Differential homotopy approach in the higher spin theory

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Presentation plan

1. Equations of higher-spin fields
2. Nonlinear generating system
3. Differential homotopy

Fronsdal equations

Fronsdal (1978) equations:

$$\partial_m\partial^m\phi^{a_1\dots a_s} - \partial^{(a_1}\partial_m\phi^{a_2\dots a_s)m} + \partial^{(a_1}\partial^{a_2}\phi^{a_3\dots a_s)m}{}_m = 0 \qquad (1)$$

on totally symmetric and double traceless fields

$$\phi^{a_1\dots a_s} := \varphi^{a_1\dots a_s} + g^{(a_1a_2}\varphi^{a_3\dots a_s)} \qquad (2)$$

$$\phi^{a_1a_2a_3a_4a_5\dots a_s}g_{a_1a_2}g_{a_3a_4} =: \phi^{a}{}_{a}{}^{b}{}_{b}{}^{a_5\dots a_s} = 0 \qquad (3)$$

where g_{ab} – space-time metric tensor.

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Gauge transformations:

$$\delta_\epsilon \phi^{a_1 \dots a_s} = \partial^{(a_1} \epsilon^{a_2 \dots a_s)}, \quad \epsilon^{b}{}_{b}{}^{a_4 \dots a_s} \text{ and symmetric} \qquad (4)$$

Spinorial representation

In $d = 4$:

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$$A^{a_1\ldots a_k, b_1\ldots b_m} \Rightarrow A^{\alpha_1\ldots\alpha_{k+m},\dot\alpha_1\ldots\dot\alpha_{k-m}} \oplus \bar A^{\alpha_1\ldots\alpha_{k-m},\dot\alpha_1\ldots\dot\alpha_{k+m}} \tag{6}$$

$$A^{a_1\ldots a_k} \Rightarrow A^{\alpha_1\ldots\alpha_k,\dot\alpha_1\ldots\dot\alpha_k} \tag{7}$$

$$A^{a_1\ldots a_k, b_1\ldots b_k} \Rightarrow A^{\alpha_1\ldots\alpha_{2k}} \oplus \bar A^{\dot\alpha_1\ldots\dot\alpha_{2k}} \tag{8}$$

Generating functions

Combine all fields in one generating function

$$\omega(y,\bar{y}|x)=\sum_{n=0,m=0}^{\infty}\frac{1}{n!m!}\omega_{\alpha_1,\ldots,\alpha_n,\dot{\alpha}_1,\ldots,\dot{\alpha}_m}(x)y^{\alpha_1}\ldots y^{\alpha_m}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_n},\quad(9)$$

where components with $n+m=2(s-1)$ connected with field of spin s;

$$C(y,\bar{y}|x)=\sum_{n=0,m=0}^{\infty}\frac{1}{n!m!}C_{\alpha_1,\ldots,\alpha_n,\dot{\alpha}_1,\ldots,\dot{\alpha}_m}(x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_n},\quad(10)$$

where components with $|n-m|=2s$ connected with field of spin s.

Klein operators

$$\{k, y_\alpha\} = [k, \bar{y}_{\dot\alpha}] = [\bar{k}, y_\alpha] = \{\bar{k}, \bar{y}_{\dot\alpha}\} = 0$$
$$k^2 = \bar{k}^2 = 1, \qquad [k, \bar{k}] = 0$$

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$$\omega(Y \quad |x) = \sum_{n=0,m=0}^{\infty} \frac{1}{n!m!} \omega_{\alpha_1,\dots,\alpha_n,\dot\alpha_1,\dots,\alpha_m}(x) \, y^{\alpha_1} \dots y^{\alpha_m} \bar{y}^{\dot\alpha_1} \dots \bar{y}^{\dot\alpha_n} \quad , \quad (11)$$

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$$y^{\alpha_1} \dots y^{\alpha_m} \bar{y}^{\dot\alpha_1} \dots \bar{y}^{\dot\alpha_n} (k\bar{k})^j, \quad (11)$$

$$C(Y, K|x) = \sum_{j=0}^{1} \sum_{n=0,m=0}^{\infty} \frac{1}{2n!m!} C^{(j,1-j)}_{\alpha_1,\dots,\alpha_n,\dot\alpha_1,\dots,\dot\alpha_m}(x)$$
$$y^{\alpha_1} \dots y^{\alpha_n} \bar{y}^{\dot\alpha_1} \dots \bar{y}^{\dot\alpha_n} k^j \bar{k}^{1-j}. \quad (12)$$

Star-multiplication

For introduced Moyal star-product

$$f(Y) \star g(Y) = \int dS dT f(Y+S) g(Y+T) \exp iS_\alpha T^\alpha \qquad (13)$$

Central on-mass-shell theorem

$$\mathrm{d}_x \omega_0 = -\omega_0 \star \omega_0$$
$$\mathrm{d}_x \omega = -\{\omega_0, \omega\}_\star + \mathcal{V}(\omega_0, \omega_0, C)$$
$$\mathrm{d}_x C = -[\omega_0, C]_\star$$

Nonlinear system

Auxiliary variables are introduced: $z_\alpha, \quad \mathrm{d}z_\alpha := \theta_\alpha$

$$\{k, z_\alpha\} = [k, \bar{z}_{\dot\alpha}] = [\bar{k}, z_\alpha] = \{\bar{k}, \bar{z}_{\dot\alpha}\} = 0; \qquad \gamma = \exp(iz_\alpha y^\alpha) k\theta^\alpha\theta_\alpha$$

$$(f*g)(Z;Y) = \int dSdT f(Z+S;Y+S)g(Z-T;Y+T)\exp iS_A T^A. \quad (14)$$

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$$\begin{aligned} &W(Z,Y,K|x) \sim dx, \qquad && W(0,Y,K|x) = \omega(Y,K|x) \\ &S(Z,Y,K|x) \sim \theta && \\ & && B(0,Y,K|x) = C(Y,K|x) \end{aligned} \tag{14}$$

Vasiliev (1992) proposed nonlinear equations

$$\mathrm{d}_x W + W * W = 0, \tag{15}$$

$$\mathrm{d}_x B + [W, B]_* = 0, \tag{16}$$

$$-2i\mathrm{d}_z W + \mathrm{d}_x S + \{W, S\}_* = 0, \tag{17}$$

$$-2i\mathrm{d}_z S + S * S = iB * (\eta\gamma + \bar\eta\bar\gamma), \tag{18}$$

$$-2i\mathrm{d}_z B + [S, B]_* = 0. \tag{19}$$

Solution methods

Perturbation theory

$$W(Z,Y) = \omega(Y) + W_1(Z,Y) + W_2(Z,Y) + \dots,$$
$$B(Z,Y) = C(Y) + B_2(Z,Y) + B_3(Z,Y) + \dots,$$
$$S(Z,Y) = S_1(Z,Y) + S_2(Z,Y) + \dots.$$

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$$\mathrm{d}_z\omega = 0, \quad \mathrm{d}_z C = 0,$$
$$\mathrm{d}_z S_1 = -\tfrac{1}{2} C * (\eta\gamma + \bar{\eta}\bar{\gamma}),$$
$$\dots$$

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- Didenko, Gelfond, Korybut, Vasiliev (2018):
  $\Delta_q g(z;y;\theta) = (z+q)^\alpha \frac{\partial}{\partial\theta^\alpha} \int_0^1 \frac{d\tau}{\tau} g(\tau z - (1-\tau)q; y; \theta\tau)$

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- Didenko, Gelfond, Korybut, Vasiliev (2019):
  $\Delta_{q,\beta} g =$
  $\int \frac{d^2u d^2v}{(2\pi)^2} e^{iu_\beta v^\beta} (z+q-\beta v)^\alpha \frac{\partial}{\partial\theta^\alpha} \int_0^1 \frac{d\tau}{\tau} g(\tau z - (1-\tau)[q-\beta v]; y+u; \theta\tau)$

Differential homotopy approach

$\mathrm{d}_z f(Y,Z,\theta) = g(Y,Z,\theta)$ is solved by the shifted homotopy method:

$$\triangle_a g(Y,Z,\theta) = \int_0^1 (Z+a)^A \frac{\partial}{\partial\theta^A}\frac{\mathrm{d}t}{t} g(Y, tZ-(1-t)a, t\theta)$$

Differential homotopy implies:

- Integration is postponed until the last step
- Shift parameters a, integration variables $\longrightarrow$ space coordinates $M = (t^a)$
- Extended differential: $\mathrm{d} = \mathrm{d}_z + \mathrm{d}_t$
- Connection with the standard approach: integration $F(Z,Y,t,\theta,\mathrm{d}t) \to \int_{\{t\}} F$. Weak terms. Weak equality "$\cong$"
- Generalization of the based equation:

$$\mathrm{d}f(Z,Y,t,\theta,\mathrm{d}t) = g(Z,Y,t,\theta,\mathrm{d}t), \qquad \mathrm{d}g = 0.$$

Vasiliev (2023)

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FIN

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Frame-like formulation of HS

$$\phi^{a_1 \ldots a_s} \quad \Rightarrow \quad \mathrm{d}x^n e_n{}^{a_1 a_{s-1}}, \quad g_{a_1 a_2} e^{a_1 a_2 a_3 \ldots a_{s-1}} \tag{25}$$

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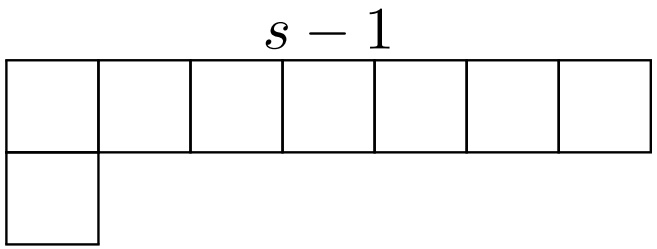

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Curvature:

$$\begin{aligned} R^{a_1\dots a_{s-1}} &= \mathrm{d}_x e^{a_1\dots a_{s-1}} + e_n \wedge \omega^{a_1\dots a_{s-1},n} \\ \delta_\epsilon \omega^{a_1\dots a_{s-1},b_1} &= \mathrm{d}_x \epsilon^{a_1\dots a_{s-1},b_1} + e_{b_2} \wedge \epsilon^{a_1\dots a_{s-1},b_1 b_2} \end{aligned} \tag{27}$$

Frame-like formulation of HS

$$\phi^{a_1\dots a_s} \quad \Rightarrow \quad \mathrm{d}x^n e_n{}^{a_1 a_{s-1}}, \quad g_{a_1 a_2} e^{a_1 a_2 a_3 \dots a_{s-1}} \tag{25}$$

Cartan gravity

$e^a = \mathrm{d}x^n e_n{}^a$ – vielbein one-form (Weyl 1929)
$\omega^{a,b}$ – spin connection, $R^a = \mathrm{d}_x e^a + \omega^{a,}{}_b \wedge e^b$ – torsion tensor

Gauge transformations:

$$\delta_\epsilon e^{a_1\dots a_{s-1}} = \mathrm{d}_x \epsilon^{a_1\dots a_{s-1}} + e_n \wedge \epsilon^{a_1\dots a_{s-1},n} \tag{26}$$

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Frame-like formulation of HS

The full set of the linearized higher-spin curvatures (for $t < s-1$)

$$R^{a_1\dots a_{s-1},b_1\dots b_t} = \mathrm{d}_x\omega^{a_1\dots a_{s-1},b_1\dots b_t} + e_c\omega^{a_1\dots a_{s-1},b_1\dots b_t c} \tag{28}$$

$$R^{a_1\dots a_{s-1},b_1\dots b_{s-1}} = \mathrm{d}_x\omega^{a_1\dots a_{s-1},b_1\dots b_{s-1}} \tag{29}$$

is invariant under the higher-spin gauge transformations

$$\delta_\epsilon\omega^{a_1\dots a_{s-1},b_1\dots b_t} = \mathrm{d}_x\epsilon^{a_1\dots a_{s-1},b_1\dots b_t} + e_c\epsilon^{a_1\dots a_{s-1},b_1\dots b_t c} \tag{30}$$

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$R^{a,b} = \mathrm{d}_x\omega^{a,b} + \omega^{a,}{}_c \wedge \omega^{c,b}$ – curvature tensor

Frame-like formulation of HS

The full set of the linearized higher-spin curvatures (for $t < s-1$)

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Equations

$$R^{a_1\dots a_{s-1},b_1\dots b_t} = 0, \tag{32}$$

$$R^{a_1\dots a_{s-1},b_1\dots b_{s-1}} = e_{a_s}\wedge e_{b_s}C^{a_1\dots a_s,b_1\dots b_s}. \tag{33}$$

Frame-like formulation of HS

Integrability condition for Weyl tensor

$$\Rightarrow \mathrm{d}_x C^{a_1\dots a_s, b_1\dots b_s} = e_c \left(C^{a_1\dots a_s c, b_1\dots b_s} + \frac{1}{2} C^{a_1\dots a_s b_1, b_2\dots b_s c} \right) \tag{34}$$

Frame-like formulation of HS

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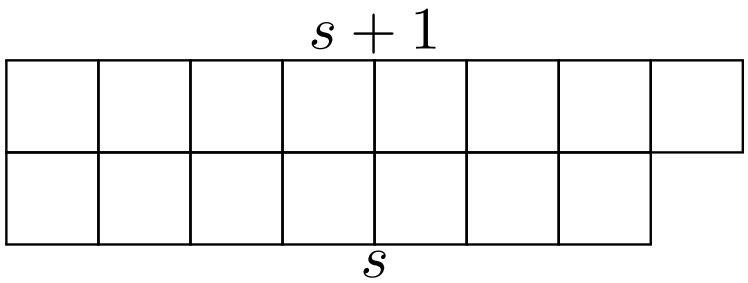

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$$\mathrm{d}_x C^{a_1\dots a_{s+i},b_1\dots b_s} = e_c \left(C^{a_1\dots a_{s+i} c,b_1\dots b_s} + \frac{1}{2} C^{a_1\dots a_{s+i} b_1,b_2\dots b_s c} \right) \tag{35}$$

Frame-like formulation of HS

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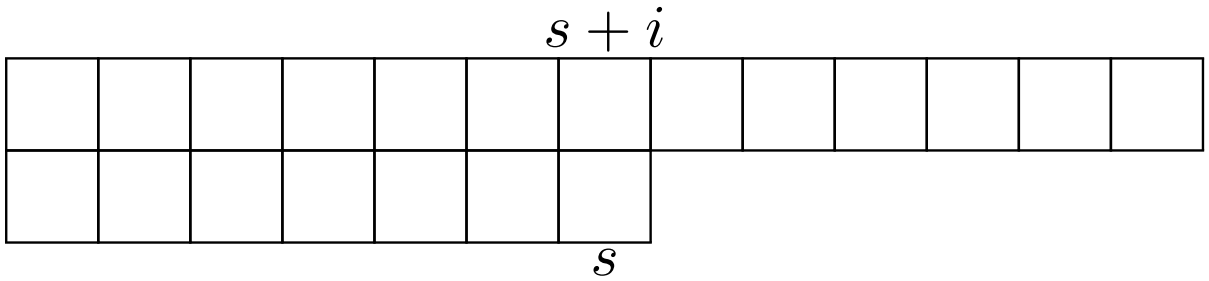

Locality in HS theory

HS equations are not local in the usual sense due to the presence of an infinite set of fields of all spins.

Consider field equations of the form:
$\sum_{k=0,l=1}^{\infty} a^{n_1,n_2\ldots n_k}_{A_0A_1\ldots A_l}(s_0,s_1,\ldots,s_l)\partial_{n_1}\ldots\partial_{n_k}\varphi^{A_0}\ldots\varphi^{A_l}=0.$

Spin-locality implies that $\forall\{s_0,\ldots,s_l\}\exists k_{max}(s_0,\ldots,s_l)$:
$a^{n_1,n_2\ldots n_k}_{A_0A_1\ldots A_l}s_0,s_1,\ldots,s_l)=0$ at $k>k_{max}(s_0,\ldots,s_l)$.

As we work with spinor variables, there two types of localities. The first is in terms of space-time variables and the second — in terms of spinor variables. These types may be related with the aid of equations of motions (The Central On-shell Theorem and corrections).

Spinor spin-locality implies space-time one if the vertex is projectively-compact (Vasiliev, 2022).

Compatibility

$$\mathrm{d}_z F = G; \quad \mathrm{d}_z G = 0.$$

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$$\mathrm{d}_z \int_{\{h\}} F = \int_{\{h\}} G; \quad \mathrm{d}_z \int_{\{h\}} G = 0 \quad \Leftrightarrow \quad \int_{\{h\}} \mathrm{d}_z G = 0.$$

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A naive substitution leads to an inconsistent equation $\mathrm{d}F = G$, because $\mathrm{d}G = \mathrm{d}_z G + \mathrm{d}_h G \neq 0$.

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A naive substitution leads to an inconsistent equation $\mathrm{d}F = G$, because $\mathrm{d}G = \mathrm{d}_z G + \mathrm{d}_h G \cong 0$.

However, we can always add weak term $\mathcal{V} \cong 0$, $\mathrm{d}\mathcal{V} = -\mathrm{d}G$ to the R.H.S. of the equation

$$\mathrm{d}F = G + \mathcal{V} \quad \Leftrightarrow \quad \mathrm{d}F \cong G$$