FROM BAKER-AKHIEZER TO B-ITZYKSON-ZUBER INTEGRAL

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Itzykson-Zuber

General scheme

Trigonometric Calogero system

Rational Calogero system

From Baker-Akhiezer to Itzykson-Zuber

Further research

Itzykson-Zuber integral

- Integrable systems
- Matrix models (2d CFT, 2d gravitation)
- Calogero-Mozer-Sutherland systems
- Building block for Kazakov-Migdal model (QCD)

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Itzykson-Zuber integral

- Original definition – angular matrix integral

$$I(X,Y) = \int dU \exp[UXU^{+}Y]$$

- Harish-Chandra form:

$$I(X,Y) = \frac{\det \exp[x_i y_j]}{\Delta(x)\Delta(y)} \prod_{k=1}^{N-1} k!$$

- IZ integral as eigenfunction of Calogero-Mozer hamiltonian

$$H_{CM}^{\beta=1} I(X,Y) = -\sum y_i^2 \, I(X,Y)$$

- Polynomial form:

$$I_\beta(X,Y) = \sum_R \xi_R^{-1} S_R(X) S_R(Y)$$

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β-Itzykson-Zuber integral

- There are three matrix integral over angular variables – symplectic, unitary and orthogonal ($\beta = \frac{1}{2}, 1, 2$)
- More thick structures
- Physical meaning (revers temperature at Ising model)

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β-Itzykson-Zuber integral

- IZ integral as eigenfunction of Calogero-Mozer hamiltonian

$$H_{CM} I^{\beta}(X,Y) = \sum y_i^2 \, I^{\beta}(X,Y)$$

- Polynomial form:

$$I^{\beta}(X,Y) = \sum_R \frac{J_R(\delta_{k,1})}{J_R(N)} \frac{J_R(X) J_R(Y)}{||J_R||^2}$$

- There is the recurrent formula for $\tilde{I}^{\beta}(X, Y_\sigma)$ with positive integer β (see Brezin, Hikami)

$$I^{\beta}(X,Y) = \sum_\sigma \tilde{I}^{\beta}(X, Y_\sigma) Exp[\sum_i x_i \, y_{\sigma(i)}]$$

Result: the explicit formula for arbitrary β

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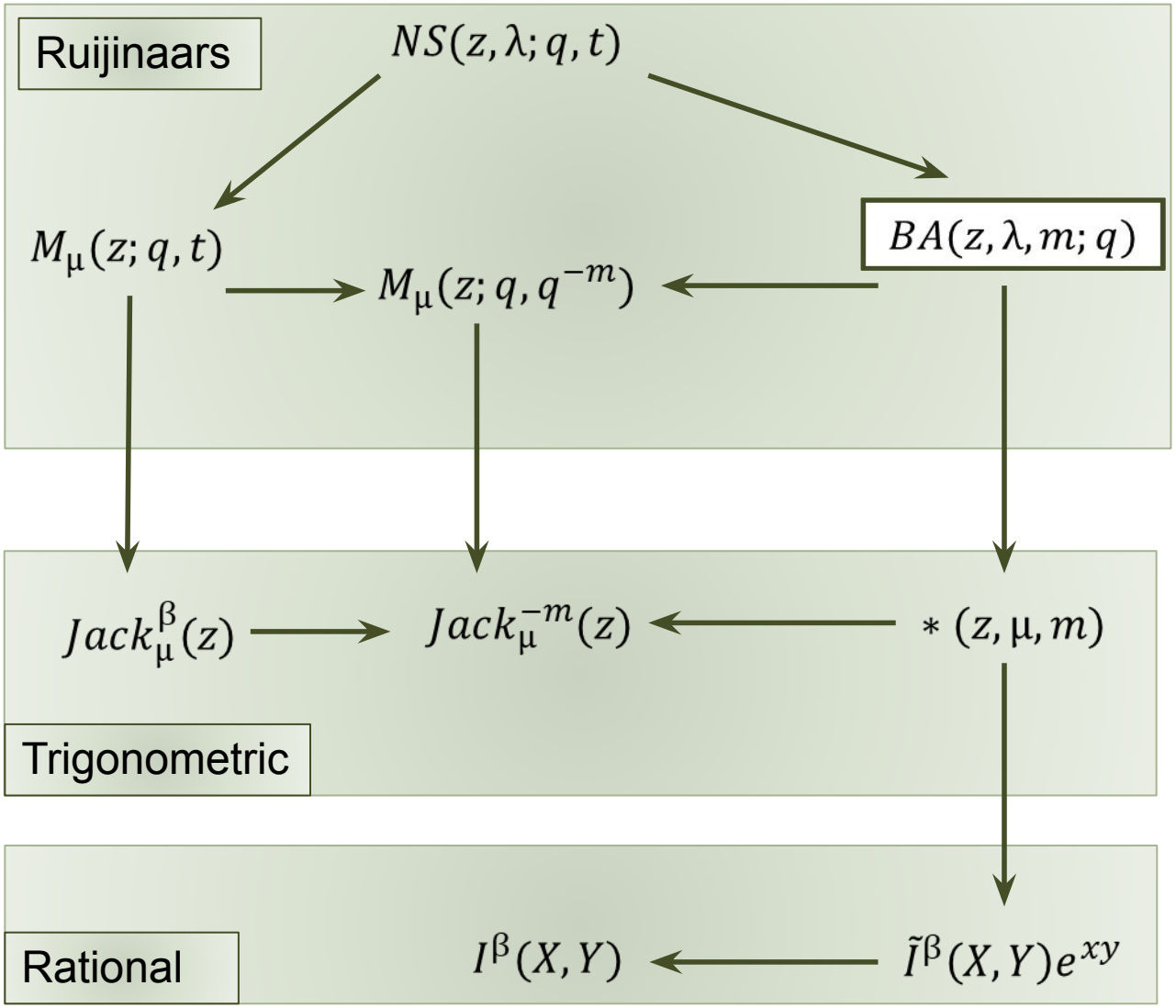
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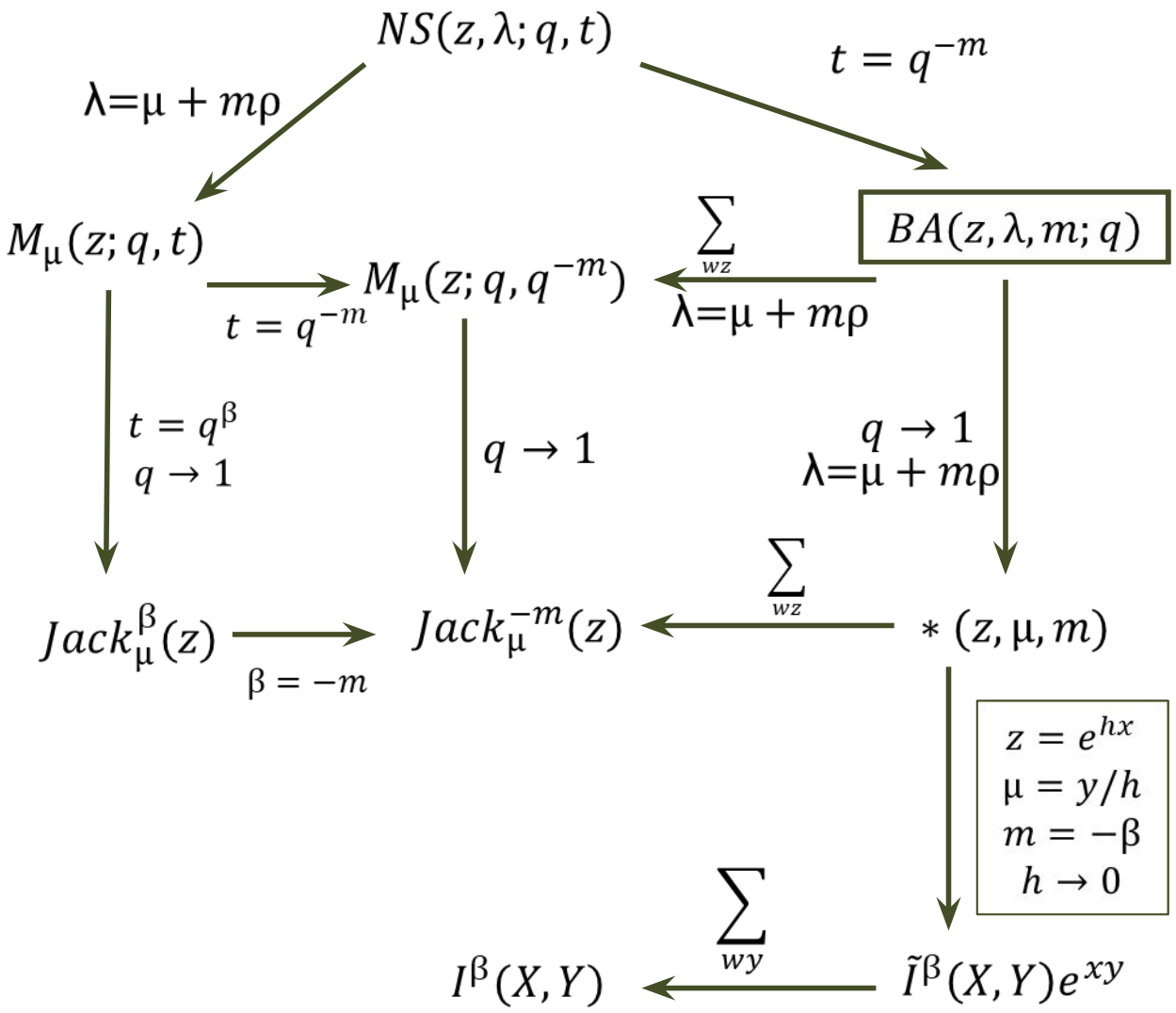
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Trigonometric Calogero system

- Trigonometric case of the Calogero system:

$$H_{trig}\psi(k,x) = k^2\psi(k,x)$$

- Hamiltonian operator:

$$H_{trig} = \sum \frac{\partial^2}{\partial x_i^2} - \sum_{i<j} \frac{2m\,(m+1)}{\sinh^2(x_i - x_j)}$$

- Boundary conditions (on hyperplane (k, α)=0 for $s = 1, \ldots, m$):

$$\psi(k + s\,\alpha, x) = \psi(k - s\,\alpha, x)$$

- $q \to 1$ limit of Baker-Akhiezer functions are the eigenfunctions:

$$\psi(k,x) = \Delta(x)^{-m} BA(z = e^x, \lambda{=}k + m\rho\,, m, q \to 1)$$

there $\rho_i = N + 1 - 2i$, $\Delta(x) = \prod_{i<j}(x_i - x_j)$

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Rational Calogero system

- Rational case of the Calogero system:

$$H_{rat}\psi(k,x) = k^2\psi(k,x)$$

- Hamiltonian operator:

$$H_{rat} = \sum \frac{\partial^2}{\partial x_i^2} - \sum_{i<j} \frac{2m\,(m+1)}{(x_i - x_j)^2}$$

- Boundary conditions (on hyperplane (k,α)=0 for $s = 1, \dots, m-1$):

$$\partial_\alpha \left((k,\alpha)^{-1}\partial_\alpha\right)^s \psi(k,\alpha) = 0$$

Itzykson-Zuber integral (multiplied by $\Delta(x)^\beta$) is the eigenfunction of the rational Calogero system. Moreother,

$$H_{rat}\Delta(x)^\beta \tilde{I}^\beta(X,Y)e^{xy} = \mathrm{y}^2\Delta(x)^\beta \tilde{I}^\beta(X,Y)e^{xy}$$

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- The explicit expression of the BA function:

$$BA(z,\lambda,m;q)=\bar{z}^{\bar{\mu}}\sum_{k_{i,j}}\varphi\left(\lambda,k_{i,j};q,q^{-m}\right)\prod_{i<j}\left(\frac{z_j}{z_i}\right)^{k_{i,j}}$$

- Obtaining $\tilde{I}^{\beta}(X,Y)$ from BA:

$$\lim_{h\to 0}h^{-\beta}\,BA\left(e^{h\,x},\frac{y}{h}-\beta\,\rho,-\beta;e^{h}\right)=e^{xy}\,\tilde{I}^{\beta}(X,Y)\,\Delta(x)^{\beta}\Delta(y)^{\beta}$$

there $\rho_i=N+1-2i$, $\Delta(x)=\prod_{i<j}(x_i-x_j)$,
$\mu=\lambda-m\,\rho$

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1. IZ integral as eigenfunction of Calogero-Mozer hamiltonian:

$$(\Delta(x)^{-\beta} H_{rat} \Delta(x)^{\beta})\, I^{\beta}(X,Y) = \sum y_i^2\, I^{\beta}(X,Y)$$

2. Polynomial form:

$$I^{\beta}(X,Y) = \sum_R \frac{J_R(\delta_{k,1})}{J_R(N)} \frac{J_R(X) J_R(Y)}{||J_R||^2}$$

3. IZ integral as a symmetrized limit of Baker-Akhieser function:

$$I^{\beta}(X,Y) = \sum_{\sigma} \tilde{I}^{\beta}(X,Y_{\sigma}) Exp[\sum_i x_i\, y_{\sigma(i)}]$$

There

$$\lim_{h\to 0} h^{-\beta}\, BA\left(e^{h\,x}, \frac{y}{h} - \beta\,\rho, -\beta; e^{h}\right) = e^{xy}\, \tilde{I}^{\beta}(X,Y)\, \Delta(x)^{\beta} \Delta(y)^{\beta}$$

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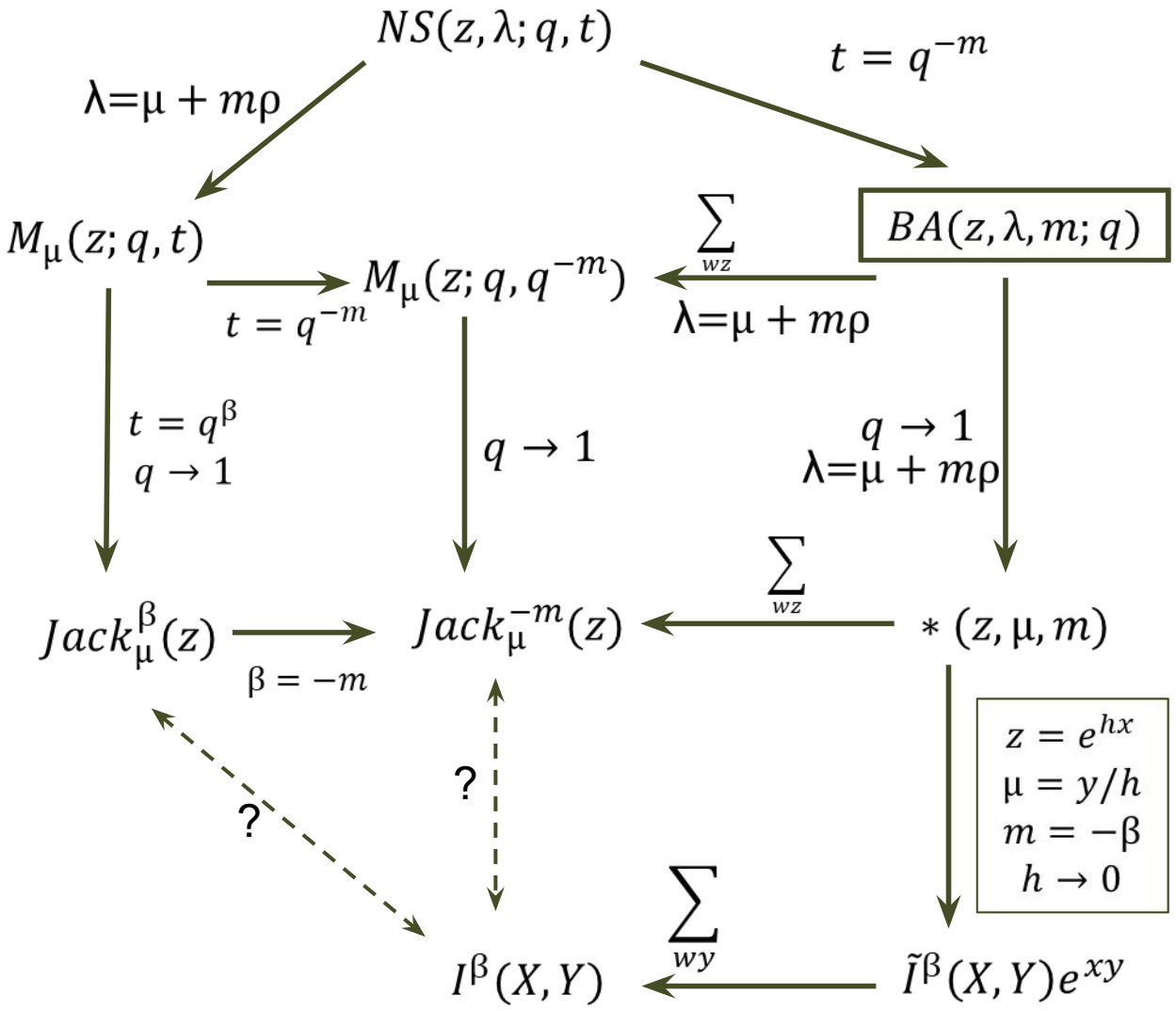
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Calogero system

$$H = \overline{\Delta}(z)^{-\beta} H_{trig} \overline{\Delta}(z)^{\beta} = \sum \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$$

$$H^{\beta} = \Delta(x)^{-\beta} H_{rat} \Delta(x)^{\beta} = \sum \frac{\partial^2}{\partial x_i^2} + 2\beta \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

$$\overline{\Delta}(x) = \prod_{i<j} \left(1 - \frac{z_i}{z_j} \right)^{\frac{1}{2}}$$

Thank you for your attention