Large colour Wilson loops in $\mathcal{N} = 4$ SYM and Fredholm determinants

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- Wilson loops are important observables in gauge theory and play a special role in dualities, specifically *holography* or *gauge-string duality*

- Holographic prediction is that Wilson loops are dual to certain configurations of strings.

- This talk: $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory

- In $\mathcal{N} = 4$ SYM theory *circular* (SUSY) Wilson loops are $\frac{1}{2}$-BPS and via localization are equal to an *average in the Gaussian matrix model*.

- It can solved be (in principle) at any coupling, hence allowing an *explicit* check of the duality beyond planar and strong coupling limit.

- This is rare as the duality is between *weak* and *strong* coupling: typically only quantities that don't recieve corrections can be compared

The fundamental Wilson loop

- Supersymmetric Wilson loop in rep. R:

$$W_R = \mathrm{Pexp}\left(\oint_{\mathcal{C}} ds\,\left(A_\mu \dot{x}^\mu + |\dot{x}| n^k(s)\Phi_k\right)\right)$$

Consider: $\mathcal{C}$ is a circle and $n^k(s) = \text{const}$,

- Localization Gaussian matrix model: fundamental representation [Ericson,Semenoff,Zarembo],[Pestun]

$$\left\langle W_\square \right\rangle^{\mathrm{QFT}} = \left\langle W_\square \right\rangle = \int \mathrm{d}X\, e^{-\frac{2N}{\lambda}\,\mathrm{Tr}\,X^2}\,\mathrm{Tr}\,e^X$$

where $\lambda = g^2_{\mathrm{YM}} N$, N is from $U(N)$.

- Exact result [Drukker, Gross]:

$$\left\langle W_\square \right\rangle = e^{\frac{\lambda}{8N}} L^1_{N-1}\left(\frac{-\lambda}{4N}\right)$$

- Planar limit (but for generic λ):

$$\left\langle W_{\square} \right\rangle \to \left\langle W_{\square} \right\rangle_{\text{planar}} = N \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

- Can be expanded at strong coupling $\lambda \gg 1$ to arbitrary order

- In holography we are at $g_s \to 0$:

$$\left\langle W_{\square} \right\rangle_{\text{planar}} = \int \mathcal{D}X^M \ldots \exp\left(-\frac{L^2_{\text{AdS}}}{2\pi\alpha'} S_{\text{string}}\left[X^M, \ldots\right] \right)$$

  where $S_{\text{string}}\left[X^M, \ldots\right]$ is the Polyakov action on the $AdS_5 \times S^5$ background:

$$S_{\text{str}} = \frac{1}{2} \int d\tau d\sigma \sqrt{h} h^{ab} \frac{1}{Z^2} \left(\partial_a X^\mu \partial_b X_\mu + \partial_a Z \partial_b Z \right) + \ldots$$

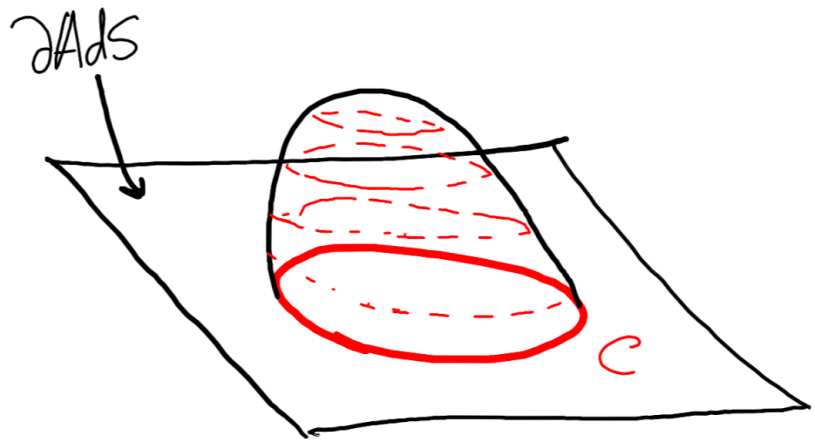
- B.c. for the string: string worldsheet ends on the Wilson loop $\mathcal{C}$.
- Matching of parameters $\alpha' = \dfrac{L^2_{\text{AdS}}}{\sqrt{\lambda}}$

- Expanding at large string tension $T = \frac{L^2_{\mathrm{AdS}}}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$ we get:

$$\int \mathcal{D}X^M e^{-T\,S_{\mathrm{string}}[X^M]} = e^{T\times\mathrm{Area}+\sum_{k\geq 0} T^{-k}\Gamma_k}$$

- The classical solution dual to the Wilson is given by a AdS_2 shaped minimal surface, that is bounded by the Wilson loop:

$$\left\langle W_{\square} \right\rangle_{\mathrm{planar}} \sim e^{\sqrt{\lambda}} = e^{2\pi T} = e^{T\times\mathrm{Area}} \left(= e^{-T\,S_{\mathrm{string}}[X^M_{\mathrm{classical}}]} \right)$$



- The key observation: holography can be compared at the level of α'-corrections!
- Corrections to the sigma model are organized in terms of $\alpha' \sim 1/T \sim 1/\sqrt{\lambda}$:

$$\langle W_{\square} \rangle_{\text{planar}} = N \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \sqrt{\frac{2}{\pi}} \frac{N}{\lambda^{3/4}} e^{\sqrt{\lambda}} \left[1 - \frac{3}{8\sqrt{\lambda}} + \cdots \right] =$$

$$= c_1 \frac{1}{g_s} \sqrt{\frac{T}{2\pi}} e^{2\pi T} \left[1 - \underbrace{\frac{3}{16\pi} \frac{1}{T}}_{[(*),2026]} + \cdots \right], \qquad .$$

$$T = \frac{L^2_{\text{AdS}}}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \quad g_s = \frac{\lambda}{4\pi N}$$

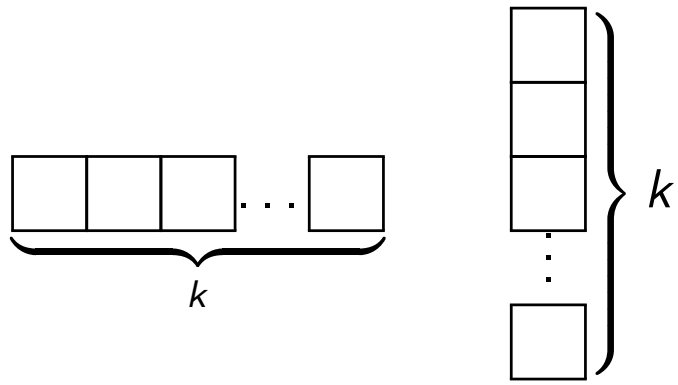
- Higly non-trivial match in loop order: [Kruczenski, Tirziu],[Bergamin, Tseytlin],[Giombi, Tseytlin] [(*) Beccaria, Kurlyand, Tseytlin]

The large color Wilson loop

- Higher representations [Ericson,Semenoff,Zarembo],[Pestun]

$$\left\langle W_R \right\rangle = \left\langle \operatorname{Tr}_R e^X \right\rangle \quad \operatorname{Tr}_{\square\square} e^X = \frac{1}{2}\left(\operatorname{Tr} e^X \operatorname{Tr} e^X + \operatorname{Tr} e^{2X}\right)$$

- We take $R = S_k = [k]$ - k-th symmetric or $R = A_k = [1^k]$ - k-th antisymmetric.



- Consider the following regime $k \sim N$ with $f = \dfrac{k}{N}$ - fixed. Similar to other *large charge* expansions.

- When $k \ll N$ – important, but boring:

$$W_{A_k/S_k} \sim \frac{1}{k!}\left(\mathrm{Tr}\, e^X\right)^k$$

which means

$$\left\langle W_{A_k/S_k} \right\rangle \sim e^{\sqrt{\lambda}k}$$

- But for finite f all terms with $\left(\frac{k^2}{N^2}\right)^n$ contribute [Drukker, Fiol]:

$$\log\left\langle W_{S_k} \right\rangle = N\left(-\frac{k}{N}\sqrt{\lambda} - \frac{\lambda^{3/2}}{48}\frac{k^3}{N^3} + \frac{\lambda^{5/2}}{10240}\frac{k^5}{N^5} + \ldots\right) + \frac{1}{N}\left(\ldots\right)$$

- When $k \sim N =$ resuming an infinite number of non-planar corrections! Piece of a non-perturbative string?

- Now the problem is hard even in the matrix model. Leading order from MM [Hartnoll, Kumar] ($k \gg 1, N \gg 1$, f-fixed)
- AS at strong coupling $\sqrt{\lambda} \to \infty$:

$$\left\langle W_{A_k} \right\rangle = \exp\left(\frac{2N\sqrt{\lambda}}{3\pi}\sin^3\theta_k\right)$$

where

$$\pi f = \theta_k - \sin\theta_k \cos\theta_k \qquad f = \frac{k}{N}$$

- S at strong coupling:

$$\left\langle W_{S_k} \right\rangle = \exp\left(2N[\,\kappa\sqrt{1+\kappa^2} + \operatorname{arcsinh}\kappa\,]\right)$$

where $\kappa = \dfrac{\sqrt{\lambda}f}{4}$ -finite, while $\sqrt{\lambda} \to \infty$.

- Holographically: String WS $\longrightarrow$ Branes of shape:

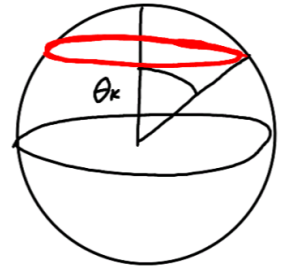
$$\underbrace{AdS_2}_{\text{String WS}} \times \underbrace{S^{p-1}}_{k \text{ units of flux}}$$

- Symmetric - D3 brane, antisymmetric - D5 brane; k - parameterizes the flux of the w/s gauge field, forcing a certain geometry

- Leading order = classical solution:

  D3 - $AdS_2 \times S^2$ geometry within AdS_5

  D5 - $AdS_2 \times S^4$, where $S^4 \subset S^5$ at angle θ_k



$$\left\langle W_{A_k/S_k} \right\rangle_{\text{lead.}} = \exp\left(-T_{D5/D3} \times (\text{Area})\right) = \exp\left(-T_{D5/D3}\, S_{\text{DBI}}^{\text{classical}}\right)$$

- Matching of parameters

$$T_{\mathrm{F1}} = \frac{\sqrt{\lambda}}{2\pi}, \quad T_{\mathrm{D3}} = \frac{N}{2\pi^2}, \quad T_{\mathrm{D5}} = \frac{N\sqrt{\lambda}}{8\pi^4}$$

- Holography:

$$\left\langle W_{A_k}/S_k \right\rangle \quad \text{vs} \quad \int DX^M \exp\left(-T_{\mathrm{Dp}}\, S_{\mathrm{DBI}}[X^M]\right)$$

- Match at leading order in T_{Dp}. Quantization?

- No proper result from the brane side, but not even in gauge theory

- Anti-symmetric: from holography [Faraggi, Mück, Pando Zayas], from MM: [Faraggi, Liu, Pando Zayas, Zhang],[Gordon],[Canazas Garay, Faraggi, Mück]

$$\frac{1}{6}\ln\sin\theta_k \quad \textbf{vs} \quad \underbrace{\frac{1}{2}\ln\sin\theta_k \; \frac{\lambda}{8\pi^2}\left[\sin^2\theta_k - \theta_k\sin 2\theta_k + \theta_k^2\right] + \frac{1}{2}\ln\sin\theta_k}_{\text{The only reliable answer}}$$

$$\int e^{-S_{D5}} \sim \exp\left(\frac{2N\sqrt{\lambda}}{3\pi}\sin^3\theta_k - \frac{1}{6}\ln\sin\theta_k\right)$$

- Symmetric: from holography [Buchbinder, Tseytlin], from MM [Faraggi, Liu, Pando Zayas, Zhang],[Chen-Lin]:

$$\frac{1}{2}\ln\frac{\kappa^3}{\sqrt{1+\kappa^2}} \qquad \textbf{vs} \qquad -\frac{1}{2}\ln\frac{\kappa^3}{\sqrt{1+\kappa^2}}$$

$$\ln\kappa^3\left(\sqrt{1+\kappa^2}\right)$$

Our approach to the matrix model

- What we are doing - derive a new reliable method to compute subleading corrections in f-fixed limit *in the matrix model*. Such that it automatically resums a certain class of non-planar terms.

- In addition we need it to treat the symmetric and the anti-symmetric case on equal footing

- We rely on exact results and methods of integrable Fredholm determinants developed recently for similar problems
  [Belitsky, Korchemsky],[Bajnok, Boldis, Korchemsky],[Its, Izergin, Korepin, Slavnov],[Tracy,Widom],[Harnad,Its]

- Introduce generating functions

$$Z_{A/S}(t) = \sum_{k=0}^{\infty} t^k \left\langle W_{A_k/S_k} \right\rangle = \left\langle \det \left(1 \pm t e^{X}\right)^{\pm 1} \right\rangle$$

  Then:

$$\left\langle W_{A_k/S_k} \right\rangle = \oint \frac{dt}{t} t^{-k} Z_{A/S}(t)$$

  The t integral is taken via a saggle point – we need to compute $Z_{A/S}(t)$ everywhere in complex t.

- Claim 1: the exact result [Fiol, Torrents],[Okuyama]:

$$Z_A(t) = \det\left(1 + t\, A(\lambda)\right)$$

$$Z_S(t) = \det\left(1 + \sum_{n=1}^{\infty} t^n A\left(\lambda n^2\right)\right)$$

where

$$A_{ij}(\lambda) = e^{\frac{\lambda}{8N}} \sqrt{\frac{(i-1)!}{(j-1)!}} \left(\sqrt{\frac{\lambda}{4N}}\right)^{j-i} L_{i-1}^{j-i}\left(-\frac{\lambda}{4N}\right).$$

- How to take large N of an $N \times N$ deerminant of a matrix that depends on N?

- Claim 2: Fredholm determinant representation

$$Z_{A/S}(t) = \det\left(1 + \mathbf{H}\boldsymbol{\chi}^{A/S}\right)$$

- Here $\mathbf{H}$ is an integral operator

$$(\mathbf{H} \cdot f)(x) = \int \mathbf{H}(x,y) f(y) dx$$

with kernel

$$\mathbf{H}(x,y) \sim \exp\left(-\frac{x^2+y^2}{2}\right) \frac{H_N(x)H_{N-1}(y) - H_{N-1}(x)H_N(y)}{x-y}$$

- $\boldsymbol{\chi}$ is:

$$\boldsymbol{\chi}^{A/S}.f = \chi^{A/S}(x)$$

$$\chi^A(x) = te^{\sqrt{\frac{\lambda}{2N}}x}$$

$$\chi^S(x) = \frac{te^{\sqrt{\frac{\lambda}{2N}}x}}{1 - te^{\sqrt{\frac{\lambda}{2N}}x}}$$

- Example of integrable Fredholm determinant - a generalization of those appearing in τ-functions. Example of *classical integrability of quantum effective actions*! (*a la* [Its, Izergin, Korepin, Slavnov],[Kojima, Korepin, Slavnov],[Harnad,Its])

- In [Belitsky, Korchemsky],[Bajnok, Boldis, Korchemsky] methods to deal with a similar kernel where developed

$$K(x_1, x_2) = \frac{1}{\sqrt{x_1 x_2}} \sum_{m=0}^{\infty} (2m+1) J_{2m+1}\left(\sqrt{x_1}\right) J_{2m+1}\left(\sqrt{x_2}\right)$$
$$= \frac{\sqrt{x_1} J_1\left(\sqrt{x_1}\right) J_0\left(\sqrt{x_2}\right) - \sqrt{x_2} J_1\left(\sqrt{x_2}\right) J_0\left(\sqrt{x_1}\right)}{2(x_1 - x_2)}.$$

- A better quantity to compute is the free energy:

$$\mathcal{F}_{A/S}(t) = \frac{1}{N} \log \det\left(1 + \mathbf{H}\boldsymbol{\chi}^{A/S}\right) =$$
$$= \mathcal{F}_0^{A/S} + \frac{1}{N}\mathcal{F}_1^{A/S} + \dots$$

- The approach is based on differential equations for a set of auxiliary functions: $Q(x,t), P(x,t), u(t), w(t), v(t)$
- These functions are matrix elements of the resolvent of $\mathbf{H}\boldsymbol{\chi}$ between:

$$\phi(x) \simeq e^{-Nx^2} H_N(\sqrt{2N}x), \quad \psi(x) \simeq e^{-Nx^2} H_{N-1}(\sqrt{2N}x)$$

of the following form

$$Q(x,t) = \langle x| \frac{1}{1+\mathbf{H}\boldsymbol{\chi}} |\phi\rangle \qquad P(x,t) = \langle x| \frac{1}{1+\mathbf{H}\boldsymbol{\chi}} |\psi\rangle$$

$$u(t) = \langle \phi| \boldsymbol{\chi} \frac{1}{1+\mathbf{H}\boldsymbol{\chi}} |\phi\rangle \ , \quad w(t) = \langle \psi| \ldots |\psi\rangle \qquad v(t) = \langle \phi| \ldots |\psi\rangle$$

- These functions satisfy a system of (integro-differential Painleve?) equations:

$$\frac{1}{2N}\mathcal{D}Q(x,t) = -xQ(x,t) + (1+2u(t))P(x,t)$$
$$\frac{1}{2N}\mathcal{D}P(x,t) = xP(x,t) + (-1+2w(t))Q(x,t)$$
$$\partial_t u(t) = \int_{-\infty}^{\infty} \mathrm{d}x Q^2(x,t)\partial_t \chi(x,t)$$
$$\partial_t w(t) = \int_{-\infty}^{\infty} \mathrm{d}x P^2(x,t)\partial_t \chi(x,t)$$

with $\mathcal{D} = \partial_x - \sqrt{\lambda}t\partial_t$ and

$$\sqrt{\lambda}t\partial_t \mathcal{F}(t) = v$$
$$2uw - u + w = -\frac{1}{2N}\sqrt{\lambda}t\partial_t v(t)$$

- We want to solve this equation in the $1/N$ expansion
- When $\chi(x) = \theta(x)$ - this gives the familiar Painleve equation for the edge scaling in the Gaussian matrix model
- Large N = WKB expansion:

$$Q(x,t) = \exp\left(N \sum_{k=0}^{\infty} N^{-k} S_k(x,t)\right)$$

- From this approach we find:

$$\mathcal{F}_0^{A/S}(t) = \frac{2}{\pi}\int_{-1}^{1} \mathrm{d}x\sqrt{1-x^2}\log(1+\chi^{A/S}(x,t)),$$

- This can now be integrated over t in the saddle point approximation, to find the k'th Wilson loop.

- $S_k(x,t)$ easily follow from WKB equations. Some difficulty is involved in the integral equations

Summary

- The problem is hard on the matrix model side, since it requires to compute "non-perturbatively".

- The holographic match at leading order is quite impressive as it matches to functions of k/N (in contrast to the string)

- A reliable result should be informative on the quantization of the D-brane configurations

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