W-algebras and integrable systems

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Plan

- Basic facts about $W_{1+\infty}$
- Construction of commutative family of operators
- Polynomial realization
- Schur basis for operators
- Generalization for another type of diagrams

$W_{1+\infty}$

$W_{1+\infty}$ algebra is a central extension of the algebra of differential operators

$$z^n D^m$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots, m = 0, 1, 2, 3, \ldots$ and

$$D = z\frac{d}{dz} \tag{1}$$

Lattice

					$\cdots$					
				$\cdots$	D^4	$\cdots$				
			$\cdots$	$z^1 D^3$	D^3	$\frac{1}{z} D^3$	$\cdots$			
		$\cdots$	$z^2 D^2$	$z^1 D^2$	D^2	$\frac{1}{z} D^2$	$\frac{1}{z^2} D^2$	$\cdots$		
	$\cdots$	$z^3 D$	$z^2 D$	$z^1 D$	D	$\frac{1}{z} D$	$\frac{1}{z^2} D$	$\frac{1}{z^3} D$	$\cdots$	
$\cdots$	z^4	z^3	z^2	z^1	1	$\frac{1}{z}$	$\frac{1}{z^2}$	$\frac{1}{z^3}$	$\frac{1}{z^4}$	$\cdots$

				$\cdots$	$\cdots$	$\cdots$				
				$\hat{e}_3$	$\hat{\psi}_4$	$\hat{f}_3$				
				$\hat{e}_2$	$\hat{\psi}_3$	$\hat{f}_2$				
				$\hat{e}_1$	$\hat{\psi}_2$	$\hat{f}_1$				
				$\hat{e}_0$	1	$\hat{f}_0$				

The algebra can be generated by finite number of generators

$\hat{e}_0$ $\hat{\psi}_3$ $\hat{f}_0$

rising diagonal lowering

Relations

$$\left[\hat{\psi}_n, \hat{\psi}_m\right] = 0 \qquad \left[\hat{e}_n, \hat{f}_m\right] = \hat{\psi}_{n+m}$$

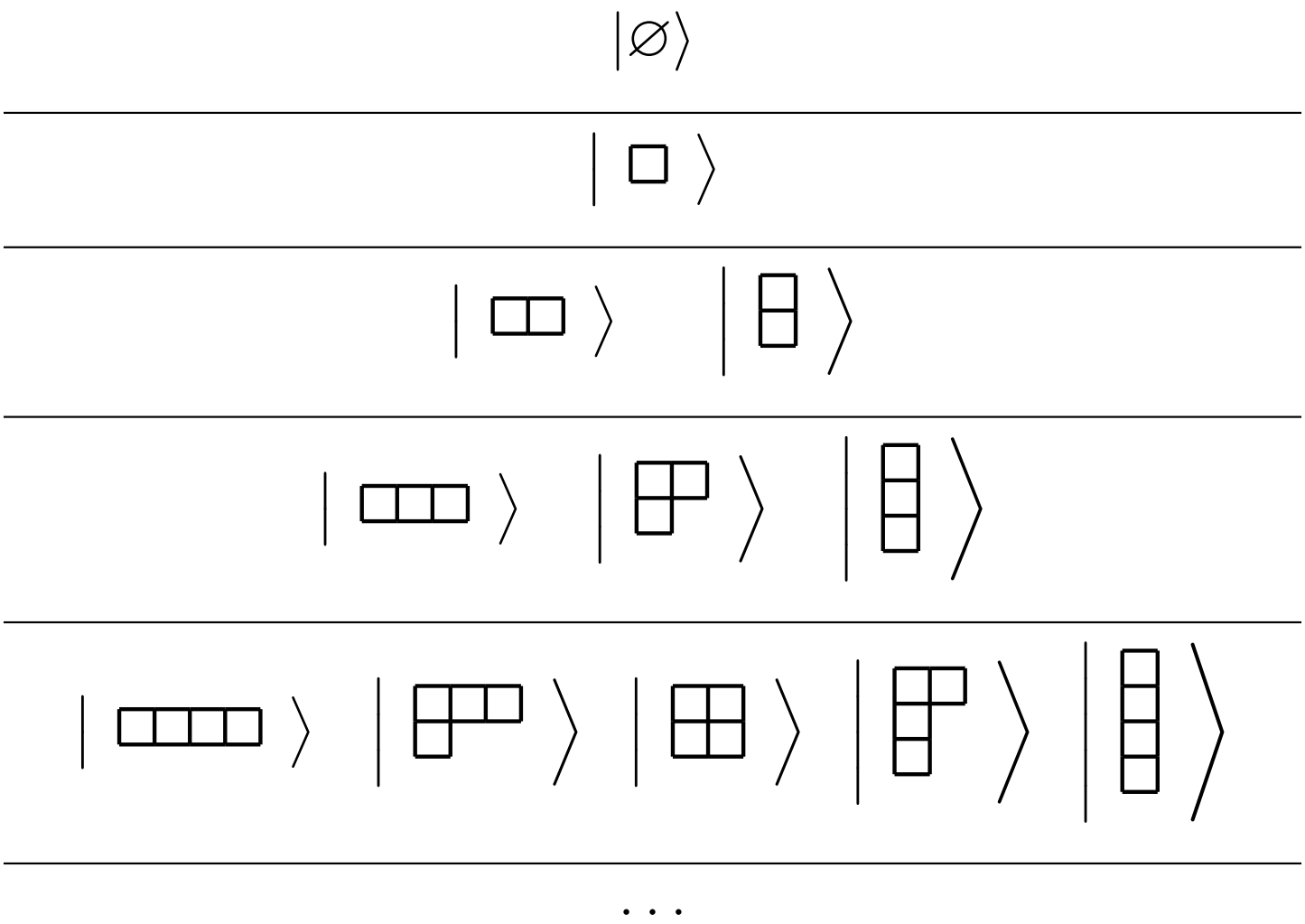
$$\left[\hat{e}_{n+3}, \hat{e}_m\right] - 3\left[\hat{e}_{n+2}, \hat{e}_{m+1}\right] + 3\left[\hat{e}_{n+1}, \hat{e}_{m+2}\right] - \left[\hat{e}_n, \hat{e}_{m+3}\right] - \\ - \left[\hat{e}_{n+1}, \hat{e}_m\right] + \left[\hat{e}_n, \hat{e}_{m+1}\right] = 0$$

$$\left[\hat{\psi}_{n+3}, \hat{e}_m\right] - 3\left[\hat{\psi}_{n+2}, \hat{e}_{m+1}\right] + 3\left[\hat{\psi}_{n+1}, \hat{e}_{m+2}\right] - \left[\hat{\psi}_n, \hat{e}_{m+3}\right] - \\ - \left[\hat{\psi}_{n+1}, \hat{e}_m\right] + \left[\hat{\psi}_n, \hat{e}_{m+1}\right] = 0$$

$$\mathrm{Sym}_{i,j,k}\left[\hat{e}_i, \left[\hat{e}_j, \hat{e}_{k+1}\right]\right] = 0$$

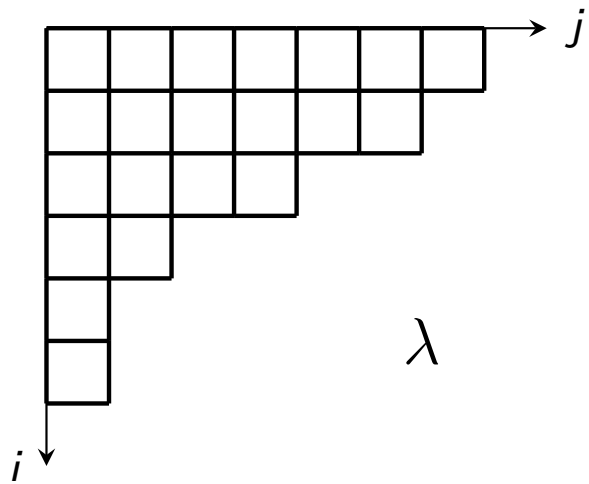
Operators on Young diagrams

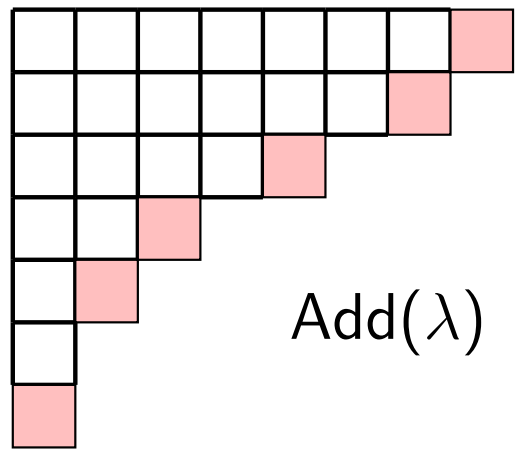
Fock representation



Adding a box

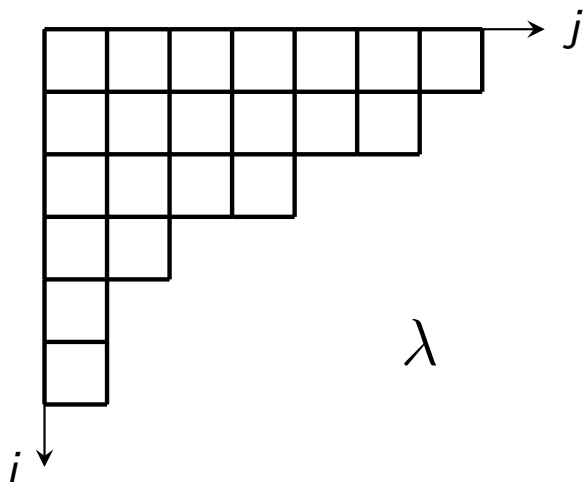
$$\hat{e}_0 \left|\lambda\right\rangle = \sum_{\square \in \mathsf{Add}(\lambda)} \left|\lambda + \square\right\rangle \tag{2}$$

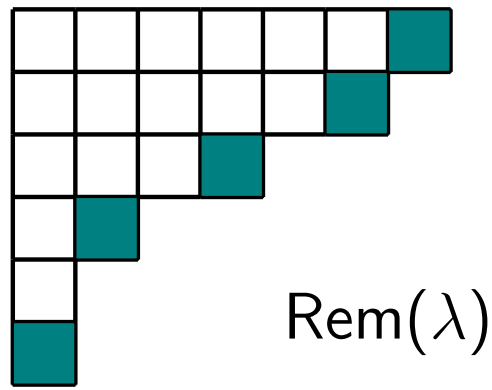




Removing a box

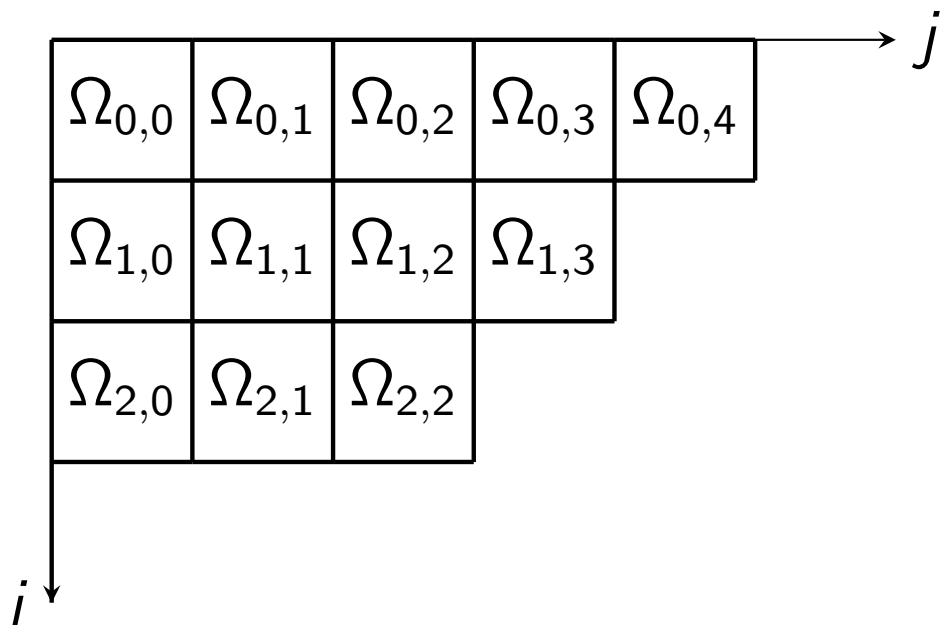
$$\hat{f}_0 |\lambda\rangle = \sum_{\square \in \mathrm{Rem}(\lambda)} |\lambda - \square\rangle \tag{3}$$





Counting boxes

$$\hat{\psi}_3 |\lambda\rangle = \left(\sum_{\square \in \lambda} \Omega_\square \right) |\lambda\rangle \tag{4}$$



$$\Omega_\square = \Omega_{i_\square, j_\square} = j_\square - i_\square \tag{5}$$

Auxiliary operators

$$\left[\hat{\psi}_3, \hat{e}_0\right] |\lambda\rangle = \sum_{\square \in \mathrm{Add}(\lambda)} \Omega_{\square} \; |\lambda + \square\rangle \tag{6}$$

Consider operators

$$\hat{e}_k = \left(\mathrm{ad}_{\hat{\psi}_3}\right)^k \hat{e}_0 \qquad\qquad \hat{f}_k = \left(-\mathrm{ad}_{\hat{\psi}_3}\right)^k \hat{f}_0 \tag{7}$$

Then

$$\left[\hat{e}_0, \hat{f}_0\right] |\lambda\rangle \propto |\lambda\rangle \quad \Rightarrow \quad \left[\hat{e}_a, \hat{f}_b\right] |\lambda\rangle \propto |\lambda\rangle \tag{8}$$

Commutative (diagonal) family

$$\hat{\psi}_{a+b} = \left[\hat{e}_a, \hat{f}_b\right] \tag{9}$$

Operators $\hat{\psi}_a$ form commutative family

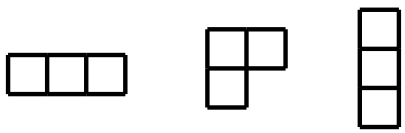
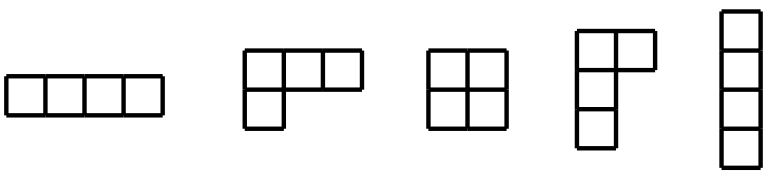
$$\left[\hat{\psi}_a, \hat{\psi}_b\right] = 0 \tag{10}$$

To construct a family of commuting operators we need:

- the set of states-diagrams
- operator of adding boxes
- operator of removing boxes
- diagonal box counting operator

Towards polynomials...

Graded polynomials $[p_a] = a$

Young diagrams	Basis monomials
	p_1
	p_2, p_1^2
	p_3, p_2p_1, p_1^3
	$p_4, p_3p_1, p_2^2, p_2p_1^2, p_1^4$

Back to symmetric functions $p_a = \sum_{i=1}^{N} (x_i)^a$.

Schur polynomials

$$|\lambda\rangle = S_\lambda \tag{11}$$

Orthogonality conditions

$$\langle\mu|\lambda\rangle = \delta_{\mu,\lambda} \tag{12}$$

are equivalent to Cauchy identity

$$\sum_\lambda S_\lambda(p)\, S_\lambda(\bar{p}) = \exp\left(\sum_{k=1} \frac{p_k\, \bar{p}_k}{k}\right) \tag{13}$$

Examples

$$
\begin{aligned}
S_{\varnothing} &= 1 \\
S_{\square} &= p_1 \\
S_{\square\square} &= \frac{p_1^2}{2} + \frac{p_2}{2} \\
S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{p_1^2}{2} - \frac{p_2}{2} \\
S_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} &= \frac{p_1^3}{3} - \frac{p_3}{3} \\
S_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} &= \frac{p_1^3}{6} - \frac{p_1 p_2}{2} + \frac{p_3}{3}
\end{aligned}
\tag{14}
$$

$$\hat{e}_0 = p_1 \qquad \hat{f}_0 = \frac{\partial}{\partial p_1} \tag{15}$$

The simplest Pieri rules

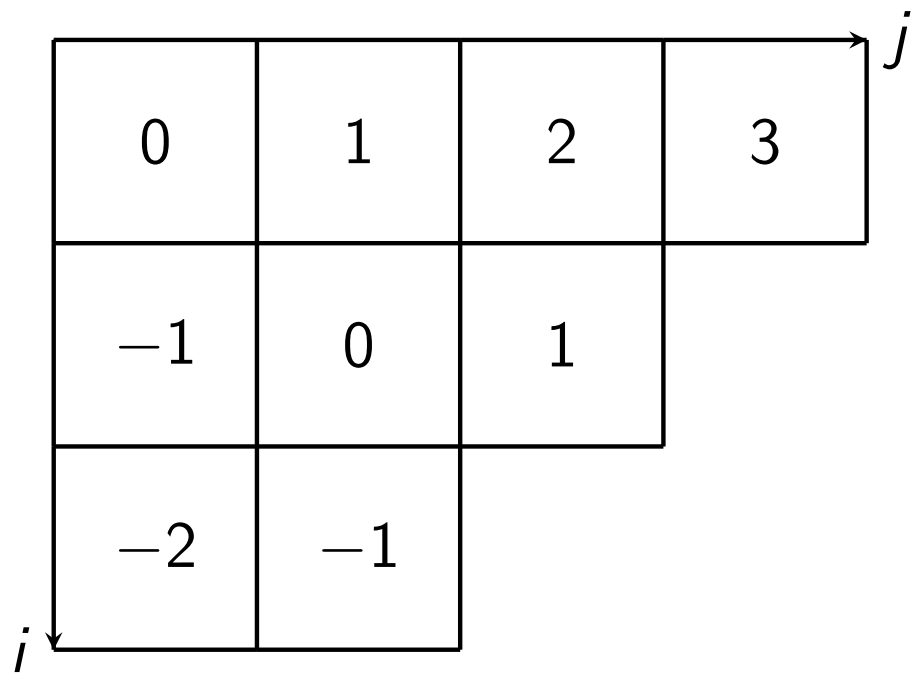
$$p_1 S_\lambda = \sum_{\square \in \mathsf{Add}(\lambda)} S_{\lambda+\square} \tag{16}$$

$$\frac{\partial}{\partial p_1} S_\lambda = \sum_{\square \in \mathsf{Rem}(\lambda)} S_{\lambda-\square} \tag{17}$$

Cut-and-join operator

$$\hat{\psi}_3 = \frac{1}{2} \sum_{a,b=1}^{\infty} (a+b)\, p_a p_b \frac{\partial}{\partial p_{a+b}} + ab\, p_{a+b} \frac{\partial}{\partial p_a} \frac{\partial}{\partial p_b} \qquad (18)$$

$$\hat{\psi}_3\, S_\lambda = \left(\sum_{\square \in \lambda} j_\square - i_\square \right) S_\lambda$$



(19)

Schur basis for operators

Operator in a Schur basis

$$\hat{\mathcal{H}} = \sum_{|\lambda|=|\mu|} A_{\lambda\mu} S_\lambda \hat{S}_\mu{}^T \tag{20}$$

where $A_{\lambda\mu}$ are coefficients and

$$\hat{S}_\lambda = S_\lambda \left(p_k \to k \frac{\partial}{\partial p_k} \right) \tag{21}$$

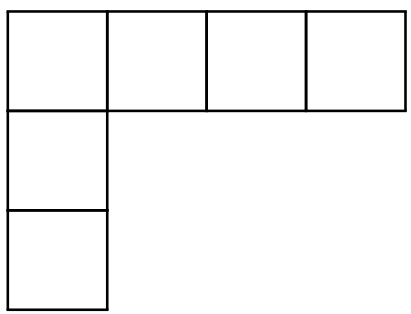
For $|\lambda| \leqslant |\mu|$

$$\hat{S}_\lambda S_\mu = S_{\mu/\lambda} \tag{22}$$

Single-hooks

$$\hat{\psi}_3 = \sum_{n=1}^{\infty} \sum_{i,j=0}^{n-1} (-1)^{n+1+i+j} \, (j-i) \, S_{(i+1|n)} \hat{S}_{(j+1|n)^T} \quad (23)$$

The above expansion includes only *single-hook* Young diagrams $(i|n) := [n-i+1, \underbrace{1, \ldots, 1}_{i-1}]$

$(3|7) =$  (24)

All $\hat{\psi}_a$ have box additivity property

$$\hat{\psi}_a S_\lambda = \left(\sum_{\square \in \lambda} \omega_a(j_\square - i_\square) \right) S_\lambda \tag{25}$$

and in Schur basis have simple form

$$\hat{\psi}_a = \sum_{n=1}^{\infty} \sum_{i,j=0}^{n-1} (-1)^{n+1+i+j} \, \omega_a(j-i) \, S_{(i+1|n)} \hat{S}_{(j+1|n)^T} \tag{26}$$

where

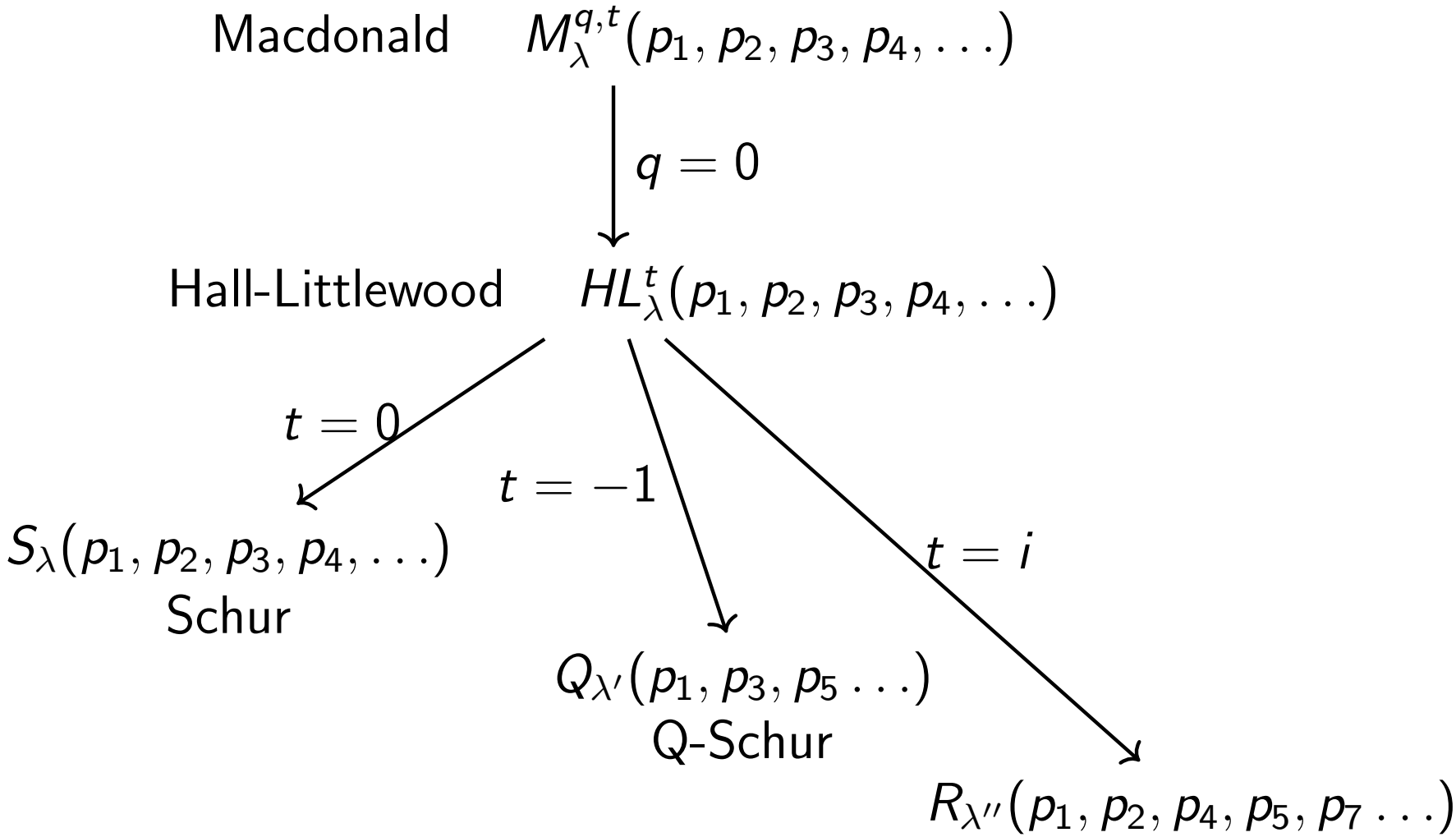
$$\omega_a(x) = (x+1)^a + (x-1)^a - 2(x)^a \tag{27}$$

General operator for Schur polynomials

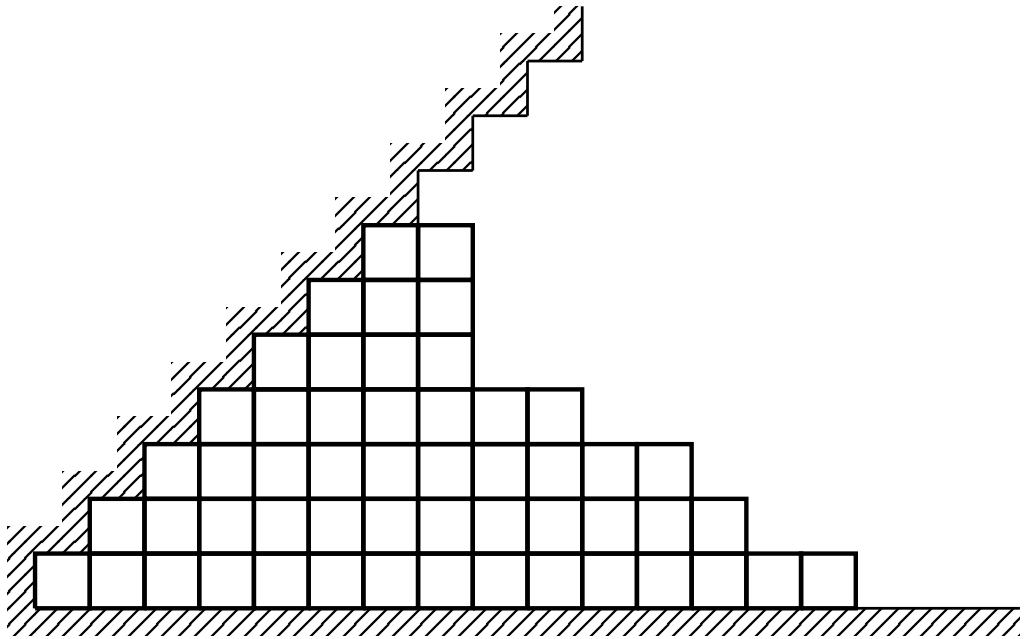
$$\hat{\mathcal{H}}^{\Omega} = \sum_{n=1}^{\infty} \sum_{i,j=0}^{n-1} (-1)^{n+1+i+j}\, \Omega(j-i)\, S_{(i+1|n)} \hat{S}_{(j+1|n)^T} \tag{28}$$

$$\hat{\mathcal{H}}^{\Omega}\, S_{\lambda} = \left(\sum_{\square \in \lambda} \Omega(j_{\square} - i_{\square}) \right)\, S_{\lambda} \tag{29}$$

Another type of diagrams

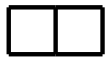
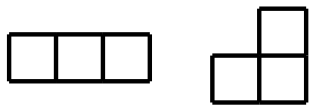
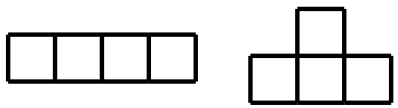
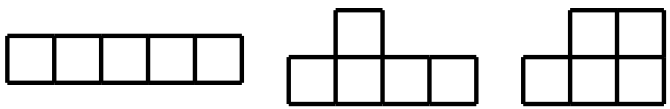
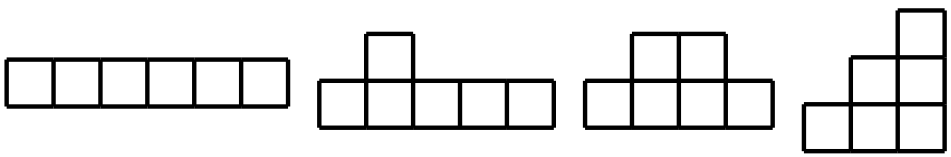

Macdonald
$M_\lambda^{q,t}(p_1, p_2, p_3, p_4, \ldots)$
$q = 0$
Hall-Littlewood
$HL_\lambda^t(p_1, p_2, p_3, p_4, \ldots)$
$t = 0$
$t = -1$
$t = i$
$S_\lambda(p_1, p_2, p_3, p_4, \ldots)$
Schur
$Q_{\lambda'}(p_1, p_3, p_5 \ldots)$
Q-Schur
$R_{\lambda''}(p_1, p_2, p_4, p_5, p_7 \ldots)$

Strict Young diagrams



(30)

Graded polynomials $[\tau_a] = 2a + 1$

Strict Young diagrams	Basis polynomials
	τ_0
	τ_0^2
	τ_1, τ_0^3
	$\tau_1\tau_0, \tau_0^4$
	$\tau_2, \tau_1\tau_0^2, \tau_0^5$
	$\tau_2\tau_0, \tau_1^2, \tau_1\tau_0^3, \tau_0^6$

Q-Schur polynomials

$$Q_\mu(\tau_0, \tau_1, \tau_2, \ldots) \tag{31}$$

Examples

$$\begin{aligned} Q_\varnothing &= 1 \\ Q_{\square} &= \tau_0 \\ Q_{\square\square} &= \tau_0^2 \\ Q_{\square\square\square} &= \frac{2\tau_0^3}{3} + \tau_1 \\ Q_{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}} &= \frac{\tau_0^3}{3} - \tau_1 \end{aligned} \tag{32}$$

Conjugated operators

$$\hat{Q}_\mu := Q_\mu\left(\tau_k \to \frac{4^k}{2k+1}\frac{\partial}{\partial \tau_k}\right) \tag{33}$$

Towards the algebra

Adding boxes

$$\tau_0 \cdot Q_\mu = \sum_{\square \in \mathsf{Add}(\mu)} Q_{\mu+\square} \tag{34}$$

Removing boxes

$$\frac{\partial}{\partial \tau_0} Q_\mu = \sum_{\square \in \mathsf{Rem}(\mu)} (2 - \delta_{i_\square, j_\square}) \cdot Q_{\mu-\square} \tag{35}$$

Diagonal operator

$$\hat{\Psi}_\Omega \, Q_\mu = \left(\sum_{\square \in \mu} \Omega(j_\square - i_\square) \right) Q_\mu \tag{36}$$

General operator for Q-Schur polynomials

$$\hat{\Psi}_{\Omega} = \sum_{n=1}^{\infty} \sum_{i,j=0}^{\text{IntPart}(\frac{n-1}{2})} A_{i,j} Q_{[n-i,i]} \hat{Q}_{[n-j,j]} \qquad (37)$$

$$A_{i,j} = (-2)^{1-n}(2-\delta_{i,0})(2-\delta_{j,0}) \left[\sum_{k=|j-i|}^{n-1-i-j} (-1)^k \, \Omega(k) \right] \qquad (38)$$

We obtain operators from $W_{1+\infty}^{B}$ for

$$\Omega(j_{\square} - i_{\square}) = (j_{\square} - i_{\square} + 1)^3 - (j_{\square} - i_{\square})^3 \qquad (39)$$

Conclusion

- The universal operator

$$\hat{\mathcal{H}}^{\Omega} = \sum_{n=1}^{\infty} \sum_{i,j=0}^{n-1} (-1)^{n+1+i+j} \, \Omega_{i,j} \, S_{(i+1|n)} \hat{S}_{(j+1|n)^T} \tag{40}$$

$\Omega(j-i)$	$j-\beta i$	$q^j t^{-i}$
Schur	Jack	Macdonald

For Q-Schurs?

- Single-hook expansions for Schur and Q-Schur cases are connected to boson-fermion correspondence.
  Generalization to Hall cases?