Vogel universality on diagrams

What is the Vogel universality?

Alexey Sleptsov

ITEP (Kurchatov institute), IITP & MIPT, Moscow
in collaboration with D.Khudoteplov

February 17, 2026

JINR, Dubna

Vogel's paper

1. Pierre Vogel. "*Algebraic structures on modules of diagrams*". In: Journal of Pure and Applied Algebra 215.6 (2011), pp. 1292–1339. Preprint 1995.

2. Pierre Vogel. "*The universal Lie algebra*". preprint available at https://webusers.imj-prg.fr/~pierre.vogel/. (1999)

Universality of Lie algebras: formula (1)

Some values of Lie algebras in the adjoint representation can be parametrized by 3 parameters $\alpha,\ \beta,\ \gamma$, computed by P.Vogel:

Lie algebra	α	β	γ
$A_n = \mathfrak{sl}_{n+1}$	-2	2	$n+1$
$B_n = \mathfrak{so}_{2n+1}$	-2	4	$2n-3$
$C_n = \mathfrak{sp}_{2n}$	-2	1	$n+2$
$D_n = \mathfrak{so}_{2n}$	-2	4	$2n-4$
G_2	-2	10/3	8/3
F_4	-2	5	6
E_6	-2	6	8
E_7	-2	8	12
E_8	-2	12	20
$D(2,1;\lambda)$	-2	λ	$2-\lambda$

$$\dim \mathfrak{g} = -\frac{(2\alpha+2\beta+\gamma)(2\alpha+2\gamma+\beta)(2\beta+2\gamma+\alpha)}{\alpha\beta\gamma} \qquad (1)$$

Attributes (properties) of the universality (1)

$$\dim \mathfrak{g} = -\frac{(2\alpha+2\beta+\gamma)(2\alpha+2\gamma+\beta)(2\beta+2\gamma+\alpha)}{\alpha\beta\gamma}$$

1. 3 parameters (α, β, γ)
2. rational function
3. permutation invariance
4. factorization as a product of linear factors

Universality of Lie algebras: formula (2)

Tensor squared of the adjoint representation can be decomposed into irreducibles in a uniform way. Dimensions are rational functions.

Lie algebra	α	β	γ
$A_n = \mathfrak{sl}_{n+1}$	-2	2	$n+1$
$B_n = \mathfrak{so}_{2n+1}$	-2	4	$2n-3$
$C_n = \mathfrak{sp}_{2n}$	-2	1	$n+2$
$D_n = \mathfrak{so}_{2n}$	-2	4	$2n-4$
$D(2,1;\lambda)$	-2	λ	$2-\lambda$

Lie algebra	α	β	γ
G_2	-2	10/3	8/3
F_4	-2	5	6
E_6	-2	6	8
E_7	-2	8	12
E_8	-2	12	20

$$\mathfrak{g} \otimes \mathfrak{g} = \underbrace{(\varnothing \oplus Y_\alpha \oplus Y_\beta \oplus Y_\gamma)}_{\mathrm{Sym}^2\mathfrak{g}} \oplus \underbrace{(\mathfrak{g} \oplus X_2)}_{\Lambda^2\mathfrak{g}} \tag{2}$$

$$\dim Y_\alpha = -\frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(3\alpha-2t)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)} \tag{3}$$

Attributes (properties) of the universality (2)

$$\mathfrak{g} \otimes \mathfrak{g} = \underbrace{(\varnothing \oplus Y_\alpha \oplus Y_\beta \oplus Y_\gamma)}_{\mathrm{Sym}^2\mathfrak{g}} \oplus \underbrace{(\mathfrak{g} \oplus X_2)}_{\Lambda^2\mathfrak{g}} \tag{4}$$

$$\dim Y_\alpha = -\frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(3\alpha-2t)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)} \tag{5}$$

$$\dim Y_\beta = \dim Y_\alpha\big|_{\alpha\leftrightarrow\beta}, \qquad \dim Y_\gamma = \dim Y_\alpha\big|_{\alpha\leftrightarrow\gamma} \tag{6}$$

1. 3 parameters (α, β, γ)
2. rational function
3. ~~permutation invariance~~
3. permutation covariance
4. factorization as a product of linear factors
5. decomposition into Symmetric and Antisymmetric
6. decomposition into irreps

Why **3** parameters?

Lie algebra	α	β	γ
$A_n = \mathfrak{sl}_{n+1}$	-2	2	$n+1$
$B_n = \mathfrak{so}_{2n+1}$	-2	4	$2n-3$
$C_n = \mathfrak{sp}_{2n}$	-2	1	$n+2$
$D_n = \mathfrak{so}_{2n}$	-2	4	$2n-4$
G_2	-2	10/3	8/3
F_4	-2	5	6
E_6	-2	6	8
E_7	-2	8	12
E_8	-2	12	20

Actually, there are only **2** parameters, because α is a constant.

$$\dim \mathfrak{g} = \frac{(2\beta+\gamma-4)(\beta+2\gamma-4)(\beta+\gamma-1)}{\beta\gamma} \tag{7}$$

Fake universality?

Lie algebra	α	β	γ	dim	fake dim
$A_n = \mathfrak{sl}_{n+1}$	-2	2	$n+1$	$n(n+2)$	$\mathbf{(n-1)(n+3)}$
$B_n = \mathfrak{so}_{2n+1}$	-2	4	$2n-3$	$n(2n+1)$	$\mathbf{(n-1)(2n+3)}$
$C_n = \mathfrak{sp}_{2n}$	-2	1	$n+2$	$n(2n+1)$	$\mathbf{(n-1)(2n+3)}$
$D_n = \mathfrak{so}_{2n}$	-2	4	$2n-4$	$n(2n-1)$	$\mathbf{(n+1)(2n-3)}$
G_2	-2	10/3	8/3	14	**11**
F_4	-2	5	6	52	**49**
E_6	-2	6	8	78	**75**
E_7	-2	8	12	133	**130**
E_8	-2	12	20	248	**245**

$$\dim \mathfrak{g} = -\frac{(2\alpha+2\beta+\gamma)(2\alpha+2\gamma+\beta)(2\beta+2\gamma+\alpha)}{\alpha\beta\gamma} \tag{8}$$

$$\textbf{fake}\dim \mathfrak{g} = -\frac{2(\alpha+2\beta+\gamma)(\alpha+\beta+2\gamma)(2\alpha+\beta+\gamma)}{\alpha\beta\gamma} \tag{9}$$

Rational function?

Simple Lie algebra $\mathfrak{g}$ is embedded into $\mathfrak{so}$ by the adjoint representation $Adj(\mathfrak{g}) \subset \mathfrak{so}(\mathfrak{g})$.

Then one can consider representation $R = \mathrm{Spin}(Adj(\mathfrak{g}))$.

Its dimension is **[P.Suprun, today, 15:30]**:

$$\dim R = 2^{\lfloor \frac{\dim \mathfrak{g}}{2} \rfloor} \tag{10}$$

So we have **non-rational** and even **discontinuous** function.

Is it universal?

Factorization?

In the paper **[M.Avetisyan, A.Isaev, S.Krivonos, R.Mkrtchyan, "The uniform structure of $\mathfrak{g}^{\otimes 4}$", arXiv:2311.05358]** it was calculated the following:

$$[(2,1,1)]\mathfrak{g} = 4\,\mathfrak{g} \oplus E \oplus E' \oplus E'' \oplus \mathbb{L}_3 \oplus 7\,X_2 \oplus I \oplus I' \oplus I'' \oplus F \oplus F' \oplus F'' \oplus F''' \oplus F'''' \oplus F''''' \oplus Y_2 \oplus Y_2' \oplus Y_2'' \oplus 4\,B \oplus 4\,B' \oplus 4\,B'' \oplus 3C \oplus 3C' \oplus 3C'' \oplus Y_3 \oplus Y_3' \oplus Y_3''$$

$$\begin{aligned}
\dim I \quad = \quad & -\frac{(\hat\alpha+\hat\beta+\hat\gamma)(\hat\alpha+2\hat\beta+\hat\gamma)(2\hat\alpha+2\hat\beta+\hat\gamma)(\hat\alpha+\hat\beta+2\hat\gamma)(2\hat\alpha+\hat\beta+2\hat\gamma)}{2\hat\alpha^4\hat\beta^2\hat\gamma^2(\hat\alpha-3\hat\beta)(\hat\alpha-2\hat\beta)(\hat\alpha-\hat\beta)(\hat\alpha-3\hat\gamma)(\hat\alpha-2\hat\gamma)(\hat\alpha-\hat\gamma)(\hat\alpha-\hat\beta-\hat\gamma)} \times \\
& \Big(4\hat\alpha^{10} - 14\hat\alpha^9\hat\beta - 14\hat\alpha^9\hat\gamma - 32\hat\alpha^8\hat\beta^2 - 8\hat\alpha^8\hat\beta\hat\gamma - 32\hat\alpha^8\gamma^2 + 116\hat\alpha^7\hat\beta^3 + 259\hat\alpha^7\hat\beta^2\hat\gamma + 259\hat\alpha^7\hat\beta\hat\gamma^2 + \\
& 116\hat\alpha^7\hat\gamma^3 + 116\hat\alpha^6\hat\beta^4 + 116\hat\alpha^6\hat\beta^3\hat\gamma + 196\hat\alpha^6\hat\beta^2\hat\gamma^2 + 116\hat\alpha^6\hat\beta\hat\gamma^3 + 116\hat\alpha^6\hat\gamma^4 - 310\hat\alpha^5\hat\beta^5 - 1963\hat\alpha^5\hat\beta^4\hat\gamma - \\
& 4762\hat\alpha^5\hat\beta^3\hat\gamma^2 - 4762\hat\alpha^5\hat\beta^2\hat\gamma^3 - 1963\hat\alpha^5\hat\beta\hat\gamma^4 - 310\hat\alpha^5\hat\gamma^5 - 296\hat\alpha^4\hat\beta^6 - 2634\hat\alpha^4\hat\beta^5\hat\gamma - 9263\hat\alpha^4\hat\beta^4\hat\gamma^2 - \\
& 13706\hat\alpha^4\hat\beta^3\hat\gamma^3 - 9263\hat\alpha^4\hat\beta^2\hat\gamma^4 - 2634\hat\alpha^4\hat\beta\hat\gamma^5 - 296\hat\alpha^4\hat\gamma^6 + 256\hat\alpha^3\hat\beta^7 + 500\hat\alpha^3\hat\beta^6\hat\gamma - 2938\hat\alpha^3\hat\beta^5\hat\gamma^2 - \\
& 10206\hat\alpha^3\hat\beta^4\hat\gamma^3 - 10206\hat\alpha^3\hat\beta^3\hat\gamma^4 - 2938\hat\alpha^3\hat\beta^2\hat\gamma^5 + 500\hat\alpha^3\hat\beta\hat\gamma^6 + 256\hat\alpha^3\hat\gamma^7 + 352\hat\alpha^2\hat\beta^8 + 4248\hat\alpha^2\hat\beta^7\hat\gamma + \\
& 20252\hat\alpha^2\hat\beta^6\hat\gamma^2 + 48516\hat\alpha^2\hat\beta^5\hat\gamma^3 + 64320\hat\alpha^2\hat\beta^4\hat\gamma^4 + 48516\hat\alpha^2\hat\beta^3\hat\gamma^5 + 20252\hat\alpha^2\hat\beta^2\hat\gamma^6 + 4248\hat\alpha^2\hat\beta\hat\gamma^7 + \\
& 352\hat\alpha^2\hat\gamma^8 + 96\hat\alpha\hat\beta^9 + 2976\hat\alpha\hat\beta^8\hat\gamma + 20712\hat\alpha\hat\beta^7\hat\gamma^2 + 65496\hat\alpha\hat\beta^6\hat\gamma^3 + 112896\hat\alpha\hat\beta^5\hat\gamma^4 + 112896\hat\alpha\hat\beta^4\hat\gamma^5 + \\
& 65496\hat\alpha\hat\beta^3\hat\gamma^6 + 20712\hat\alpha\hat\beta^2\hat\gamma^7 + 2976\hat\alpha\hat\beta\hat\gamma^8 + 96\hat\alpha\hat\gamma^9 + 576\hat\beta^9\hat\gamma + 5184\hat\beta^8\hat\gamma^2 + 19728\hat\beta^7\hat\gamma^3 + \\
& 41472\hat\beta^6\hat\gamma^4 + 52704\hat\beta^5\hat\gamma^5 + 41472\hat\beta^4\hat\gamma^6 + 19728\hat\beta^3\hat\gamma^7 + 5184\hat\beta^2\hat\gamma^8 + 576\hat\beta\hat\gamma^9\Big)\,,
\end{aligned}$$

$$\dim I' \quad = \quad (\dim I)_{\hat\alpha\leftrightarrow\hat\beta}, \quad \dim I'' = (\dim I)_{\hat\alpha\leftrightarrow\hat\gamma}. \qquad (11)$$

Vogel's algebra Λ

Definition (Algebra Λ)

Λ is an algebra over $\mathbb{Q}$, generated by 3-legged 3-valent diagrams (legs are numerated) modulo AS and IHX relations, antisymmetric with respect to permutations of legs.

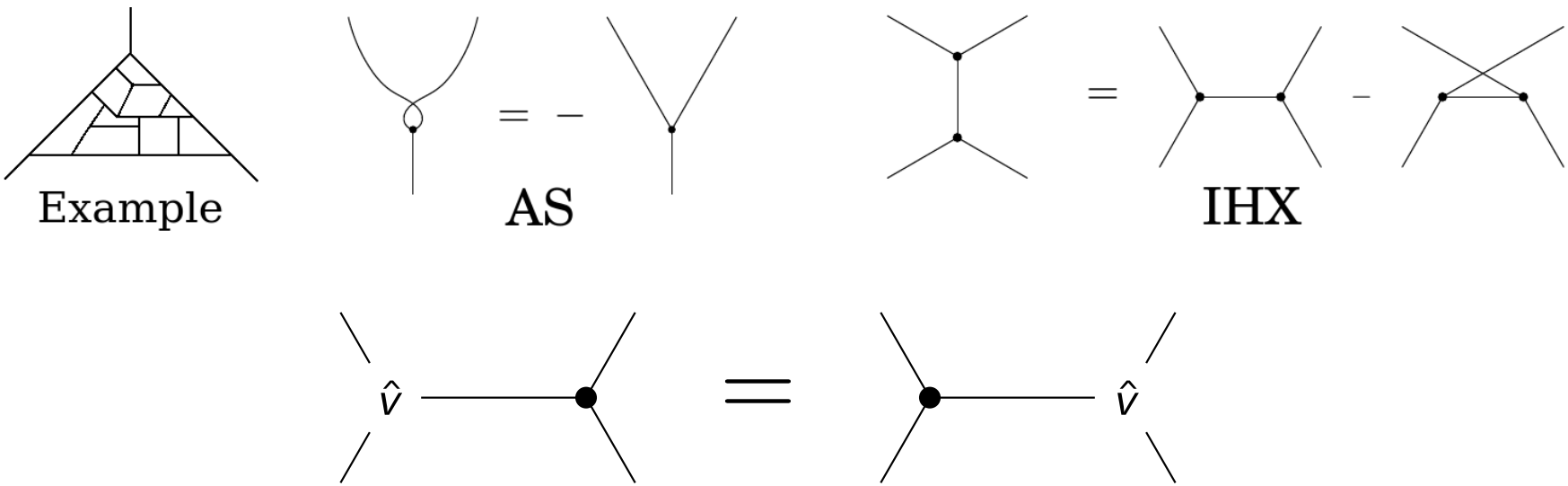


Figure: Well-defined **multiplication** in Λ is given by insertion of one factor into any vertex of the other factor.

Trivalent diagrams as Λ-module

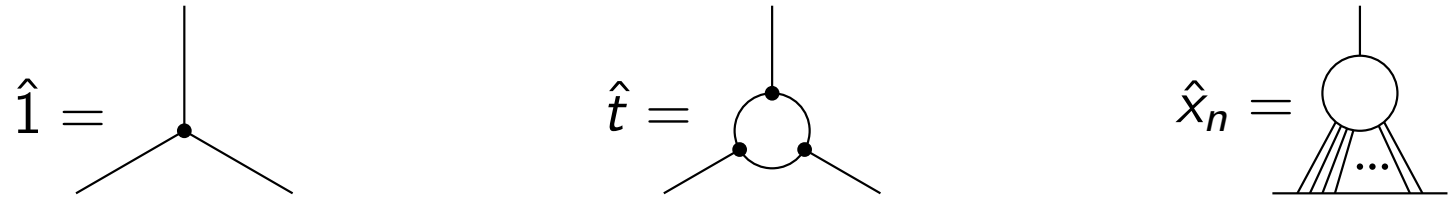


Figure: Some elements of Λ

Λ-algebra naturally acts on connected trivalent diagrams (modulo AS and IHX) with k legs and distinguished curve Γ (modulo STU):

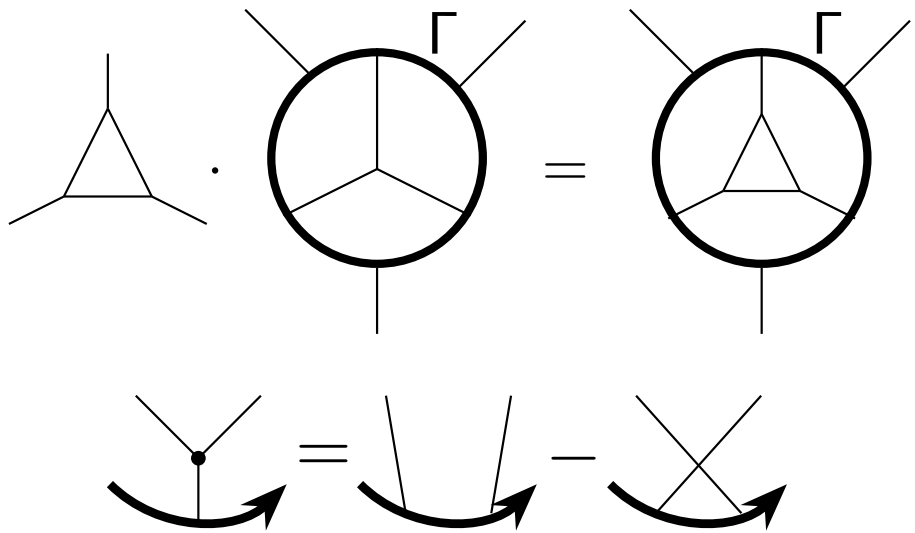


Figure: STU relation

Λ : generators

- The structure of the algebra Λ is **unknown**.
- The dimension of Λ is known up to order **10**.
- Λ-algebra is a **commutative**.
- There exists a unique graded algebra homomorphism

$$\psi : \mathbb{Q}[t] \oplus \omega\mathbb{Q}[t, \sigma, \omega] \to \Lambda,$$

where formal variables $\deg(t) = 1$, $\deg(\sigma) = 2$ and $\deg(\omega) = 3$.

- ψ is **bijective** for the degrees $\leq$ **10** and injective for the degrees ≤ 15. In degree 16 $\operatorname{Ker}\psi \neq \varnothing$.

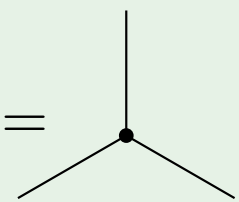
Λ : examples

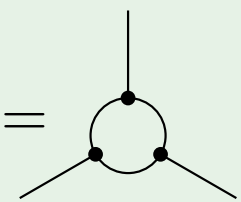
$$\psi : \mathbb{Q}[t] \oplus \omega\mathbb{Q}[t, \sigma, \omega] \to \Lambda.$$

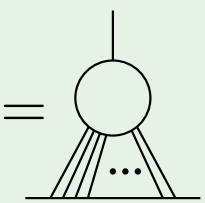
Example

$\psi(t) = \hat{t} = \frac{1}{2}\hat{x}_1, \qquad \psi(t^2) = \hat{t}^2 = \hat{x}_2,$

$\psi(4t^3 - \frac{3}{2}\omega) = \hat{x}_3, \quad \psi(12t^5 - \frac{17}{2}t^2\omega + \frac{3}{2}\sigma\omega) = \hat{x}_5.$

$\hat{1} =$ 

$\hat{t} =$ 

$\hat{x}_n =$ 

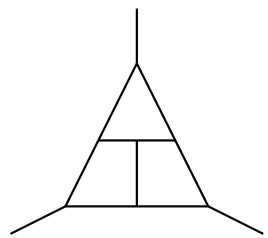
Λ : generalized weight system

Simple Lie algebra $\mathfrak{g}$ (over field K) with non-degenerate bilinear form gives a weight system on Λ by the map

$$\Phi_{\mathfrak{g}} : \Lambda \to \mathfrak{g}^{\otimes k}, \quad \Theta \mapsto f^{abc}, \quad \Theta_{\Gamma} \mapsto \rho(X^a), \qquad (12)$$

where ρ is any finite-dimensional irreducible representation of $\mathfrak{g}$.
Then AS means anticommutativity, IHX means Jacobi identity.
Any simple Lie algebra $\mathfrak{g}$ has only 1 primitive 3-tensor:

$$\Phi_{\mathfrak{g}}(\hat{x}) = \chi_{\mathfrak{g}}(\hat{x})\, f_{abc}, \quad \hat{x} \in \Lambda,$$



$\chi_{\mathfrak{g}}(\hat{x}) : \Lambda \to K$ is a well-defined character. It is a polynomial in t, σ, ω.

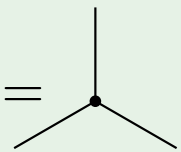
Characters I

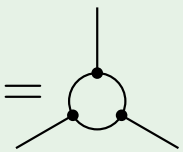
Characters $\chi_{\mathfrak{g}}(\hat{x})$ are polynomials in t, σ, ω.

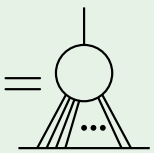
Example

$$\chi_{\mathfrak{g}}(\hat{t}) = t_{\mathfrak{g}}, \qquad \chi_{\mathfrak{g}}(\hat{x}_2) = t_{\mathfrak{g}}^2,$$

$$\chi_{\mathfrak{g}}(\hat{x}_3) = 4t_{\mathfrak{g}}^3 - \tfrac{3}{2}\omega_{\mathfrak{g}}, \qquad \chi_{\mathfrak{g}}(\hat{x}_5) = 12t_{\mathfrak{g}}^5 - \tfrac{17}{2}t_{\mathfrak{g}}^2\omega_{\mathfrak{g}} + \tfrac{3}{2}\sigma_{\mathfrak{g}}\omega_{\mathfrak{g}}.$$

$\hat{1} =$ 

$\hat{t} =$ 

$\hat{x}_n =$ 

One can introduce symmetrized variables α, β, γ:

$$\alpha + \beta + \gamma = t \qquad \alpha\beta + \beta\gamma + \alpha\gamma = \sigma - 2t^2 \qquad \alpha\beta\gamma = \omega - t\sigma \tag{13}$$

$$t = \alpha + \beta + \gamma \qquad \omega = (2\alpha + \beta + \gamma)(\alpha + 2\beta + \gamma)(\alpha + \beta + 2\gamma)$$
$$\sigma = 2\alpha^2 + 5\alpha\beta + 5\alpha\gamma + 2\beta^2 + 5\beta\gamma + 2\gamma^2 \tag{14}$$

$$\chi_{\mathfrak{g}}(\hat{x}_3) = \alpha^3 + \beta^3 + \gamma^3 + \frac{3}{2}\left(\alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2\right)$$

Characters II

$$\chi_{\mathfrak{g}}(P_{\mathfrak{sl}}) = -2t^3 + 2\sigma t - \omega$$

$$\chi_{\mathfrak{g}}(P_{\mathfrak{osp}}) = -24\sigma t^4 + 40\omega t^3 + 29\sigma^2 t^2 - 72\omega\sigma t + 4\sigma^3 + 27\omega^2$$

$$\chi_{\mathfrak{g}}(P_{\mathfrak{exc}}) = 40t^3 - 45\sigma t + 27\omega$$

$$\hat{P}_{15} := \hat{\omega} \cdot \hat{P}_{\mathfrak{sl}} \cdot \hat{P}_{\mathfrak{osp}} \cdot \hat{P}_{\mathfrak{exc}}, \quad \chi_{D(2,1;\lambda)}(\hat{P}_{15}) \neq 0$$

Lie algebra	t	σ	ω
$A_n = \mathfrak{sl}_{n+1}$	$n+1$	$2n^2+4n-2$	$2(n^2-1)(n+3)$
$B_n = \mathfrak{so}_{2n+1}$	$2n-1$	4(n+1)(2n-3)	4(n-1)(2n+3)(2n-3)
$C_n = \mathfrak{sp}_{2n}$	$n+1$	(n+2)(2n-1)	(n-1)(2n+3)(n+2)
$D_n = \mathfrak{so}_{2n}$	$2n-2$	4(2n+1)(n-2)	8(n-2)(2n-3)(n+1)
G_2	4	260/9	880/9
F_4	9	170	1470
E_6	12	308	3600
E_7	18	704	12480
E_8	30	1976	58800

Characters III

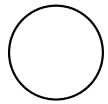
$$
\begin{aligned}
\chi_{\mathfrak{g}}(\hat{P}_{\mathfrak{sl}}) &= (\alpha+\beta)\,(\beta+\gamma)\,(\alpha+\gamma),\\
\chi_{\mathfrak{g}}(\hat{P}_{\mathfrak{osp}}) &= (\alpha+2\beta)(2\alpha+\beta)\,(\beta+2\gamma)(2\beta+\gamma)\,(\alpha+2\gamma)(2\alpha+\gamma),\\
\chi_{\mathfrak{g}}(\hat{P}_{\mathfrak{exc}}) &= (\alpha-2\beta-2\gamma)\,(\beta-2\alpha-2\gamma)\,(\gamma-2\alpha-2\beta).
\end{aligned}
$$

Lie algebra	α	β	γ
$A_n = \mathfrak{sl}_{n+1}$	-2	2	$n+1$
$B_n = \mathfrak{so}_{2n+1}$	-2	4	$2n-3$
$C_n = \mathfrak{sp}_{2n}$	-2	1	$n+2$
$D_n = \mathfrak{so}_{2n}$	-2	4	$2n-4$
G_2	-2	10/3	8/3
F_4	-2	5	6
E_6	-2	6	8
E_7	-2	8	12
E_8	-2	12	20

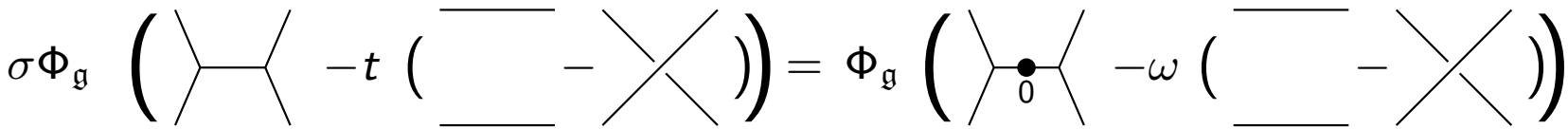
Formula (1) from diagrammatic technique

Equality $\dim_{\text{adj}} = \operatorname{tr} \hat{1}_{\text{adj}}$ in terms of the diagrams trace corresponds to a loop:

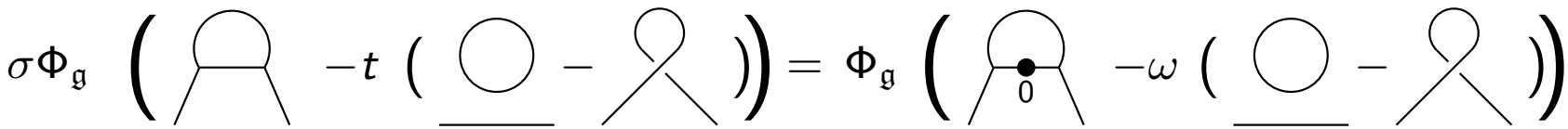
$$\dim_{\text{adj}} = \quad (15)$$



Now the goal is to find a diagrammatic formula that involves such a loop. Thankfully, such formula can be found in Vogel's paper:



We can add a loop by connecting top two legs by an edge, which corresponds to taking a half-trace over one of the tensor factors:



The bubble in the first term can be factorized as $2t$ and the diagram with a bullet marked by 0 vanishes. Hence, the whole equation can be rewritten as:

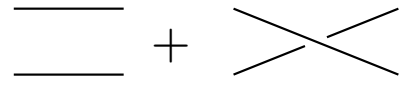
$$\sigma(2t - t(\dim_{\mathfrak{g}} - 1)) = -\omega((\dim_{\mathfrak{g}} - 1)) \quad (16)$$

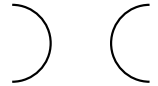
$$\dim_{\mathfrak{g}} = \frac{\omega - 3t\sigma}{\omega - t\sigma} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} \quad (17)$$

Formula (2) from diagrammatic technique

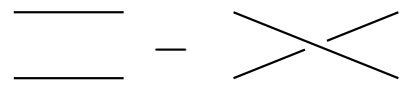
$$\mathfrak{g} \otimes \mathfrak{g} = (\varnothing \oplus Y_\alpha \oplus Y_\beta \oplus Y_\gamma) \ \oplus \ (\mathfrak{g} \oplus X_2)$$

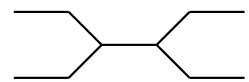
Symmetric square $Sym^2(\text{adj})$ can be obtained by the following projector:

$$P_s = \tfrac{1}{2}\Big($$  $$\Big), \quad P_s^2 = P_s \tag{18}$$

$$P_\varnothing =$$  $$, \quad P_\varnothing^2 = \bigcirc \cdot P_\varnothing \tag{19}$$

Antisymmetric square $\Lambda^2(\text{adj})$ can be obtained by the following projector:

$$P_a = \tfrac{1}{2}\Big($$  $$\Big), \quad P_a^2 = P_a \tag{20}$$

$$P_{\text{adj}} =$$  $$, \quad P_{\text{adj}}^2 = 2\hat{t} \cdot P_{\text{adj}} \tag{21}$$

Thus, tensor square of the Adj representation can be expanded into 4 terms:

$$\text{Adj}^2 = (X_{\text{adj}} \oplus X_2) \ \oplus \ (Y_\varnothing \oplus Y)\,, \tag{22}$$

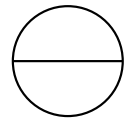
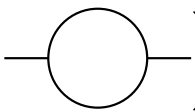
Projectors to X_2 and Y subspaces can be easily found

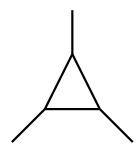
$$P_Y = \bigcirc \cdot P_s - P_\varnothing, \tag{23}$$

$$P_{X_2} = 2\hat{t} \cdot P_a - P_{\text{adj}}. \tag{24}$$

2 or 3 parameters?

There are non-trivial kernels of the map $\Phi_{\mathfrak{g}} : \mathcal{D} \to End(\mathfrak{g})$:

- **0-legged 3-diagrams:** $\Phi_{\mathfrak{g}}\Big(\hat{P}_{15} \cdot$  $\Big) = \chi_{\mathfrak{g}}(\hat{P}_{15}) \cdot \chi_{\mathfrak{g}}(2\hat{t}) \cdot \dim_{\mathfrak{g}} = 0 \ \ \forall \mathfrak{g}$
- **2-legged 3-diagrams:** $\Phi_{\mathfrak{g}}\Big(\hat{P}_{15} \cdot$  $\Big) = 0 \quad \forall \mathfrak{g}$ (25)
- **Closed Jacobi diagrams:** $\Phi_{\mathfrak{g}}\Big(\hat{P}_{15} \cdot$  $\Big) = 0 \quad \forall \mathfrak{g}$ (26)
- **Vogel Λ-algebra:** no kernel! Because diagrams $\hat{t}$ and $\hat{P}_{15}$ are zero divisors in Λ-algebra:

$$\hat{t} \cdot \hat{P}_{15} = 0, \quad \hat{t} = \text{} \neq 0, \ \hat{P}_{15} \neq 0$$

2 or 3 parameters?

Output

Diagrams	Lie algebra
universality	
3 parameters	2 parameters
invariance	covariance
polynomiality	rationality/exp-ty/?
–	factorization
symmetry	decomposition
?	irreps decomposition
zero kernel	–

Thank you for your attention!