Spinor representations in Vogel's theory

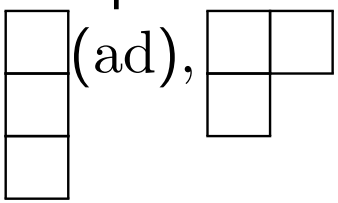
work in progress

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PMMΦ'26

Problem Statement

- In Vogel's universality theory key role is played by our ability to decompose "tensor spaces" related to adjoint representations of simple Lie algebras "from first principles" (as opposed to e.g. weights)
- For example, Vogel himself deduced a decomposition $\mathrm{ad} \otimes \mathrm{ad} = \Big[\mathrm{ad} \oplus Y\Big] \oplus \Big(\bullet \oplus X_1 \oplus X_2 \oplus X_3 \Big)$ and calculated projection operators $\mathrm{ad} \cdot \mathrm{ad} \to X_{1,2,3}$ out of properties of structure constants
- Later (in the same text) it was generalized to more complicated spaces


$(\mathrm{ad}), \quad (\mathrm{ad}), \quad (\mathrm{ad}) \subset \mathrm{ad}^{\otimes 3}$

Problem Statement

- Different tensor spaces = representations of $GL(\mathfrak{g})$
- All simple Lie algebras are inner product spaces $\Rightarrow$ we can consider finer representations of $SO(\mathfrak{g})$
- SO groups have spinor representations complementing tensorial ones; that is, to simple Lie algebras $\mathrm{Spin}(\mathrm{ad})$ vector spaces are invariantly associated
- Question: can we squeeze some universality out of these spaces?

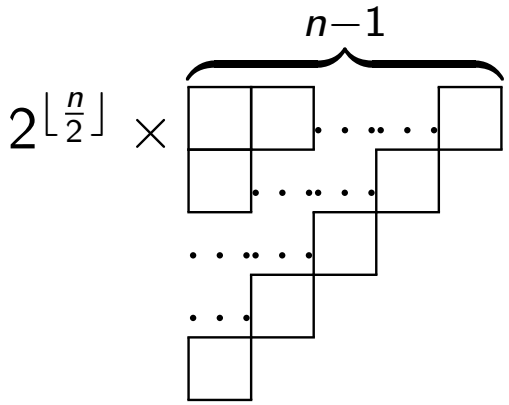
Reminder: spinors in n dimensions

Suppose (V, g) is inner product space. Depending on dimension parity, it can be decomposed either as $V = W \oplus W^*$ or $V = W \oplus W^* \oplus \langle \zeta \rangle$. Here W is non–canonical isotropic subspace: $g(w_1, w_2) = 0 \forall w_1, w_2 \in W$.
Then spinor space is basically $\bigwedge^\bullet W$ with a tricky $so(V)$ action. First, vectors of V act on $\bigwedge^\bullet W$ via Γ–matrices:

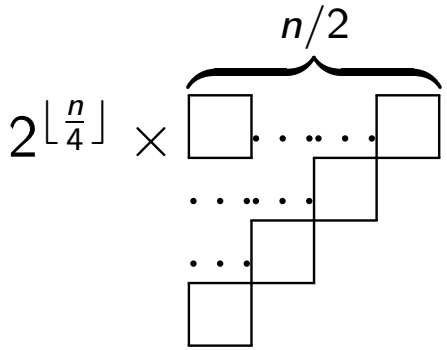
- for $w^\mu \in W$
  $\Gamma_\mu w^\mu : \; w^{\nu_1} \wedge w^{\nu_2} \wedge \ldots \wedge w^{\nu_i} \mapsto w^\mu \wedge w^{\nu_1} \wedge w^{\nu_2} \wedge \ldots \wedge w^{\nu_i}$
- for $w^\mu \in W^*$ $\Gamma_\mu w^\mu : \; w^{\nu_1} \wedge w^{\nu_2} \wedge \ldots \wedge w^{\nu_i} \mapsto$
  $\sum (-)^k g(w^\mu, w^{\nu_k}) w^{\nu_2} \wedge \ldots \wedge w^{\nu_i}$
- if ζ is present
  $\Gamma_\zeta : \; w^{\nu_1} \wedge w^{\nu_2} \wedge \ldots \wedge w^{\nu_i} \mapsto (-)^i w^{\nu_1} \wedge w^{\nu_2} \wedge \ldots \wedge w^{\nu_i}$
- $so(V)$ action is defined by $\omega_{\mu\nu} = \frac{1}{2}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu)$

Results

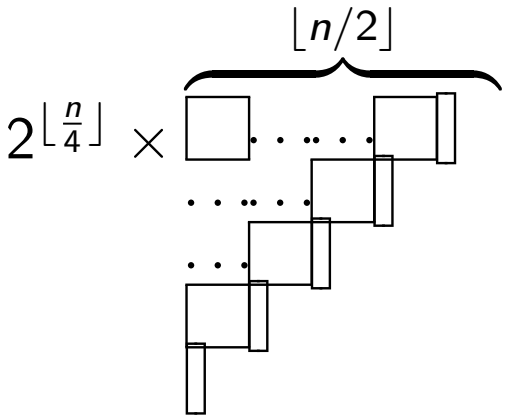
- $sl(n)$



- $so(n \text{ even})$

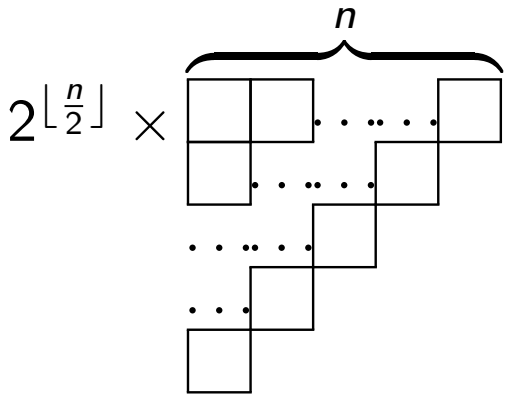


- $so(n \text{ odd})$



Results

- $sp(2n)$



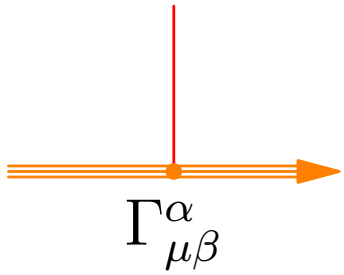
- $\mathfrak{g}_2$ $2 \times [1,1]$
- $\mathfrak{f}_4$ $2^2 \times [1,1,1,1]$
- $\mathfrak{e}_6$ $2^3 \times [1,1,1,1,1,1]$
- $\mathfrak{e}_7$ $2^3 \times [1,1,1,1,1,1,1]$ **(???)**
- $\mathfrak{e}_8$ $2^4 \times [1,1,1,1,1,1,1,1]$ **(???)**

Observations

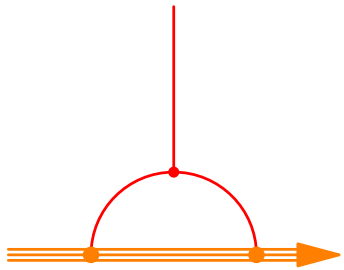
- all $\mathrm{Spin}(\mathrm{ad})$ representations have only one type of irreducible component
- this irreducible component comes with multiplicity $2^{\lfloor \frac{\mathrm{rank}}{2} \rfloor}$
- Corollary: all these representations have dimension 2^{smth}
- pretty sure highest weight space is $\bigwedge^{\bullet} \mathfrak{h}^{iso} \otimes \bigwedge^{top} \mathfrak{e}^{+}$

Explanation 1

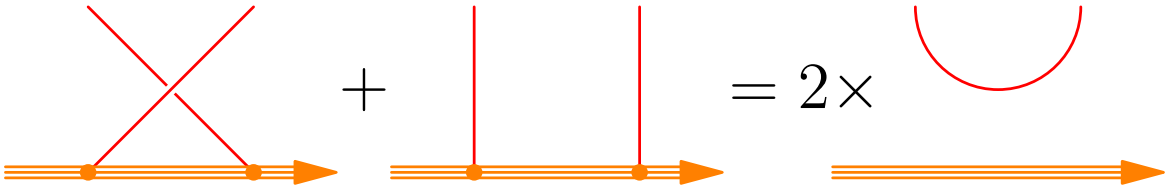
- Gamma–matrices:



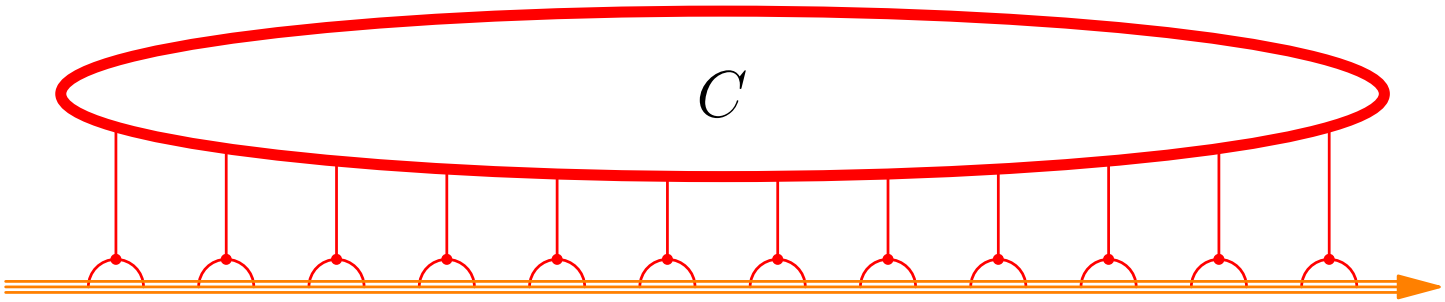
- action tensors:



- Clifford's property:



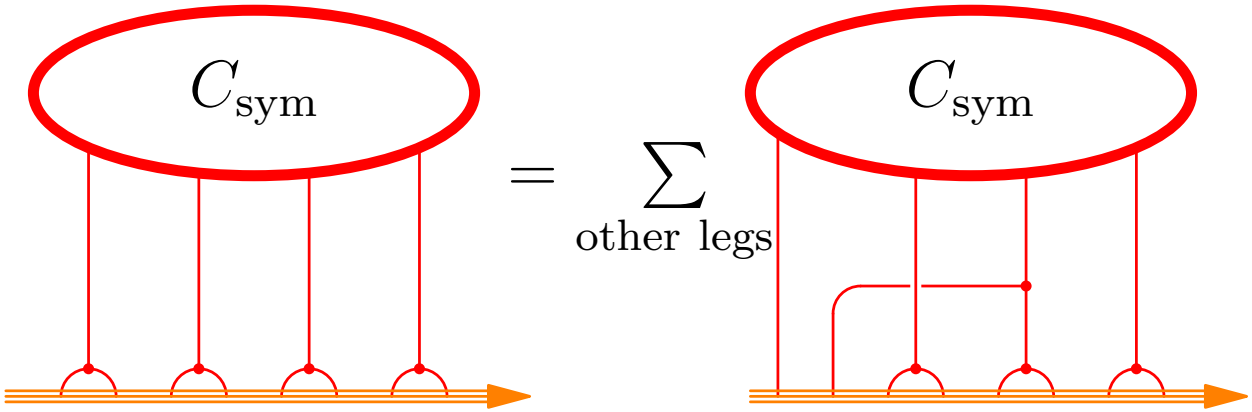
- symmetric Casimir acting on spinor space:



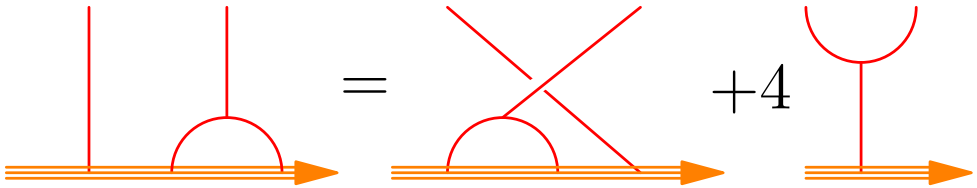
Explanation 1

single irreducible

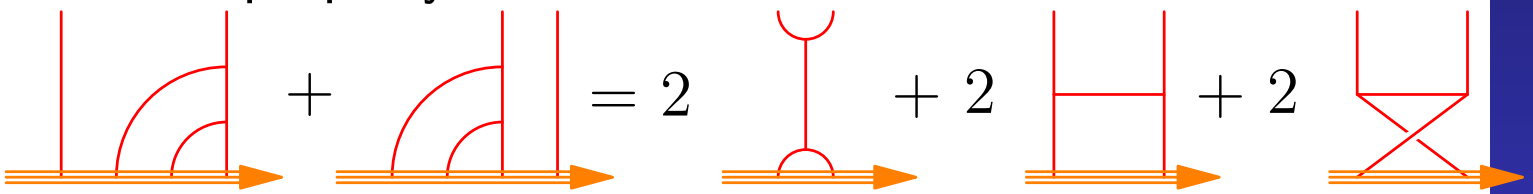
- invariance:



- Clifford's property 1 $\Rightarrow$



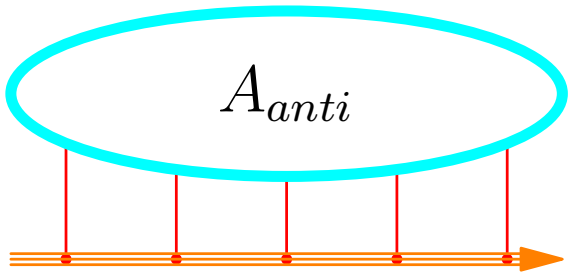
- Clifford's property 2 $\Rightarrow$



Explanation 2

multiplicity

- large multiplicity spaces $\Rightarrow$ lots of non–commutative invariant operators
- supply of invariant operators on spinor spaces:



- invariant elements of $\bigwedge^q \mathrm{ad}$ = elements of Lie algebra cohomology
- Hopf: Lie algebra cohomology is freely generated by odd–degree elements
- actually, number of generators = rank

Explanation 2

multiplicity

- $[A, B] \neq 0 \Rightarrow \dim \geqslant 2$
- for 2×2 traceless matrices A, B, C:

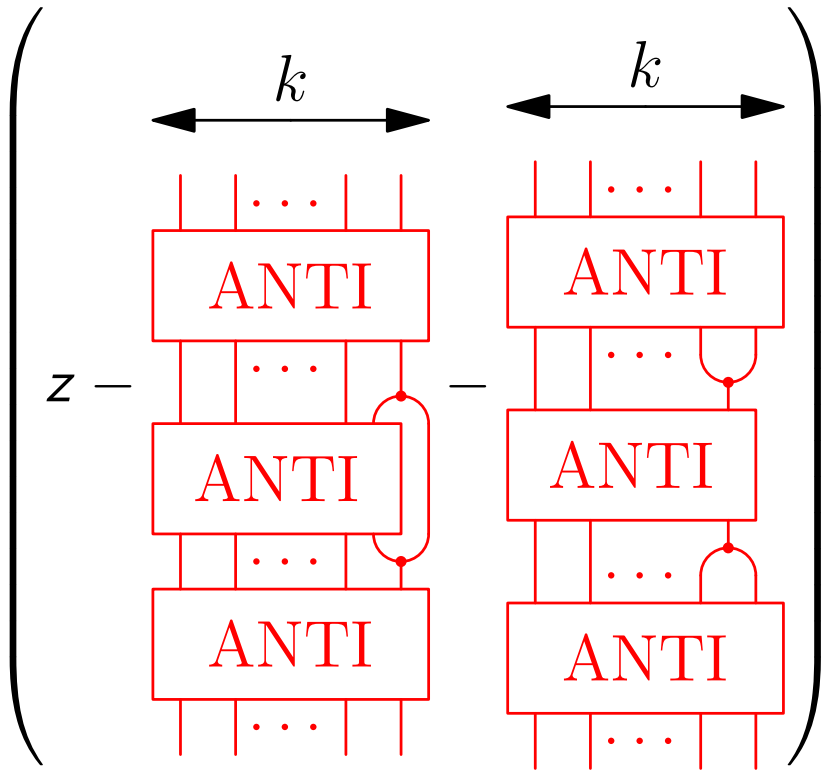
$$ABC - ACB + CAB - CBA + BCA - BAC \propto \mathrm{id}$$

- $\Rightarrow$ need at least two 2×2 blocks to anticommute with fourth matrix D:

$$\begin{bmatrix} (A_1, B_1.C_1)_{2\times 2} & D_1 \times \mathrm{id}_2 \\ D_2 \times \mathrm{id}_2 & (A_2, B_2, C_2)_{2\times 2} \end{bmatrix}$$

- there are only 2 linearly independent choices for $D \Rightarrow$ need to double dimension again to get 6 genetrators etc. etc.

Speculations on Betti numbers

- $$b_k = \oint_{C(0)} d \det \left(z - \right.$$ 

$$\left. - \right) / \det(\ldots)$$

- generator of $H^3(\mathfrak{g})$ is $f_{\mu\nu\lambda}$; maybe some else are as well

Method aka how do I know?

Compute characters! Example for $sl(n)$:
$sl(n) \xrightarrow{\mathrm{ad}} so(n^2-1)$ on Cartan subalgebras:

$$h_i \mapsto -\sum_{j=1}^{i-1} H_{ji} + \sum_{i+1}^{n} H_{ij} + \sum_{j=1}^{i} H_{j(i+1)} - \sum_{j=i+2}^{n} H_{(i+1)j}$$

where $H_{ij}(j>i) : X_{ij} \mapsto X_{ij},\ X_{ji} \mapsto -X_{ji},\ X_{\text{others}} \mapsto 0$ (there are also $\lfloor \frac{n-1}{2} \rfloor$ $so(n^2-1)$ Cartans acting on X_{ii} alone)

$$\begin{pmatrix} & -1 & 1 & & \\ & -1 & 1 & & \\ & * & 2 & 1 & 1 \\ & & -* & -1 & -1 \\ & & & & \\ & & & & \end{pmatrix}$$

Method aka how do I know?

In spinor representation all $so(V)$ Cartans act with $\pm\frac{1}{2}$ eigenvalues in every possible combination $\Rightarrow$

$$\mathrm{Tr}_{\mathrm{Spin(ad)}}(e^{\phi_i h_i}) = 2^{\lfloor \frac{n-1}{2} \rfloor} \prod_{i>j} \left(\exp\left(\frac{1}{2}(\phi_i - \phi_{i-1} - \phi_j + \phi_{j-1}) \right) + \exp\left(-\frac{1}{2}(\phi_i - \phi_{i-1} - \phi_j + \phi_{j-1}) \right) \right)$$

$$\chi_{\mathrm{Spin(ad)}} = 2^{\lfloor \frac{n-1}{2} \rfloor} \prod_{j>i} \left(\sqrt{\frac{x_i}{x_j}} + \sqrt{\frac{x_j}{x_i}} \right)$$

Thank you for your attention!



Figure: Ein Vogel drehender