# Matrix elements of DIM-algebra intertwining operators.

based on work with A. Mironov, Y. Zenkevich and I. Ryzhkov

Mikhail Sharov

INR RAS, ITEP, ITMP MSU

17 Febraury 2026

# Macdonald polynomials

- **(Symmetric) Macdonald polynomials**

$$P_\lambda(x;q,t) = \mathfrak{m}_\lambda(x) + \sum_{\mu<\lambda} K_{\lambda\mu}\mathfrak{m}_\mu(x)$$

- Orthogonality:

$$\langle P_\lambda, P_\mu\rangle = z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

- Eigenfunctions of the (**Macdonald-Ruijsenaars**) difference operator:

$$D_{MR} = \sum_{i=1}^{N} \prod_{i\neq j} \frac{tx_i - x_j}{x_i - x_j} q^{x_i\partial_i}$$

where $q^{x_i\partial_i} f(x_1,...,x_n) = f(x_1,\ldots,qx_i,\ldots,x_n)$

# Macdonald difference operators

- $N$ difference operators:

$$D_m = \sum_{I \subset \{1,N\}, |I|=m} \prod_{i \in I, j \notin I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} q^{x_i \partial_i}$$

- Commutative set:

$$\forall m, n \in 1 \ldots N : [D_m, D_n] = 0$$

- Macdonald polynomials are eigenfunctions.

# Noumi-Shiraishi functions

- Eigenfunctions of $D_{MR}$: polynomials $\to$ functions

$$p_n(x; s \mid q, t) = \sum_{\theta \in M_n} c_n(\theta; s \mid q, t) \prod_{1 \le i < j \le n} (x_j/x_i)^{\theta_{i,j}} ,$$

  + constraints on $c_n$

- Example of $N = 2$

$$NS_{q,t}(\vec{z}, \vec{\lambda}) = q^{\lambda_1 z_1 + \lambda_2 z_2} \cdot t^{\frac{z}{2}} \sum_{k=0}^{\infty} \left(\frac{x_2}{x_1}\right)^k \frac{\left(t\frac{y_2}{y_1}; q\right)_k}{\left(q\frac{y_2}{y_1}; q\right)_k} \frac{\left(q^{-k}\frac{q}{t}; q\right)_k}{\left(q^{-k}; q\right)_k},$$

  where $(x; q)_k := \prod_{j=0}^{k-1}(1 - xq^j)$

# Ding-Iohara-Miki algebra

- Quantum affinization of $U_q(\widehat{\mathfrak{gl}}_1)$ deformed by t.
- 2 loops $\rightarrow$ quantum toroidal $U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_1)$.
- Two gradings $(d, d_\perp)$, two central charges $(C_1, C_2)$
- Generated by Drinfield currents

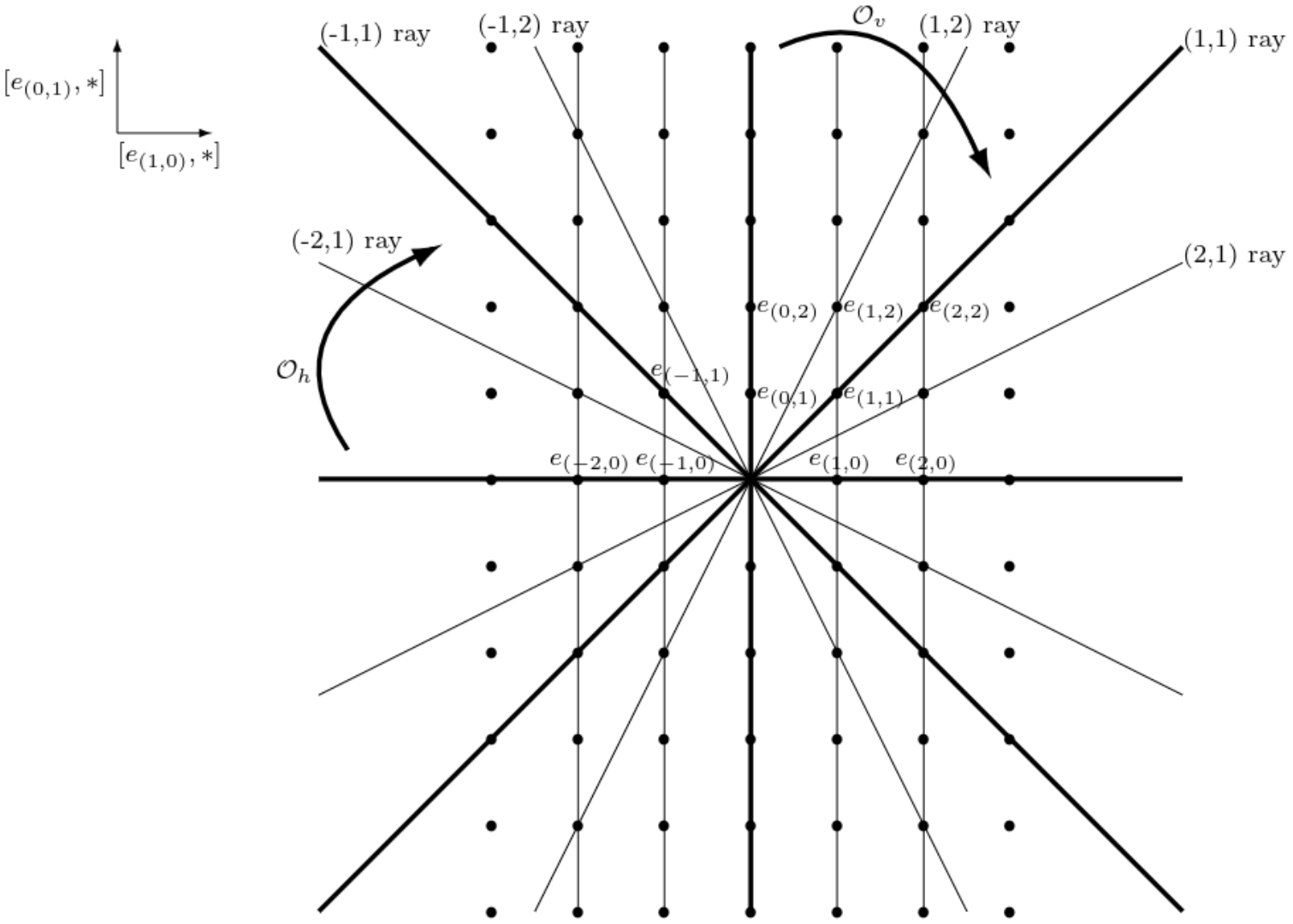
$$x^{\pm}(z) = \sum_{n\in\mathbb{Z}} e_{(\pm 1,n)} z^{-n}, \quad \psi^{\pm}(z) = C_2^{\pm\frac{1}{2}} \exp\left[-\sum_{n\geq 1} \frac{\kappa_n}{n} e_{(0,\pm n)} z^{\mp z}\right]$$

- Structure function

$$g(z,w) = (z - q_1 w)(z - q_2 w)(z - q_3 w),$$

where $q_1 = q$, $q_2 = t^{-1}$, $q_3 = \frac{t}{q}$

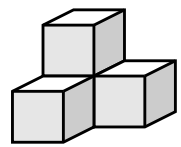
# Elliptic Hall Algebra




- $SL(2,\mathbb{Z})$ automorphisms

# DIM algebra representations
## MacMahon representation

- MacMahon representation - central charge vector $(1, K)$, arbitrary $K$
- $3d$ Young diagram:



- Three $q_i$ $\quad \rightarrow \quad$ Three axes $(q, t^{-1}, t/q)$
- Irreducible subrepresentation of MacMahon: labeled by $1d$, $2d$ and $3d$ Young diagrams.

# DIM algebra representations
## Vector representation

- Vector representation $\mathcal{V}_w^q$
- Trivial central charges $(\gamma^2, \gamma_\perp^2) = (1, 1)$
- Evaluation representation for the currents:

$$x^+(z)|w\rangle = -\frac{1}{1-q^{-1}}\delta\left(\frac{w}{z}\right)|qw\rangle,$$

$$\psi^+(z)|w\rangle = \frac{\left(1-\frac{t}{q}\frac{w}{z}\right)\left(1-\frac{1}{t}\frac{w}{z}\right)}{\left(1-\frac{w}{z}\right)\left(1-\frac{1}{q}\frac{w}{z}\right)}|w\rangle,$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$

- Two more: $\mathcal{V}_w^{t^{-1}}$ and $\mathcal{V}_w^{t/q}$.

# DIM algebra representations
## Vector representation

- Tensor product

$$\mathcal{V}^N(u) = \mathcal{V}(u) \otimes \mathcal{V}\left(ut^{-1}\right) \otimes \cdots \otimes \mathcal{V}\left(ut^{-N+1}\right),$$

Set

$$\mathcal{P}^N = \left\{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N\right\}$$

- Let $W^N(u) \hookrightarrow V^N(u)$ be subspace spanned by the

$$|\lambda\rangle_u = [u]_{\lambda_1} \otimes \left[ut^{-1}\right]_{-1+\lambda_2} \otimes \cdots \otimes \left[ut^{-N+1}\right]_{-N+1+\lambda_N},$$

where $[u]_{\lambda_1} = |uq^{\lambda_1}\rangle$

- $W^N(u)$ is a level $(1,1)$ submodule of $V^N(u)$

# DIM algebra representations
## Fock representation

- Semi-infinite construction. Inductive limit

$$\mathcal{F}(u) = \lim_{N\to\infty} W^{N,+}(u),$$

where $\lambda_N \geq 0$

- Endow $\mathcal{F}(u)$ with module structure

$$x(z)|\lambda\rangle = \lim_{N\to\infty} x^{[N]}(z)\,|\lambda_1, \ldots, \lambda_N\rangle$$

- $\mathcal{F}(u)$ is a level $(1, q_2)$ module.
- In Fock representation basis labeled by $2d$ Yound diagrams.
- Fock representation admits free field realization with:

$$[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}$$

# Intertwining operators

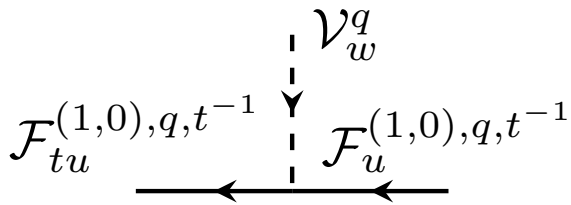
- Vector-Fock intertwiner:

$$\Phi(w) : \mathcal{V}_w^q \otimes \mathcal{F}_u^{(1,0),q,t^{-1}} \to \mathcal{F}_{tu}^{(1,0),q,t^{-1}}$$

- Intertwining property $\to$ $\Phi(w)$ exists uniquely

$$\Phi(w) \circ \Delta(g) = g \circ \Phi(w), \quad \forall g$$

- Diagrammatical depiction:

$\Phi(w) =$ 


- Free field realization:

$$\Phi(w) = e^{-\epsilon_2 Q} w^{\frac{P}{\epsilon_1}} \exp\left[-\sum_{n\geq 1} \frac{w^n}{n} \frac{1-t^{-n}}{1-q^n} a_{-n}\right] \exp\left[\sum_{n\geq 1} \frac{w^{-n}}{n} \frac{1-t^n}{1-q^{-n}} a_n\right]$$

# Gluing rules

- Gluing dual intertwiners:

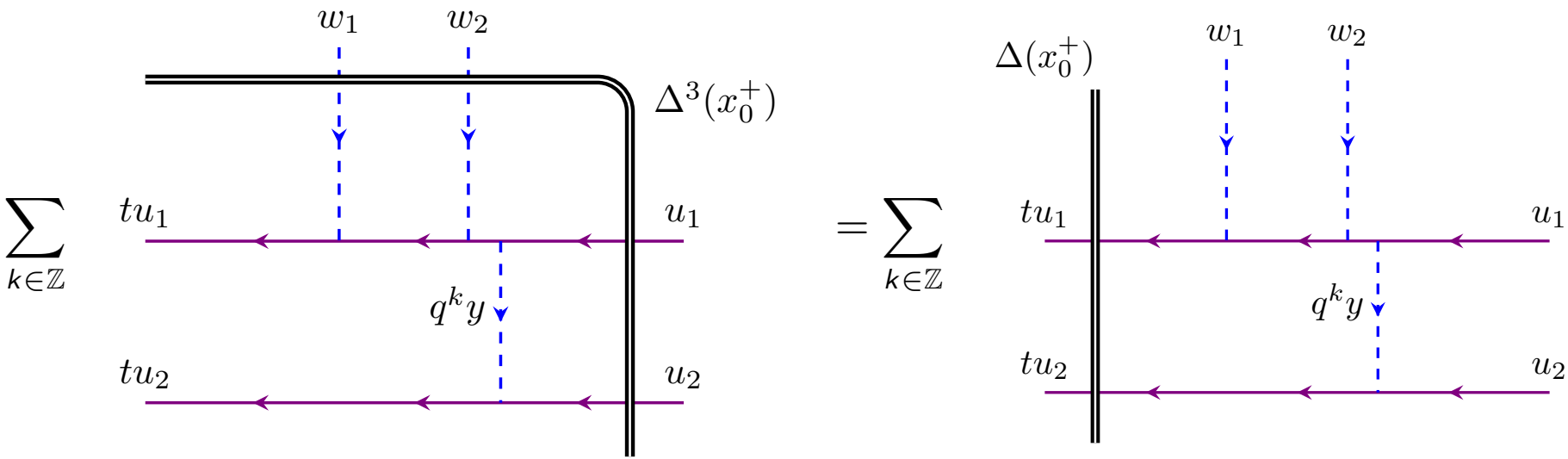
$$\mathcal{F}^{q,t^{-1}}_{u_1} \otimes \mathcal{F}^{q,t^{-1}}_{u_2} \to \mathcal{F}^{q,t^{-1}}_{t^{-1}u_1} \otimes \mathcal{F}^{q,t^{-1}}_{tu_2} \ :$$

- Summation over the complete basis of states in the vector representation:

$$\mathcal{Q}^{q,t^{-1}}_{q} = \sum_{k\in\mathbb{Z}} \mathcal{F}^{q,t^{-1}}_{t^{-1}u_1} \ \mathcal{F}^{q,t^{-1}}_{u_1} \ \mathcal{V}^{q}_{q^k w} \ \mathcal{F}^{q,t^{-1}}_{tu_2} \ \mathcal{F}^{q,t^{-1}}_{u_2} = \sum_{k\in\mathbb{Z}} \Phi^*(q^k w) \otimes \Phi(q^k w)$$

# Noumi-Shiraishi function

- Intertwining property for DIM element $g = x_0^+$:




- Coproduct:

$$\Delta^3\left(x_0^+\right)\Big|_{\mathcal{V}^q_{w_1}\otimes\mathcal{V}^q_{w_2}\otimes\mathcal{F}^{q,t^{-1}}_{w_1}\otimes\mathcal{F}^{q,t^{-1}}_{w_1}} |w_1\rangle\otimes|w_2\rangle\otimes|\varnothing\rangle\otimes|\varnothing\rangle,$$

where

$$\Delta^3| = (\Delta\otimes\Delta)\circ\Delta$$

# Noumi-Shiraishi function

- Eigenequation:

$$\mathcal{G}(w_1, w_2)\psi(\vec{w}, \vec{u}) =$$
$$\left( q^{w_1 \partial_{w_1}} + \frac{\left(1 - t\frac{w_2}{w_1}\right)\left(1 - \frac{q}{t}\frac{w_2}{w_1}\right)}{\left(1 - \frac{w_2}{w_1}\right)\left(1 - q\frac{w_2}{w_1}\right)} q^{w_2 \partial_{w_2}} \right) \psi(\vec{w}, \vec{u}) = (u_1 + u_2)\,\psi(\vec{w}, \vec{u})$$

- Towards RS Hamiltonians:

$$F^{-1}(\vec{w})\mathcal{G}(\vec{w})F(\vec{w}) = H_1^{RS,\ N=2}$$

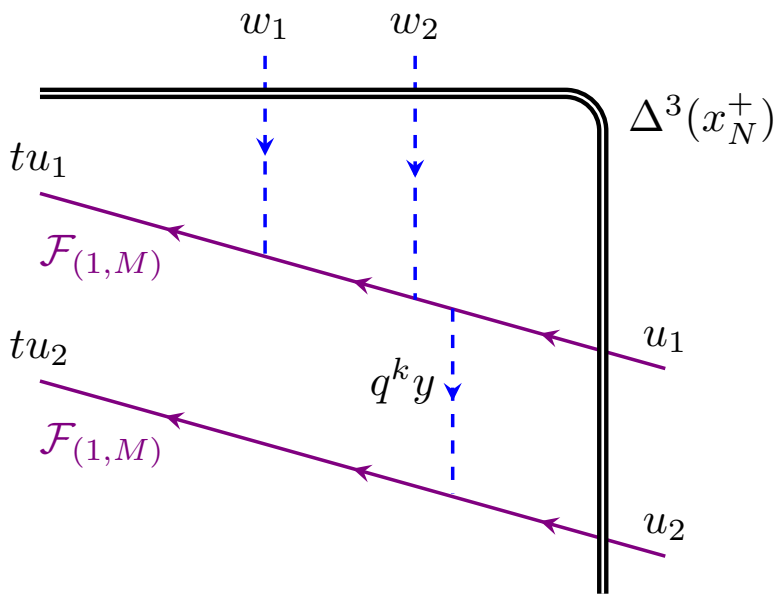
- Matrix element (VEV):

$$\begin{matrix} \langle\varnothing| \\ \otimes \\ \langle\varnothing| \end{matrix} \Big(Net\Big) \begin{matrix} |\varnothing\rangle \\ \otimes \\ |\varnothing\rangle \end{matrix} = \left(\frac{q}{t}\right)^{\frac{1}{2}\log_q \frac{u_2}{u_1} + \beta} w_1^{\log_q u_1} w_2^{\log_q u_2} \sum_{k \in \mathbb{Z}} \left(\frac{t u_2}{u_1}\right)^k \frac{\left(\frac{q}{t}\frac{w_2}{w_1}; q\right)_k}{\left(q\frac{w_2}{w_1}; q\right)_k} \frac{\left(\frac{q}{t}; q\right)_k}{(q; q)_k},$$

# Twisted NS-functions

- EHA's element:

$$e_{(\pm n,1)} \overset{(N-body)}{=} \frac{q^{\frac{1\pm n}{2}}}{q-1} \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} x_i^{\pm n} q^{\hat{D}_i}$$




- Simplest twisted ray of Hamiltonians, ray (a,1)

$$H_1^{(N,1)} = \left( \frac{x_2 - tx_1}{x_2 - x_1} x_1^N q^{x_1 \partial_{x_1}} + \frac{x_1 - tx_2}{x_1 - x_2} x_2^N q^{x_2 \partial_{x_2}} \right)$$

# Higher order (Twisted) Hamiltonians

- EHA's element:

$$e^{(-1,a+1)} \equiv \hat{\mathcal{H}}_1^{(-1,a+1)} = \left[\hat{\mathcal{H}}_1^{(-1,a)}, \hat{\mathcal{H}}_{MR}\right] \equiv \left[e^{(-1,a)}, e^{(0,1)}\right]$$

- Operator, emerges from the coproduct:

$$\Delta([x_N^+, x_M^+])|_{\mathcal{V}_{w_1}\otimes\mathcal{V}_{w_2}} = (1-q)\left(\frac{(t-q)(t-1)(w_1+w_2)}{(qw_1-w_2)(qw_2-w_1)} q^{w_1\partial_{w_1}} q^{w_2\partial_{w_2}} + \right.$$
$$\left. + \frac{(w_1-tw_2)(w_1-qtw_2)}{(qw_2)(w_1-w_2)(w_1-qw_2)} q^{2w_2\partial_{w_2}} + \frac{(tw_1-w_2)(qtw_1-w_2)}{(qw_1)(w_1-w_2)(qw_1-w_2)} q^{2w_1\partial_{w_1}}\right)$$

- Possible construction of an appropriate network:

$$\Delta([x_N^+, x_M^+])|_{\mathcal{V}_{w_1}\otimes\mathcal{V}_{w_2}\otimes\mathcal{F}_{u_1}^{(1,N)}\otimes\mathcal{F}_{u_2}^{(1,M)}}.$$

# Twisted Hamiltonians

- All Twisted Hamiltonians appear as operators in the vector representation of DIM algebra.

$$H_{twisted} = \Delta(g)$$

- The EF of twisted Hamiltonians cannot be built as DIM algebra matrix elements of networks.

# Twisted Hamiltonians

- All Twisted Hamiltonians appear as operators in the vector representation of DIM algebra.

$$H_{twisted} = \Delta(g)$$

- The EF of twisted Hamiltonians cannot be built as DIM algebra matrix elements of networks.

- Another coproduct and $SL(2,\mathbb{Z})$ automorphisms.

$$\Delta_{\theta'}(g) = F^{-1}_{\theta,\theta'}\Delta_{\theta}(g)F_{\theta,\theta'}.$$

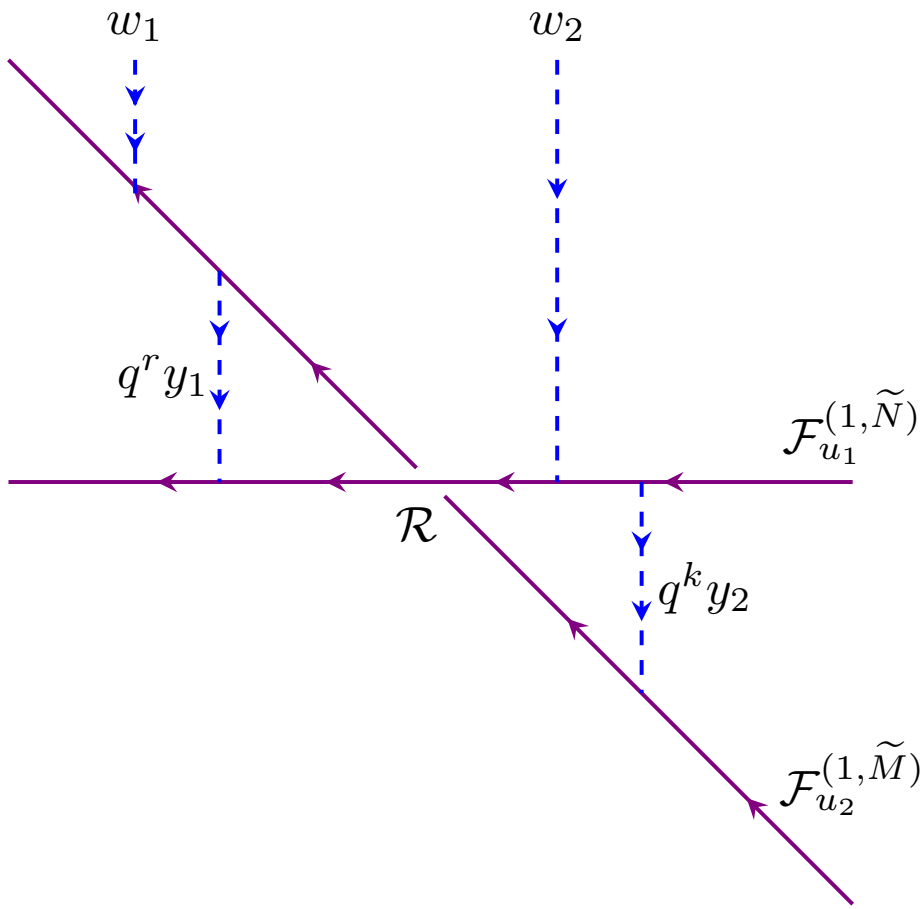
- Previously considered: $\theta = -\frac{\pi}{2} + \epsilon$

- $H_{twisted} = \Delta_{\theta'}(x_0^+)$

# Higher order Hamiltonians

- Possible construction of an appropriate network:

$$\Delta([x_N^+, x_M^+])|_{\mathcal{V}_{w_1} \otimes \mathcal{V}_{w_2} \otimes \mathcal{F}_{u_1}^{(1,N)} \otimes \mathcal{F}_{u_2}^{(1,M)}}.$$




- Does not lead to an eigenfunction equation.