Towards the τ-function of the quantum groups.

based on the work with M.Sharov [arXiv:2508.20966]

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2TDL hierarchy

Set of solutions of Hirota-like equations can be solved by a single function called τ -function, which, in a broad sense, can be defined as a generating function for matrix elements:

$$\tau_V(t,\bar{t}|g) \equiv \sum_{m,\bar{m}\in V} s^V_{m,\bar{m}}(t,\bar{t})\,\langle m|g|\bar{m}\rangle_V\,,$$

In the case of 2TDL these solutions are parameterized by elements of the group $GL(\infty)$.

Difference between cases of classical and quantum case

To obtain bilinear identity on τ-function it is common to use group elements, which satisfies

$$\Delta(g) = g \otimes g.$$

Then in the classical case we use the operator:

$$\Gamma = \sum_i \psi_i \otimes \psi_i^*,$$

Which commutes with coproduct:

$$(g \otimes g)\Gamma = \Gamma(g \otimes g).$$

But in the case of quantum groups there are no elements, which satisfies group elements property, except Cartan subalgebra. Coproduct in q-deformed case is defined as follows

$$\Delta(T^{(\alpha)}) \equiv D^{\alpha}_{\beta\gamma} T^{(\beta)} \otimes T^{(\gamma)}$$

Construction of group element in q-deformed case

To build group element which satisfies coproduct property it is necessary to construct universal T-matrix

$$\mathbf{T} \equiv \sum_{\alpha} X^{(\alpha)} \otimes T^{(\alpha)} \in A(G) \otimes U_q(\mathcal{G}).$$

Such T should still satisfy group-like element comultiplication, which reads

$$\Delta_U(\mathbf{T}) = \mathbf{T} \otimes_U \mathbf{T} \in A(G) \otimes U_q(\mathcal{G}) \otimes U_q(\mathcal{G})$$

Using this T-matrix it is possible to build bilinear quantum case, but the property of commutativity of flows will be lost, because variables are no longer c-numbers, but elements of non-commutative algebra of functions.

Intertwining operator approach

In the case of group elements it is convenient to use intertwining operator method. There are two intertwining operators:

$$E_R : \hat{V} \longrightarrow V \otimes W$$

$$E'_L : \hat{V}' \longrightarrow W' \otimes V'$$

$$E_R a = \Delta(a) E_R$$

$$E'_L a = \Delta(a) E'_L$$

And projection operator

$$\pi : \quad W \otimes W' \longrightarrow I$$

Bilinear identity using intertwining operator

We can define new operator

$$\Gamma: \quad \widehat{V} \otimes \widehat{V}' \overset{E_R \otimes E_L'}{\longrightarrow} V \otimes W \otimes W' \otimes V' \overset{I \otimes \pi \otimes I}{\longrightarrow} V \otimes V',$$

which obey

$$\Gamma(g \otimes g) = (g \otimes g)\Gamma$$

Taking matrix elements allows to obtain bilinear identity:

$$\left\langle 0 \middle| U(t) \otimes \left\langle 0 \middle| U(t')\Gamma\Delta(g)\bar{U}(\bar{t}) \middle| 0 \right\rangle \otimes \bar{U}(\bar{t}') \middle| 0 \right\rangle =$$

$$= \left\langle 0 \middle| \mathrm{U(t)} \otimes \left\langle 0 \middle| U(t')\Delta(g)\Gamma\bar{U}(\bar{t}) \middle| 0 \right\rangle \otimes \bar{U}(\bar{t}') \middle| 0 \right\rangle,$$

Non-group element

There is new approach, which allow to avoid restriction of group element coproduct. Instead of that we use coproduct of arbitary q-deformed UEA and consider τ-functions based on this

$$\Delta(X) = \sum_{\alpha} X'_{\alpha} \otimes X''_{\alpha}$$

Split Casimir

In this approach instead of the operator Γ we use split casimir operator.

$$\mathcal{C} = \Delta(C)$$

Property of split Casimir

$$\Delta(X)\mathcal{C} = \mathcal{C}\Delta(X)$$

Taking matrix elements we obtain bilinear identity

$$\left\langle 0 \middle| U(t) \otimes \left\langle 0 \middle| U(t')\mathcal{C}\Delta(X)\bar{U}(\bar{t}) \middle| 0 \right\rangle \otimes \bar{U}(\bar{t}') \middle| 0 \right\rangle =$$

$$= \left\langle 0 \middle| \mathrm{U(t)} \otimes \left\langle 0 \middle| U(t')\Delta(X)\mathcal{C}\bar{U}(\bar{t}) \middle| 0 \right\rangle \otimes \bar{U}(\bar{t}') \middle| 0 \right\rangle$$

Evolution operators and τ-function

$$U(t) = \exp\left(\sum_{k=1}^{N-1} t_k \mathbf{E}_k^{(n)}\right), \quad \bar{U}(\bar{t}) = \exp\left(\sum_{k=1}^{N-1} \bar{t}_k \mathbf{F}_k^{(n)}\right)$$

.

$$\mathbf{E}_k^{(n)} := \left(\sum_i \mathbf{e}_i\right)^k, \quad \mathbf{F}_k^{(n)} := \left(\sum_i \mathbf{f}_i\right)^k$$

τ-function, which corresponds to the n-th fundamental representation:

$$\tau_n(t; g) = {}_n\langle 0| \exp_q\left(\sum_{k=1}^{N-1} t_k \mathbf{E}_k^{(n)}\right) X \exp_q^{-1}\left(\sum_{k=1}^{N-1} \bar{t}_k \mathbf{F}_k^{(n)}\right) |0\rangle_n$$

$U_q(\mathfrak{sl}_3)$ calculations

Evolution operators of $U_q(\mathfrak{sl}_3)$ satisfy the following property

$$\mathbf{E}_1^{(n)}\mathbf{E}_2^{(n)} = 0, \quad \mathbf{F}_\mathbf{1}^{(\mathbf{n})}\mathbf{F}_\mathbf{2}^{(\mathbf{n})} = 0$$

So the flows are both commutative and q-commutative For q-commutative $\mathbf{E_1}$ and $\mathbf{E_2}$

$$e_q^{t_1\mathbf{E_1}} e_q^{t_2\mathbf{E_2}} = e_q^{t_1\mathbf{E_1}+\mathbf{t_2}\mathbf{E_2}}$$

q-difference operator:

$$D_{q,x}f(x) = \frac{f(q^2x) - f(x)}{(q^2-1)x}$$

$U_q(\mathfrak{sl}_3)$ calculations

Property of difference operator:

$$D_{q,x} e_q^{xT} = T e_q^{xT}$$

q-BCH formula for bringing central elements through evolution operators.

$$e_{q^{-1}}^{A} B e_q^{-A} = B + [A, B] + \frac{1}{[2]_q!}[A, [A, B]_q] + \frac{1}{[3]_q!}[A, [A, [A.B]_q]_{q^2}] + \ldots,$$

where $[A, B]_{q^k} = q^k AB - q^{-k} BA$.
Then we obtain bilinear equation with commutative flows.

Fermionic realisation of $U_q(\mathfrak{sl}_n)$

Fermionic algebra

$$\{\psi_i, \psi_j^*\} = \delta_{ij} \quad \{\psi_i, \psi_j\} = 0 \quad \{\psi_i^*, \psi_j^*\} = 0,$$

The correspondence between the simple root generators of $U_q(sl_n)$ reads as

$$e_i = \psi_i \psi_{i+1}^*$$

$$f_i = \psi_{i+1} \psi_i^*$$

$$h_i = \psi_i \psi_i^* - \psi_{i+1} \psi_{i+1}^*$$

Fermionic realisation of $U_q(\mathfrak{sl}_n)$

Now let us introduce the vector $|0\rangle$ such that

$$\psi_i^*|0\rangle = 0 \quad \text{for } i \geq 1.$$

Then, the highest weight vector in the n-th fundamental representation is given by

$$|0\rangle_n = \psi_n\psi_{n-1}\dots\psi_2\psi_1|0\rangle := |n\rangle$$

We choose the evolution operators in the same manner as before while in terms of free fermions flows $\mathbf{E}_i, \mathbf{F}_i$ read as follows

$$\mathbf{E}_k := \sum_i \psi_i\psi_{i+k}^*, \quad \mathbf{F}_k := \sum_i \psi_{i+k}\psi_i^*$$

Approach to $U_q(\mathfrak{sl}_n)$, using intertwining operator

We redefine operators

$$e_i \to e_i q^{-\frac{h_i}{2}}$$

$$f_i \to q^{\frac{h_i}{2}} f_i$$

New coproduct:

$$\Delta e_i = e_i \otimes K_i^{-2} + I \otimes e_i,$$

$$\Delta f_i = f_i \otimes I + K_i^2 \otimes f_i,$$

We introduce q-deformed analogue of fermions

$$\Phi_i^R = q^{2\sum_{j=1}^{i-1} \psi_j \psi_j^*} \psi_i. \quad \Phi_i^{R*} = q^{2\sum_{j=1}^{i-1} \psi_j \psi_j^*} \psi_i^*$$

$$\Phi_i^L = q^{-2\sum_{j=1}^{i-1} \psi_j \psi_j^*} \psi_i \quad \Phi_i^{L*} = q^{-2\sum_{j=1}^{i-1} \psi_j \psi_j^*} \psi_i^*$$

where Φ^L,$\Phi^{*,L}$:$F_1 \otimes F_n \leftrightarrows F_{n+1}$ and $\Phi^R, \Phi^{*,R} : F_n \otimes F_1 \leftrightarrows F_{n+1}$

Approach to $U_q(\mathfrak{sl}_n)$, using intertwining operator

Using an analogy with the classical case

$$\Gamma = \sum_i \Phi_i^L \otimes \Phi_i^{R*}$$

This operator commute with new coproduct

$$\Gamma\Delta(X) = \Delta(X)\Gamma$$

Taking the matrix element we can again obtain bilinear identity

Approach to $U_q(\mathfrak{sl}_n)$, using intertwining operator

$$\Gamma = \oint \frac{dz}{z} \Phi(z) \otimes \Phi^*(z), \qquad \Phi(z) = \sum_{k=1}^{N} \Phi_k^L z^k.$$

$$\Phi^*(z) = \sum_{k=1}^{N} \Phi_k^{R*} z^{-k}.$$

After applying the q-BCH formula, we obtain the explicit expressions for the vertex operators for the fundamental representations:

$$\sum_{\alpha} \oint \frac{dz}{z} \left(\left[V_3\left(\bar{t}, z\right) \tau_n\left(t, \bar{t}; X'_\alpha\right) \right] \cdot \left[V_4\left(\bar{t}', z\right) \tau_m\left(t', \bar{t}'; X''_\alpha\right) \right] - \right.$$
$$\left. - \left[V_2(t, z) \tau_{n-1}\left(t, \bar{t}; X'_\alpha\right) \right] \cdot \left[V_1\left(-t', z\right) \tau_{m+1}\left(t', \bar{t}'; X''_\alpha\right) \right] \right) = 0$$

Conclusion

Results

- Bilinear identity for $U_q(\mathfrak{sl}_3)$ with commutative flows was derived using intertwining operator and split Casimir for non-group element.
- It was suggested new way to calculate bilinear identities for $U_q(\mathfrak{sl}_n)$ using intertwining operator for non-group element.

Problems

- We did not obtain bilinear identity with commutative flows in the case of $U_q(\mathfrak{sl}_n)$