On symmetry breaking in the self-dual higher-spin theory

Ilgam Faliakhov[1, 2]
Vyacheslav Didenko[2]

[1]Moscow institute of physics and technology

[2]Lebedev Physical Institute, RAS

Problems of Modern Mathematical Physics – PMMP 26
Based on arXiv:2509.01477 [hep-th]

План

1. HS equations
2. Trivial vacuum
3. Vertices in the self-dual sector
4. Symmetry breaking vacuum
5. Linearization
6. Dual formulation
7. Conclusions

Unfolded formulation

Higher-spin theory can be given via the *unfolded equations*:

$$d_x\omega = -\omega \star \omega + \mathcal{V}(\omega,\omega,C) + \mathcal{V}(\omega,\omega,C,C) + ... \tag{1a}$$
$$d_x C = -\omega \star C + C \star \pi(\omega) + \Upsilon(\omega,C,C) + \Upsilon(\omega,C,C,C) + ... \tag{1b}$$

- Fields are generating functions of auxiliary variables $(y,\bar{y})$, e.g.:

$$C(x|y,\bar{y}) = \sum_{m,n} \frac{1}{n!m!} C_{\alpha(m),\dot{\beta}(n)}(x) y^{\alpha(m)} \bar{y}^{\dot{\beta}(n)} \tag{2}$$

- *Star-product* governs the higher-spin algebra:

$$f(y,\bar{y}) \star g(y,\bar{y}) = \int f(y+u,\bar{y}+\bar{u}) g(y+v,\bar{y}+\bar{v}) e^{iuv+i\bar{u}\bar{v}} \tag{3}$$

- Equations must be consistent in a sense of $\mathrm{d}_x^2 = 0$. This constraint defines *vertices* $\mathcal{V}, \Upsilon$ up to an exact form.

Generating system

One can reproduce local form of the vertices of the self-dual sector using generating system [Didenko, 2022] (Vasiliev's approach)

$$\mathrm{d}_x W + W * W = 0\,, \tag{4a}$$

$$\mathrm{d}_z W + \{W, \Lambda\}_* + \mathrm{d}_x \Lambda = 0\,, \tag{4b}$$

$$\mathrm{d}_x C + (W(z'; y, \bar{y}) * C - C * W(z'; -y, \bar{y}))\Big|_{z'=-y} = 0\,. \tag{4c}$$

$$\Lambda = \mathrm{d}z^\alpha \int_0^1 \mathrm{d}\tau \tau z_\alpha C(-\tau z, y) e^{i\tau z y} \tag{5}$$

Here algebra has been enlarged by new variable z and hence $W = W(x|y, z)$. Zero-form C does not depend on z.

Trivial vacuum

- Trivial vacuum is

$$\omega_0 = \omega_{AdS},\ C_0 = 0. \tag{6}$$

One can linearize (1) for $\mathbf{w}, \mathcal{C}$. For trivial vacuum this procedure gives *free* higher-spin fields in AdS (Fronsdall equations). This result is known as *Central On-Mass-Shell Theorem*:

$$d_x \mathbf{w} + \{\omega_{AdS}, \mathbf{w}\}_\star = \frac{1}{2}\mathbf{e}^{\alpha\dot\alpha} \wedge \mathbf{e}_\alpha{}^{\dot\beta} \bar\partial_{\dot\alpha}\bar\partial_{\dot\beta}\mathcal{C}(0,\bar y) \tag{7a}$$

$$d_x \mathcal{C} = -\omega_{AdS} \star \mathcal{C} + \mathcal{C} \star \pi(\omega_{AdS}) \tag{7b}$$

- Spin s *primaries* are singled out by

$$(y\partial + \bar y\bar\partial)\mathbf{w} = 2(s-1)\mathbf{w},\ (y\partial - \bar y\bar\partial)\mathcal{C} = \pm 2s\mathcal{C} \tag{8}$$

One-forms sector

[Didenko, Povarnin 2024]

$$\mathcal{V}(\omega,\omega,C^n)=\sum_{0\leqslant k_1\leqslant k_2\leqslant n}(-)^{k_2-k_1+1}\int_{\mathcal{D}_{n+1}^{[k_1,k_2]}}d\xi d\eta\times$$
$$\times\int\frac{d^2ud^2u'd^2vd^2v'}{(2\pi)^4}\,e^{iuv+iu'v'}\left(\prod_{j=1}^{k_1}\star C\Big(\xi_j v+\eta_j v',\bar{y}\Big)\right)\star$$
$$\star\,(\partial_\alpha)^n\omega\Big(\xi_{n+1}y+u-\frac{S_n^{[k_1,k_2]}}{2}v',\bar{y}\Big)\star\left(\prod_{j=k_1+1}^{k_2}\star C\Big(\xi_j v+\eta_j v',\bar{y}\Big)\right)\star \tag{9}$$
$$\star\,(\partial^\alpha)^n\omega\Big(\eta_{n+1}y+u'+\frac{S_n^{[k_1,k_2]}}{2}v,\bar{y}\Big)\star\left(\prod_{l=k_2+1}^{n}\star C\Big(\xi_l v+\eta_l v',\bar{y}\Big)\right),$$

One-form sector

$$S_n^{[k_1,k_2]} = 1 - \sum_{s=1}^{k_2} \xi_s + \sum_{s=k_2+1}^{n} \xi_s + \sum_{s=1}^{k_1} \eta_s - \sum_{s=k_1+1}^{n} \eta_s + \sum_{i<j} (\xi_i \eta_j - \xi_j \eta_i)\,, \quad (10)$$

and the compact (ξ, η)-integration domain is

$$\mathcal{D}_{n+1}^{[k_1,k_2]} = \begin{cases} \eta_1 + \ldots + \eta_{n+1} = 1, & \eta_i \geq 0\,, \\ \xi_1 + \ldots + \xi_{n+1} = 1, & \xi_i \geq 0\,, \\ \eta_i \xi_{i+1} - \eta_{i+1} \xi_i \geqslant 0, & i \in [0, k_1 - 1]\,, \\ \eta_i \xi_{i+1} - \eta_{i+1} \xi_i \leqslant 0, & i \in [k_1 + 1, k_2 - 1]\,, \\ \eta_i \xi_{i+1} - \eta_{i+1} \xi_i \geqslant 0, & i \in [k_2 + 1, n]\,, \end{cases} \quad (11)$$

Zero-form sector

$$\Upsilon(\omega, C^n) =$$

$$= (-i)^{n-1} \sum_{k=0}^{n} (-)^{k+1} \int_{\mathcal{D}_n^{[k]}} d\xi d\eta \int \frac{d^2u d^2v}{(2\pi)^2} e^{iuv} \left(\prod_{j=1}^{k} \star C(\xi_j y + \eta_j v, \bar{y}) \right) \star$$

$$\star (y^\alpha \partial_\alpha)^{n-1} \omega(S_n^{[k]} y + u, \bar{y}) \star \left(\prod_{j=k+1}^{n} \star C(\xi_j y + \eta_j v, \bar{y}) \right) \tag{12}$$

with

$$S_n^{[k]} = -\sum_{s=1}^{k} \xi_s + \sum_{s=k+1}^{n} \xi_s + \sum_{i<j}^{n} (\xi_i \eta_j - \xi_j \eta_i) \,. \tag{13}$$

Zero-form sector

$$\mathcal{D}_n^{[k]} = \begin{cases} \eta_1 + ... + \eta_n = 1\,, \quad \eta_i \geq 0\,, \\ \xi_1 + ... + \xi_n = 1\,, \quad \xi_i \geq 0\,, \\ \eta_i \xi_{i+1} - \eta_{i+1}\xi_i \leqslant 0\,, \quad i \in [1\,,\, k-1]\,, \\ \eta_i \xi_{i+1} - \eta_{i+1}\xi_i \geqslant 0\,, \quad i \in [k+1\,,\, n-1]\,, \end{cases} \tag{14}$$

The integration domains (14) enjoy the following important property:

$$\bigsqcup_{k-\text{odd}} \mathcal{D}_n^{[k]} = \bigsqcup_{k-\text{even}} \mathcal{D}_n^{[k]}\,, \tag{15}$$

Scalar vacuum

Generating system (4) admits simple yet non-trivial solution:

$$W_0 = W_{AdS}, C_0 = \nu \cdot \mathrm{z} e^{y\bar{y}} \tag{16}$$

- Vacuum (16) describes free scalar field in AdS.

$$\phi(x) := C(x|0,0) = \nu \cdot \mathrm{z} \tag{17}$$

Rest of the components of C are *on-shell* derivatives of ϕ, i.e.

$$\partial_{\mu_1} ... \partial_{\mu_n} \phi = C_{\mu_1 ... \mu_n} \tag{18}$$

- ν is a free parameter of deformation.
- (16) breaks HS symmetry, space-time part breaks down to the Poincaré in 3 dimensions. C_0 actually is a projector

$$C_0 \star C_0 \sim C_0 \tag{19}$$

It annihilates the states $y^{\pm} = y \pm i\bar{y}$

$$y^+ \star C_0 = C_0 \star y^- = 0 \tag{20}$$

Linearized equations

$$\mathrm{d}_x \mathbf{w} + \{\omega_{AdS}, \mathbf{w}\}_* = \frac{1}{2}\mathbf{e}^{\alpha\dot{\alpha}} \wedge \mathbf{e}_\alpha{}^{\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \mathcal{C}(0, \bar{y}) \,, \tag{21a}$$

$$\mathrm{d}_x \mathcal{C} + \omega_{AdS} * \mathcal{C} - \mathcal{C} * \pi(\omega_{AdS}) = \Upsilon(\Omega, \mathcal{C}, C_0) \,. \tag{21b}$$

- Fluctuation of connection has a specific argument: $\mathbf{w} = \mathbf{w}(y^+)$. This leads to a vanishing of almost all the vertices in (21) and imposes a constraint on the $\mathcal{C}$

$$\mathcal{C}(0, \bar{y}) = \mathcal{C}_0 + \mathcal{C}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} + \frac{1}{2}\mathcal{C}_{\dot{\alpha}\dot{\alpha}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\alpha}} \tag{22}$$

- Equations (21a) and (21b) yield (22) independently. This is nontrivial result which allows further study of (21). General case is for the general $w(y^+, y^-)$.
- The source for the gauge fields on the right side of equation (21a) is zero for all spins except for $s = 1$. This means that symmetry breaking leads to a collapse of higher-spin states leaving us with only one gauge field of spin $s = 1$, along with matter fields of spins $s = 0$ and chiral $s = \frac{1}{2}$.

Dual formulation

We introduce Vasiliev's current module T [Vasiliev, 2012]

$$\mathcal{C} = \mathrm{z} e^{y\bar{y}} \cdot T(\sqrt{\mathrm{z}} y, \sqrt{\mathrm{z}} \bar{y}) \tag{23}$$

Linear equations (21) become

$$\mathrm{d_x} T - \frac{i}{2} \mathrm{dx}^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta T = \frac{i\nu}{2} \mathrm{dx}^{\alpha\beta} w_\alpha \bar{\partial}_\beta \int_0^1 d\tau (1-\tau)(T^- - T^+)\,, \tag{24a}$$

$$\frac{\partial}{\partial \mathrm{z}} T - \frac{1}{2} \partial \bar{\partial}\, T = -\frac{\nu}{2} w^\alpha \bar{\partial}_\alpha \int_0^1 d\tau (1-\tau)(T^- - T^+)\,, \tag{24b}$$

where

$$T^\pm = T(\tau w, \bar{w} \pm i(1-\tau) w)\,. \tag{25}$$

Dual formulation

- Symmetry unbroken case arises when $\nu = 0$ [Vasiliev, 2012].

$$\mathrm{d_x} T - \frac{i}{2} \mathrm{dx}^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta T = 0 \tag{26a}$$

$$\frac{\partial}{\partial \mathrm{z}} T - \frac{1}{2} \partial \bar{\partial} T = 0 \tag{26b}$$

(26a) is an unfolded equation describing conserving currents of all spins

$$j(y) = T(y, 0), \ \bar{j} = T(0, \bar{y}), j_0 = w\bar{w} \ \partial\bar{\partial} T(0, 0) \tag{27}$$

(26b) gives the z-dependence. Equations are solved by the following T

$$T = C_+(y^+) \cdot C_-(y^-) \tag{28}$$

$$\mathrm{d_x} C_\pm \pm \frac{1}{2} \mathrm{dx}^{\alpha\beta} \partial_\alpha \partial_\beta C_\pm = 0 \tag{29}$$

Latter is nothing but unfolded equation for a massless scalar in $3d$ Minkowski. This is known as a Flato-Fronsdall decomposition.

Dual formulation

$$\mathrm{d_x} T - \frac{i}{2}\mathrm{dx}^{\alpha\beta}\partial_\alpha\bar\partial_\beta T = \frac{i\nu}{2}\mathrm{dx}^{\alpha\beta} w_\alpha \bar\partial_\beta \int_0^1 d\tau(1-\tau)(T^- - T^+)\,, \tag{30a}$$

$$\frac{\partial}{\partial \mathrm{z}} T - \frac{1}{2}\partial\bar\partial\, T = -\frac{\nu}{2} w^\alpha \bar\partial_\alpha \int_0^1 d\tau(1-\tau)(T^- - T^+)\,, \tag{30b}$$

- Symmetry broken case describes a set of non-conserved currents $s > 1$. Currents of spins $s = -1, \pm\frac{1}{2}, 0$ don't change and hence conserve, spin $s = 1$ current changes but remains conserved, and currents of $s < -1$ are prohibited as a consistency condition.

Conclusions

- Two-parametric solution of the self-dual higher spin theory that breaks space-time symmetry of $3+1$ AdS down to a Poincare in $2+1$ was obtained.
- Linear dynamics around this vacuum were studied. The source for the gauge fields is zero for all helicities except for $s=1$. Symmetry breaking leads to a collapse of higher-spin states leaving us with only one gauge field of helicity $s=1$, along with matter fields of spins $s=0$ and $s=\frac{1}{2}$.
- Holographic dual was examined. While undeformed system is dual to a set of conserved currents, in the symmetry broken phase higher-spin currents ($s>1$) do not conserve. Currents of low spins remain conserved, $s=1$ current deforms and others are not.