On three-loop divergences in $6D, \mathcal{N} = (1, 1)$ SYM

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Aims:

- Construction of the background superfield method for $6D, \mathcal{N} = (1,0)$ interacting non-Abelian vector multiplet with hypermultiplet
- Analysis of divergences in the $\mathcal{N} = (1,1)$ SYM theory up to three loops

The talk is based on:

I.L. Buchbinder, E.A. Ivanov, B.M., arXiv:2026.xxxxx

I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., *JHEP 2305 (2023); Phys.Lett.B 820 (2021); Nucl.Phys.B 921 (2017); JHEP 1701 (2017); Phys.Lett.B 763 (2016).*

Plan

- General motivations
- $\mathcal{N} = (1,1)$ SYM theory in terms of $\mathcal{N} = (1,0)$ harmonic supersfields
- Background field method
- Divergent part of one-loop effective action
- Two-loop results
- Three-loop analysis
- Summary

General motivation

The modern interest to $6D$ supersymmetric gauge theories is stipulated by the following reasons:

- The problem of describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings and the connection of effective action for the D5-branes at low energies with maximally supersymmetric Yang-Mills theory in six dimensions. [**N.Seiberg (1996), E. Witten (1996); N. Seiberg, (1997)**].

- Lagrangian description of the interacting multiple $M5$-branes is related to $6D$, $\mathcal{N} = (2, 0)$ supersymmetric gauge theory. The theory includes self-dual non-Abelian antisymmetric tensor and it is not constructed still (see e.g. review [**J. Bagger, N. Lambert, S. Mikhu, C. Papageorgakis (2013)**]).

Motivations

▶ The problem of miraculous cancellation of on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop [**M.B. Green, J.H. Schwarz, L. Brink, (1982)**].
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions [**P.S. Howe, K.S. Stelle, (1984), (2003); G. Bossard, P.S. Howe, K.S. Stelle, (2009)**].
- Direct one-loop and two-loop component calculations (mainly in bosonic sector and mainly on-shell) [**E.S. Fradkin, A.A. Tseytlin, (1983); N. Marcus, A. Sagnotti, (1984), (1985)**].
- Direct calculations of scattering amplitudes in $6D$ theory up to five loops and in $D8, 10$ theories up to four loops [**L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, (2015).**]

Results: On-shell divergences in $6D$ theory start at three loops.

Purpose

To study one-, two- and three-loop divergences of the superfield effective action in $\mathcal{N} = (1,1)$ SYM theory.

Properties

$6D, \mathcal{N} = (1,1)$ SYM theory possesses some properties close or analogous to $4D, \mathcal{N} = 4$ SYM theory.

- The $6D, \mathcal{N} = (1,1)$ SYM theory can be formulated in harmonic superspace as well as the $4D, \mathcal{N} = 4$ SYM theory.
- The $6D, \mathcal{N} = (1,1)$ SYM theory possesses the manifest $\mathcal{N} = (1,0)$ supersymmetry and additional hidden $\mathcal{N} = (0,1)$ supersymmetry analogous to $4D, \mathcal{N} = 4$ SYM theory where there is the manifest $\mathcal{N} = 2$ supersymmetry and additional hidden $\mathcal{N} = 2$ supersymmetry.
- The $6D, \mathcal{N} = (1,1)$ SYM theory is anomaly free as well as the $4D, \mathcal{N} = 4$ SYM theory and satisfies some non-renormaization theorems.

Harmonic superspace

$4D$

A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, (1984).

A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic Superspace*, **(2001).**

General purpose:
to formulate $\mathcal{N}=2$ models in terms of unconstrained $\mathcal{N}=2$ superfields.
General idea:
to use the parameters $u^{\pm i}(i=1,2)$ (harmonics) related to $SU(2)$ automorphism group of the $\mathcal{N}=2$ superalgebra and parameterizing the 2-sphere, $u^{+i}u^{-}_{i}=1$.

It allows to introduce the $\mathcal{N}=2$ superfields and formulate the theory with manifest $\mathcal{N}=2$ supersymmetry in harmonic superspace. Price for this is a presence of extra bosonic variables, harmonics $u^{\pm i}$.

$6D$

P.S. Howe, K.S. Stelle, P.C. West, (1985).
B.M. Zupnik, (1986); (1999).

G. Bossard, E. Ivanov, A. Smilga, (2015).

Note! Pure spinor approach to describe $6D$ SYM theories, [**M. Cederwall, (2018)**].

Action

- The classical action of $6D$, $\mathcal{N}=(1,0)$ SYM model $S_0[q^+, V^{++}]$ in the harmonic superspace is written as

$$S_0 = \frac{1}{\mathrm{f}^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int d^{14}z\, du_1\ldots du_n \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}$$
$$-\frac{1}{2\mathrm{f}^2}\int d\zeta^{(-4)}\, q^{+A}\nabla^{++}q^+_A.$$

- Gauge transformations

$$V^{++\prime} = -ie^{i\lambda}D^{++}e^{-i\lambda} + e^{i\lambda}V^{++}e^{-i\lambda}, \qquad q^{+\prime} = e^{i\lambda}q^+ \tag{1}$$

Addition supersymmetry

The $\mathcal{N} = (1,0)$ SYM theory is manifestly $\mathcal{N} = (1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N} = (0,1)$ supersymmetry if the hypermultiplet align in adjoint representation.

- Action

$$S[V^{++}, q^{+}] = S_{SYM}[V^{++}] + S_{HYP}[q^{+}, V^{++}]$$

- The action is manifestly $\mathcal{N} = (1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta V^{++} = \epsilon^{+} q^{+}, \qquad \delta q^{+} = -(D^{+})^{4}(\epsilon^{-} V^{--})$$

where the transformation parameter $\epsilon^{\pm}_{A} = \epsilon_{aA}\theta^{\pm A}$.

Background field method

Aim:gauge invariant effective action, (see, e.g., [**B.DeWitt (1965)**]).

- Background-quantum splitting

$$V^{++} \to V^{++} + \mathrm{f}v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The gauge-fixing function

$$\mathcal{F}^{(+4)} = \nabla^{++} v^{++}$$

- Faddev-Popov procedure
  Analogous to one in $4D, \mathcal{N} = 2$ SYM theory[**I.L.Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

- The effective action $\Gamma[V^{++}, Q^{+}]$ is written in terms of path integral

$$e^{i\Gamma[V^{++},Q^{+}]} = \mathrm{Det}^{1/2}\,\widehat{\Box}\int \mathcal{D}v^{++}\,\mathcal{D}q^{+}\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\; e^{iS_{quant}}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S + S_{GF}[v^{++}, V^{++}] + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] + S_{NK}[\varphi, V^{++}].$$

- Gauge fixing action $S_{GF}[v^{++}, V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}]$, Nelson-Kalosh ghost action $S_{NK}[\varphi, V^{++}]$
- Operator $\widehat{\Box} = \frac{1}{2}(D^{+})^{4}(\nabla^{--})^{2}$

$$\widehat{\Box} = \eta^{MN}\nabla_{M}\nabla_{N} + W^{+a}\nabla^{-}_{a} + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

- The gauge fixing action

$$S_{GF}[v^{++},V^{++}] = -\frac{1}{2\xi}\mathrm{tr}\int d^{14}zdu_1du_2\,\frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+u_2^+)^2} + \frac{1}{4\xi}\mathrm{tr}\int d^{14}zdu\,v_\tau^{++}(D^{--})^2v_\tau^{++}. \quad (2)$$

- Faddeed-Popov ghosts and Nelson-Kalosh ghost actions

$$S_{FP} = \mathrm{tr}\int d\zeta^{(-4)}du\,b\nabla^{++}(\nabla^{++}+iv^{++})c\,, \quad (3)$$

$$S_{NK} = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\,\varphi(\nabla^{++})^2\varphi\,. \quad (4)$$

In what follows we assume gauge fixing parameter $\xi = 1$. The case $\xi \neq 1$ can be considered separately [**I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., Phys.Lett.B (2019)**]

Supergraphs

- Perturbation theory can be given in terms of Feynman diagrams formulated in superspace
- Vector multiplet propagator

$$G^{(2,2)}(1|2) = -2\frac{(D_1^+)^4}{\widehat{\Box}_1}\delta^{14}(z_1 - z_2)\delta^{(-2,2)}(u_1, u_2)$$

- Hypermultiplet propagator

$$G^{(1,1)}(1|2) = \frac{(D_1^+)^4(D_2^+)^4}{\widehat{\Box}_1}\frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}$$

- Ghost propagators have the analogous structure
- Superspace delta-function

$$\delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$

- The vertices are taken from the superfield action as usual

One-loop loop divergences

The one-loop approximation is given by the quadratic action S_2:

$$
\begin{aligned}
S_2 \;=\; & \frac{1}{2}\int d\zeta^{(-4)} du\, v^{++\,A}\, \widehat{\Box}^{AB}\, v^{++\,B} + \int d\zeta^{(-4)} du\, \mathbf{b}^A (\nabla^{++})^{2\,AB} \mathbf{c}^B \\
& + \frac{1}{2}\int d\zeta^{(-4)} du\, \varphi^A (\nabla^{++})^{2\,AB} \varphi^B - \int d\zeta^{(-4)} du\, \tilde{q}^{+\,m} (\nabla^{++})_m{}^n q_n^+ \\
& - i\mathrm{f}\int d\zeta^{(-4)} du \Big\{ \widetilde{Q}^{+\,m} (v^{++})^C (T^C)_m{}^n q_n^+ + \tilde{q}^{+\,m} (v^{++})^C (T^C)_m{}^n Q_n^+ \Big\},
\end{aligned}
\tag{5}
$$

We consider the special change of hypermultiplet variables [**I.L. Buchbinder, N.G. Pletnev,** *JHEP* 0704; **S. M. Kuzenko, S. J. Tyler,** *JHEP* 0705] in the one-loop effective action

$$
q_n^+(1) \to q_n^+(1) - \mathrm{f}\int d\zeta_2^{(-4)} du_2\, G^{(1,1)}(1|2)_n{}^p\, i v^{++\,C}(2)\, (T^C)_p{}^l\, Q_l^+(2)\,, \tag{6}
$$

where $G^{(1,1)}$ is the hypermultiplet Green function.

Structure of one-loop counterterm

According to the general analysis[**G. Bossard, E. Ivanov, A. Smilga,**(2015)] the one-loop logarithmic divergences have a structure

$$\Gamma^{(1)}_{\rm div} = \int d\zeta^{(-4)}\, du \left[c_1 (F^{++A})^2 + i c_2 F^{++A} (\widetilde{q}^{+})^m (T^A)_m{}^n (q^+)_n + c_3 \Big((\widetilde{q}^{+})^m (q^+)_m \Big)^2 \right], \tag{7}$$

where c_1, c_2, and c_3 are numerical real coefficients.

Superficial degree of divergence ω

- One can prove that any supergraph for effective action can be written through the integrals over full $\mathcal{N} = (1,0)$ superspace and contains only a single integral over $d^8\theta$ (non-renormalization theorem).
- $\omega_{\rm L-loop}(G) = 2L - N_Q - \frac{1}{2} N_D$
- The possible one-loop divergences correspond to $\omega_{1-\rm loop} = 2$ and $\omega_{1-\rm loop} = 0$

Calculations of ω are analogous to ones in $4D, \mathcal{N} = 2$ gauge theory [**I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

The one-loop quantum correction $\Gamma^{(1)}[V^{++}, Q^+]$ to the classical action, which has the following formal expression

$$\begin{aligned}\Gamma^{(1)}[V^{++}, Q] &= \frac{i}{2}\mathrm{Tr}\ln\Big\{\widehat{\Box}^{AB} - 4\mathrm{f}^2\widetilde{Q}^{+\,m}\big(T^A G T^B\big)_m{}^n Q^+_n\Big\} - \frac{i}{2}\mathrm{Tr}\ln\widehat{\Box}_{\mathrm{Adj}} \\ &\quad -i\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\mathrm{Tr}\ln\nabla^{++}_{\mathrm{R}}\,. \qquad (8)\end{aligned}$$

The divergent contributions read

$$\Gamma^{(1)}_{F^2} = \frac{C_2 - T(R)}{6(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du\,(F^{++A})^2\,. \qquad (9)$$

$$\begin{aligned}\Gamma^{(1)}_{QFQ}[V^{++}, Q^+] &= -\frac{2i\mathrm{f}^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)}du \\ &\quad \widetilde{Q}^{+\,m}(C_2\delta^l_m - C(R)_m{}^l)(F^{++})^A\,(T^A)_l{}^n\,Q^+_n. \qquad (10)\end{aligned}$$

In case of non-minimal gauge

$$\begin{aligned}\Gamma^{(1)}_{\mathcal{N}=(1,1)} &= \frac{2(\xi-1)C_2}{(4\pi)^3\varepsilon}\,\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2 \\ &\quad -\frac{\mathrm{f}^2(\xi-1)C_2}{(4\pi)^3\varepsilon}\int d^{14}z\,du\,\Big(\widetilde{Q}^+Q^- - \widetilde{Q}^-Q^+\Big). \qquad (11)\end{aligned}$$

Two-loop loop divergences

In general, the off-shell two-loop divergent contributions to effective action may include following terms

$$\Gamma^{(2)}_{\rm div} = {\rm tr}\int d\zeta^{(-4)}\Big(c_1 F^{++}\,\widehat{\Box}\,F^{++} + c_2 i\,F^{++}\,\widehat{\Box}\,[Q^{+A},Q^+_A]$$
$$+c_3\,[Q^{+A},Q^+_A]\,\widehat{\Box}\,[Q^{+B},Q^+_B]\Big) + \text{terms vanishing on eq.o.m. for } Q^+ . \quad (12)$$

Inexplicit $\mathcal{N}=(0,1)$ supersymmetry restricts the structure of the divergent contribution (12) and leads to the absence of two-loop divergences on-shell. Hence we obtain non-trivial constraints on the coefficients:

$$2c_2 + 4c_3 = c_1\,.$$

In case $Q=0$, the calculation was provided in [**I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., (2021)**]

$$c_1 = \frac{{\rm f}^2 (C_2)^2}{8(2\pi)^6\varepsilon^2}\,, \qquad \varepsilon\to 0\,, \quad (13)$$

Vertices

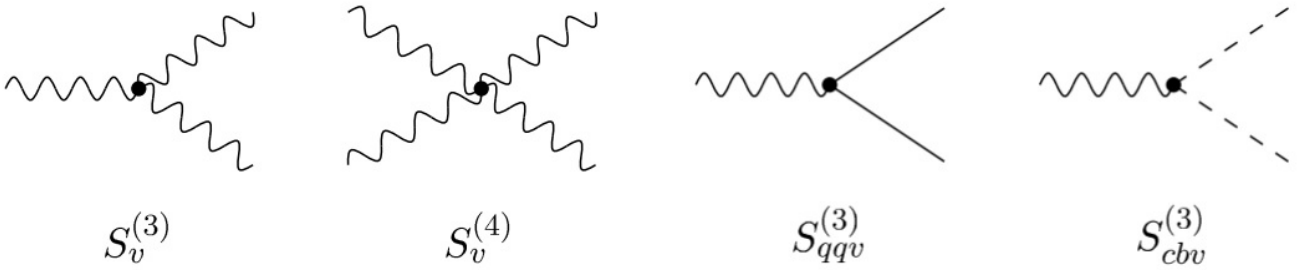


Figure: Standard vertices

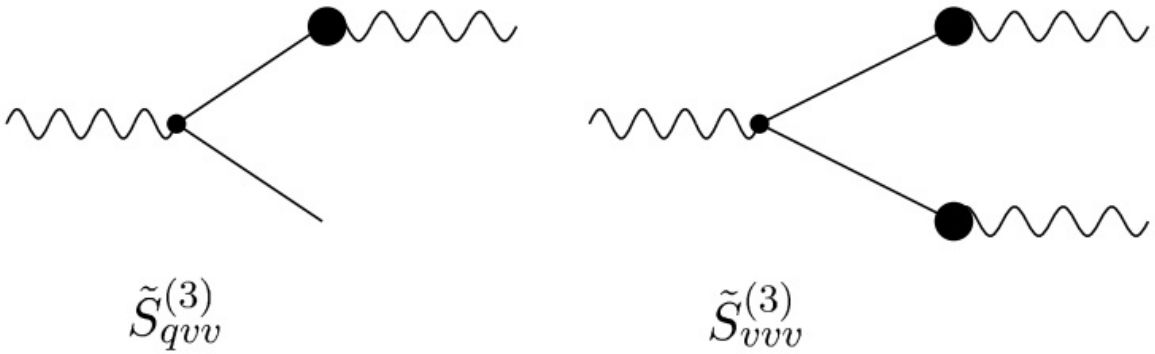


Figure: New type vertices

Two-loop diagrams

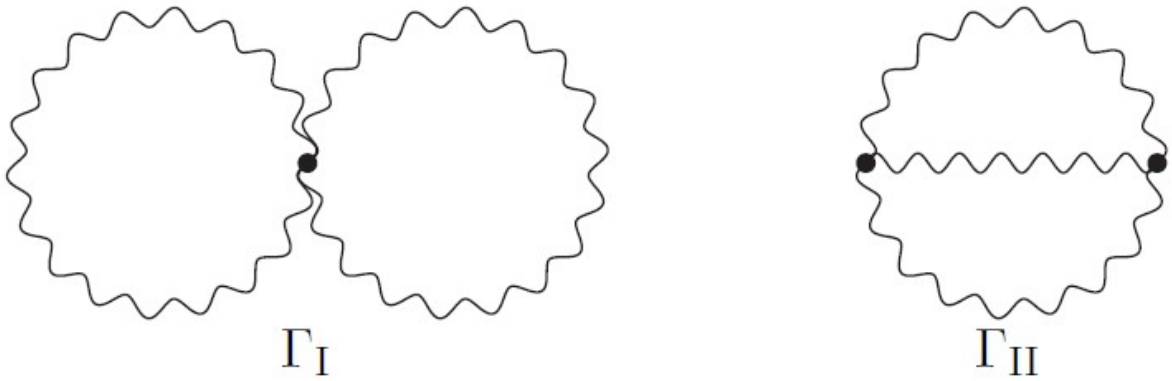


Figure: Two-loop Feynman supergraphs with the gauge self-interactions vertices.



Figure: Two-loop Feynman supergraphs with the hypermultiplet and ghosts vertices.

Two-loop diagrams

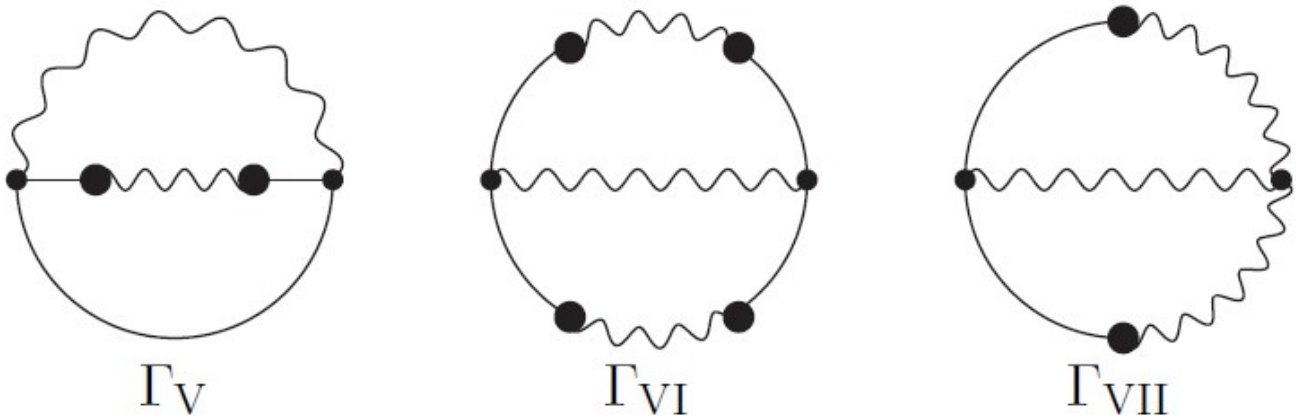


Figure: Two-loop Feynman supergraphs with the new 'non-local' vertices.

Summing up the divergent contributions which contain $F^{++}Q^+Q^+$ from $\Gamma_{\rm I}$, $\Gamma_{\rm V}$ and $\Gamma_{\rm VII}$ we obtain

$$\Gamma^{\rm FQQ}_{\rm I,\,div}+\Gamma^{\rm FQQ}_{\rm V,\,div}+\Gamma^{\rm FQQ}_{\rm VII,\,div} \tag{14}$$

$$\begin{aligned}
&=-\frac{i\mathrm{f}^2(C_2)^2}{8(2\pi)^6\,\varepsilon^2}\mathrm{tr}\int d^{14}z\,\frac{du_1du_2du_3}{(u_1^+u_2^+)(u_3^+u_1^+)}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{3A,\tau}]\\
&+\frac{2i\mathrm{f}^2(C_2)^2}{(4\pi)^6\varepsilon^2}\mathrm{tr}\int d^{14}z\frac{du_1du_2}{(u_1^+u_2^+)^2}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{2A,\tau}]\\
&-\frac{4i\mathrm{f}^2(C_2)^2}{(4\pi)^6\varepsilon^2}\mathrm{tr}\int d^{14}z\frac{du_1du_2du_3(u_1^-u_2^+)}{(u_1^+u_2^+)(u_2^+u_3^+)}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{3A,\tau}]
\end{aligned} \tag{15}$$

Here we also use the property

$$\begin{aligned}
G^{(2,2)}(z_1,u_1|z_2,u_2)\Big|_{z_2\to z_1} \;=\; &\frac{2i}{(4\pi)^3}\,\frac{1}{\varepsilon}\Big(F^{++}_{1,\tau}\,(u_1^-u_2^+)(u_1^+u_2^+)^3(D_1^{--})^2\delta^{(2,-2)}(u_1,u_2)\\
&+2(u_1^+u_2^+)\,Q^{+A}_{1,\tau}Q^+_{2A,\tau}\Big)+\text{finite terms}\,,\quad \varepsilon\to 0\,.
\end{aligned} \tag{16}$$

Final expression for the divergent part of the two-loop effective action in the $\mathcal{N}=(1,1)$ SYM theory reads

$$\Gamma^{(2)}_{\rm div} = \frac{\mathrm{f}^2(C_2)^2}{8(2\pi)^6\varepsilon^2}\mathrm{tr}\int d\zeta^{(-4)}\Big(F^{++}\,\widehat{\Box}\,F^{++} - \frac{i}{2}\,F^{++}\,\widehat{\Box}\,[Q^{+A},Q^+_A] + \frac{1}{2}\,[Q^{+A},Q^+_A]\,\widehat{\Box}\,[Q^{+B},Q^+_B]\Big) + \text{terms proportional to eq.o.m. for } Q^+.$$

The last two coefficient have the form

$$c_2 = -\frac{\mathrm{f}^2(C_2)^2}{16(2\pi)^6\varepsilon^2}\,, \qquad c_3 = \frac{\mathrm{f}^2(C_2)^2}{16(2\pi)^6\varepsilon^2}\,, \qquad \varepsilon\to 0\,. \tag{17}$$

Summary on one-and two-loop divergences in gauge superfield sector

For one-and two-loop divergences we have

$$\Gamma^{(1)}_{\rm div} \sim \frac{1}{\varepsilon}{\rm tr}\int d\zeta^{(-4)} F^{++}F^{++} + \text{terms with}\, q^+$$
$$\Gamma^{(2)}_{\rm div} \sim \frac{1}{\varepsilon^2}{\rm tr}\int d\zeta^{(-4)} F^{++}\,\widehat{\Box}\, F^{++} + \text{terms with}\, q^+$$
$$+\frac{1}{\varepsilon}{\rm tr}\int d\zeta^{(-4)} F^{++}g^{++}\,, \qquad (18)$$

where $g^{++} = G^{(2,2)}(z,u|z,u)$. All such contributions vanish **on-shell**!

Three-loop analysis

As we mentioned above, the UV divergences in $\mathcal{N} = (1,1)$ starts from three loops [**L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, (2015).**].
We will consider the simplest background superfields

$$F^{++} = 0 \qquad q^{+} = 0\,. \tag{19}$$

The only possible non-vanishing on-shell divergent contribution is

$$\Gamma^{(3)}_{\rm div} \sim \frac{1}{\varepsilon}{\rm tr}\int d^6x(F^4 + \dots), \qquad \varepsilon \to +0\,. \tag{20}$$

Here we have used the notation

$$F^4 \equiv \frac{1}{4}(F_{MN}F^{MN})^2 + \frac{1}{8}F_{MN}F_{PQ}F^{MN}F^{PQ} - \frac{1}{2}F^{NM}F_{MR}F^{RS}F_{SN} - F^{NM}F_{MR}F^{RS}F_{SN}$$

with F_{MN} being the gauge field strength.
The superfield form of the contribution (20) ia also known [**G. Bossard, E. Ivanov, A. Smilga,**(2015)] and has the form

$$\Gamma^{(3)}_{\rm div} \sim \frac{1}{\varepsilon}{\rm tr}\int d\zeta^{(-4)}(W^{+})^4 \qquad \varepsilon \to +0\,. \tag{22}$$

The superfield $W^{+a} = -\frac{i}{6}\varepsilon^{abcd}D^{+}_{b}D^{+}_{c}D^{+}_{d}V^{--}$ is analytic under the condition $F^{++} = 0$.

Three-loop diagrams

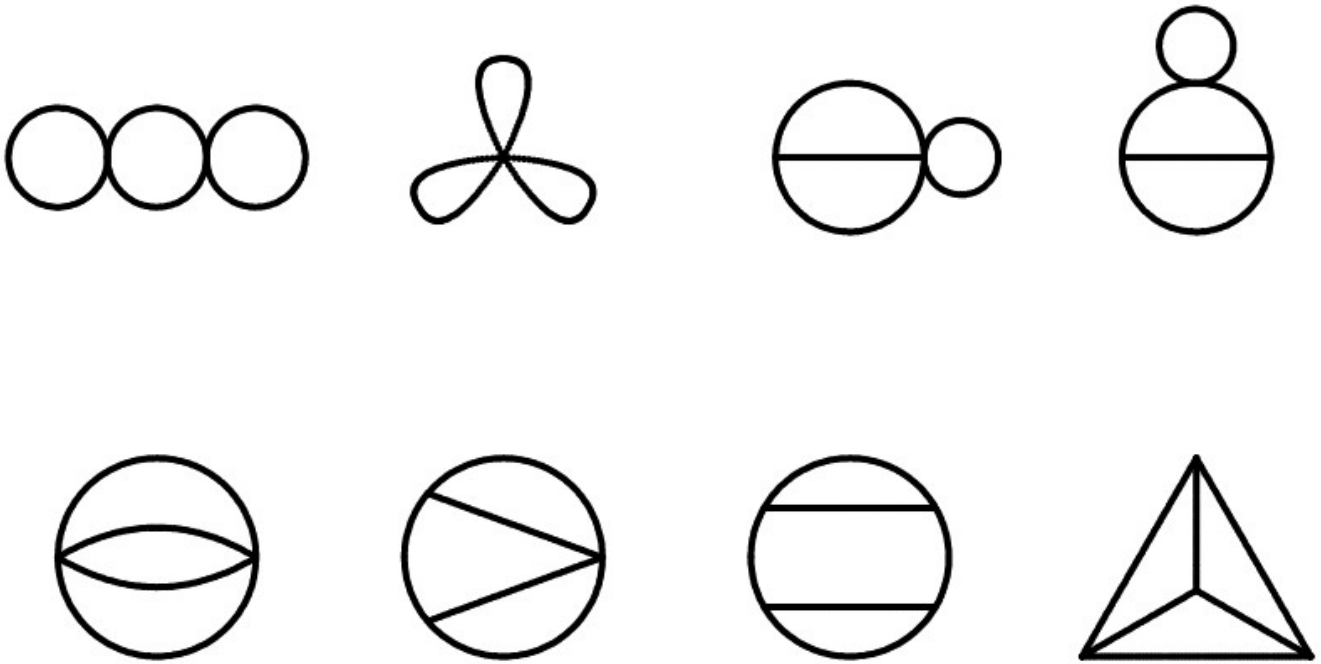


Figure: Topology of three-loop diagrams

Three-loop divergences

$$\Gamma^{(3)}_{\rm div} \sim \frac{1}{\varepsilon}{\rm tr}\int d^{14}z\,du\,(W^+)^4 I^{(-4)}(u)\,, \tag{23}$$

where we have introduced

$$I^{(-4)}(u) = \prod_{i=1}^{4}\frac{(u_2^+u_3^+)^2}{(u^+u_1^+)(u^+u_2^+)(u^+u_3^+)^2} \tag{24}$$

Summary

- One-loop effective action in $\mathcal{N} = (1,0)$ harmonic superspace was developed.
- Superficial degree of divergence was evaluated and structure of one and two-loop counterterms were studied.
- The one-loop divergences in the $6D, \mathcal{N} = (1,0)$ SYM theory were calculated *off-shell.*
- Two-loop divergences in the $6D, \mathcal{N} = (1,1)$ SYM theory are calculated.
- Three-loop analysis is provided.

Thank you for your attention!