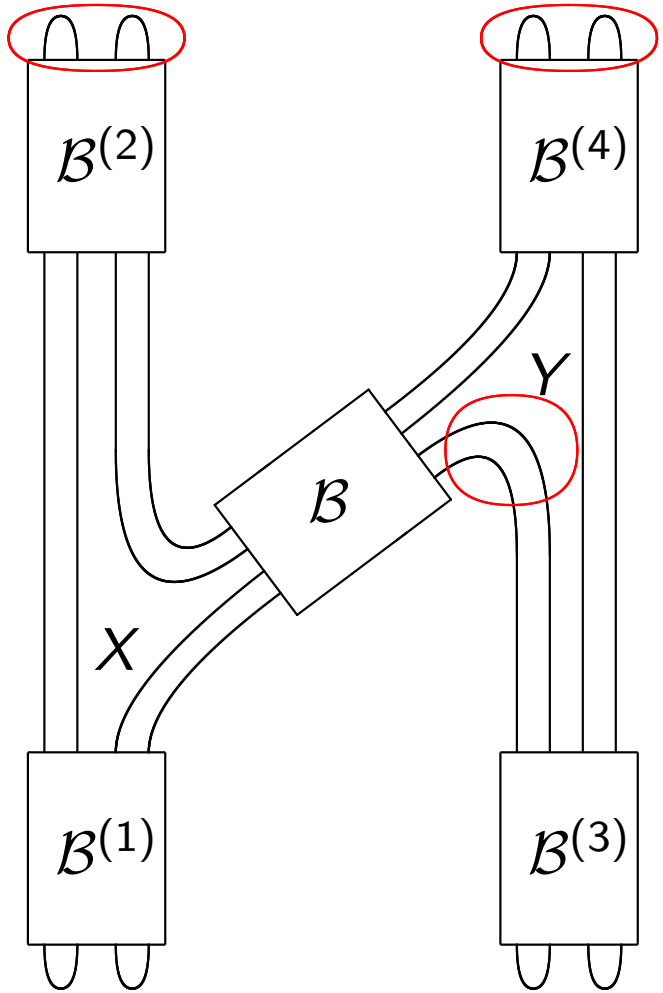
$P = 0.99$

$\phi = 0.29\pi$

Arborescent operations



Arborescent operations



THANK YOU FOR YOUR ATTENTION!