On a family of Poisson brackets on $\mathfrak{gl}_n$ compatible with the Sklyanin bracket

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- We study compatible linear bracket determined by the argument shift for the family of quadratic brackets and explore the bi-Hamiltonian formalism for some pencils from this family. We provide some interesting examples of families of this type. Some of them are related with the algebraic question in invariant theory.
- We get explicit description for the centers of the corresponding quadratic brackets using the anti-diagonal principal minors of the Lax matrix.

1 Introduction

2 The family of quadratic brackets

3 4-parameter specialization

4 Centers of the quadratic and linear brackets and applications

General family of quadratic brackets

We consider the vector space $V = \mathbb{C}^n$, E_{ij} are matrix units in $End(V)$, e_{ij} are functions on the phase space (Lie group $GL(n)$ or its Lie algebra, depending on the context), the Lax operator is given by the following generating matrix

$$L = \sum_{ij} E_{ij} \otimes e_{ij}, \tag{1}$$

we consider also the tensor copies of the Lax operator:

$$L_1 = L \otimes 1; \qquad L_2 = 1 \otimes L.$$

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We focus on Poisson structures of the form:

$$\{L \otimes L\} = AL_1L_2 + L_1BL_2 + L_2CL_1 + L_1L_2D,$$

where $A, B, C, D \in End(V) \otimes End(V)$.

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We work with the famous R-matrix due to Babelon-Cherednik-Drinfeld-Jimbo. This is the expression:

$$R = \sum_{i<j} (q - q^{-1}) E_{ii} \otimes E_{jj} + \sum_{i,j} q^{\delta(i,j)} E_{ij} \otimes E_{ji} \in Mat_n \otimes Mat_n.$$

It provides a solution of the quantum Yang-Baxter equation

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An RTT algebra is defined as a quotient algebra of a free algebra $\mathbb{C}\langle t_{ij} \rangle$ where $i, j \in (1, \ldots, n)$, by the quadratic relations

$$R T_1 T_2 = T_1 T_2 R.$$

Here T is a generating matrix for RTT-algebra elements

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RE-algebras were first introduced by Cherednik (1984) when studying scattering processes with a boundary. Such an algebra is defined by quadratic relations of the form:

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where L is a generating matrix of a_{ij}:

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Classical limit

The R-matrix can be expanded in the parameter $\hbar = 2\log(q)$ at zero:

$$R = P + \hbar\rho + O(\hbar^2),$$

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Let us introduce an involution operation on the elements $a \in End(V) \otimes End(V)$ by

$$a^* = PaP,$$

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Then

$$r^* = PrP = P\rho; \qquad (\rho + \rho^*) = E, \qquad (r + r^*) = P.$$

A similar limit for the RE-algebra gives a generalized quadratic bracket

$$\{L \otimes L\} = rL_1L_2 + L_2r^*L_1 - L_1rL_2 - L_1L_2r^*.$$

Argument shift method and linear brackets

Together with a quadratic brackets one could consider a linear one, which are obtained by the argument shift. This procedure consists in changing the coordinates $L \mapsto L + \lambda E$ in the expression

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Remark

For example, the linear bracket generated by the argument shift of the bracket given by the classical limit of the RE algebra coincides with the standard $\mathfrak{gl}_n$ Lie-Poisson-Kirillov-Kostant bracket.

Involutive families

Let us recall the general construction of involutive Poisson families by the argument shift method (Magri). Let $c(L)$ be an element of the center of a quadratic Poisson bracket $\{\cdot,\cdot\}$.

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Lemma

Suppose that $c(L+\lambda E)$ is a polynomial in λ:

$$c(L+\lambda E) = \sum_{i=0}^{m} c_i(L)\lambda^i.$$

Then, the coefficients c_i of this family commute with each other with respect to both quadratic and linear brackets. In this case, the function c_m belongs to the center of the linear bracket $\{\cdot,\cdot\}_{lin}$

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Remark

There are several refinements to the Lemma: actually the highest non-constant coefficient of the expansion is central, and moreover if there is a family of central elements of the generic bracket polynomial in λ, then the entire set of their coefficients generates a commutative Poisson subalgebra with respect to all brackets of the pencil.

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Poisson-Lie structure on $SL(n)$

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In case $n = 2$ with the notations

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

it produces the Poisson structure

$$\begin{aligned} &\{a, b\} = ab, \quad \{a, c\} = ac, \quad \{a, d\} = 2bc, \\ &\{b, d\} = bd, \quad \{c, d\} = cd, \quad \{b, c\} = 0. \end{aligned}$$

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The compatible linear structure in this case is

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Description of the general family

Consider the following specialization of quadratic brackets:

$$\{L \otimes L\} = (A+S)L_1L_2 + L_1(B-S)L_2 + L_2(-B-S)L_1 + L_1L_2(-A+S),$$

where coefficients have the form

$$\begin{aligned} A &= -w\sum_{i>j}(E_{ij}\otimes E_{ji} - E_{ji}\otimes E_{ij}) + \sum_{i>j} a_{ij}(E_{ii}\otimes E_{jj} - E_{jj}\otimes E_{ii}); \\ B &= \sum_{i>j} b_{ij}(E_{ii}\otimes E_{jj} + E_{jj}\otimes E_{ii}) + \sum_{i} b_{ii}E_{ii}\otimes E_{ii}; \\ S &= \sum_{i>j} s_{ij}(E_{ii}\otimes E_{jj} - E_{jj}\otimes E_{ii}). \end{aligned}$$

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Theorem

These brackets for any parameter values are Poisson brackets.

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The linear bracket has the following properties:

- *being restricted to $\mathfrak{n}_+$ and $\mathfrak{n}_-$ it coincides with the standard Lie-Poisson-Kirillov-Kostant bracket $\{X, Y\}_{\mathfrak{gl}}$, multiplied by $2w$ and $-2w$, respectively;*

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$$h_{k,i} = a_{k,i} + b_{k,i} - a_{k,i+1} - b_{k,i+1}.$$

The linear bracket has a natural interpretation in terms of Lie algebras extension:

Theorem

The linear bracket has the following properties:

- *being restricted to* $\mathfrak{n}_+$ *and* $\mathfrak{n}_-$ *it coincides with the standard Lie-Poisson-Kirillov-Kostant bracket* $\{X, Y\}_{\mathfrak{gl}}$*, multiplied by* $2w$ *and* $-2w$*, respectively;*
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- *the action of the Cartan subspace* $\mathfrak{h}$ *on* $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ *is given by the formulas*

$$\begin{aligned} \{u_{k,k}, u_{i,i+1}\}_{lin} &= (w(\delta_{k,i} - \delta_{k,i+1}) + h_{k,i})u_{i,i+1}, \\ \{u_{k,k}, u_{i+1,i}\}_{lin} &= (w(\delta_{k,i} - \delta_{k,i+1}) - h_{k,i})u_{i+1,i}, \end{aligned}$$

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The family of corresponding linear r-matrix brackets specializes as follows:

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Remark

This Lie algebra is solvable. This structure does not provide a Lie-bialgebra structure on $\mathfrak{gl}$*.*

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The formulas (3) argue the existence of central elements of the form

$$\prod_{k=1}^{n} \left(\Delta_k^+\right)^{p_k} \left(\Delta_k^-\right)^{q_k}, \qquad p_k, q_k \in \mathbb{C}$$

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The polynomials

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1 Introduction

2 The family of quadratic brackets

3 **4-parameter specialization**

4 Centers of the quadratic and linear brackets and applications

4-dimensional family

We consider a specialization of our family:

$$\{e_{ij}, e_{kl}\} = \frac{z_1}{2}\Big(\operatorname{sign}(k-i)+\operatorname{sign}(l-j)\Big)e_{il}e_{kj} + \frac{z_2}{2}\Big(\delta_{il}-\delta_{jk}\Big)e_{ij}e_{kl} + \tag{4}$$

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$$A = -D = \sum_{i>j} E_{ii}\otimes E_{jj} - E_{jj}\otimes E_{ii}, \qquad B = -C = \sum_{i,j} E_{ii}\otimes E_{jj}.$$

Linear bracket

For any z_1, z_2, z_4 we have $\{\mathfrak{n}_+, \mathfrak{n}_-\}_{lin} = 0$. The dependence of the bracket on the parameters z_2, z_4 is described by the formulas for the action of a commutative subalgebra e_{ii}, $i = 1, \ldots, n$ on $\mathfrak{n}_+$ and $\mathfrak{n}_-$.

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For some parameter sets, such linear brackets are well known. For example, in the case of $z_1 = z_2 = 1$, $z_4 = 0$ the bracket is related to the decomposition

$$\mathfrak{gl}_n = \mathfrak{n}_+ \oplus \mathfrak{b}_-$$

and is defined by the formula

$$\{X, Y\} = \frac{1}{2}\Big(\{R(X), Y\}_{gl} + \{X, R(Y)\}_{gl}\Big),$$

where R is the difference of the projectors corresponding to this decomposition.

1 Introduction

2 The family of quadratic brackets

3 4-parameter specialization

4 **Centers of the quadratic and linear brackets and applications**

Centers of the quadratic brackets

The centers of rings of rational functions on $\mathfrak{gl}_n^*$ with respect to the Poisson structures from our family with integer values of z_1, z_2, z_3 and $z_4 = 0$, can be explicitly found using the multiplicative ansatz. Here are some examples.

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If $z_1 = z_2 = z_3 = 1$, $z_4 = 0$ *then* $\dfrac{\Delta_k^+}{\Delta_{n-k+1}^+}$, Δ_1^+, Δ_n^+ *and* $\dfrac{\Delta_k^-}{\Delta_{n-k}^-}$ *belong to the center. In this case one could obtain a bi-Hamiltonian structure for the Kostant-Toda system by some Poisson reduction procedure.*

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If $k > n/2$ the elements of the center of the quadratic bracket described in Theorem above after shifting the argument, become polynomials in λ of the form

$$\frac{\Delta_k^-}{\Delta_{n-k}^-}(L + \lambda E) = \lambda^{2k-n} + \lambda^{2k-n-1}\Delta_k^* + \ldots, \qquad \Delta_k^+\Delta_{n-k}^+(L + \lambda E) = \lambda^{2k-n}(\Delta_{n-k}^+)^2 + \ldots.$$

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All coefficients of these decompositions with different k form a commutative Poisson subalgebra for linear and quadratic brackets. Note that for $k = n$, the first decomposition yields the coefficients of the characteristic polynomial of the matrix L.

Center of the linear bracket in the Sklyanin case

We introduce the following notation for the expansion coefficients for $k^n/2$:

$$\frac{\Delta^+_{n-k}}{\Delta^-_k}(L+\lambda E) = \lambda^{n-2k}\frac{\Delta^+_k}{\Delta^-_k} + \lambda^{n-2k-1}\delta^+_k + \dots$$

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Proof.

The statement for expressions Δ^+_k/Δ^-_k follows from Lemma 1. The fact that $\delta^-_k\Delta^+_k/\Delta^-_k + \delta^+_k\Delta^-_k/\Delta^+_k$ belongs to the center is a consequence of the Remark above applying to the decomposition of

$$\frac{\Delta^+_{n-k}}{\Delta^-_k}(L+\lambda E)\frac{\Delta^-_{n-k}}{\Delta^+_k}(L+\lambda E).$$

□

Gelfand-Tsetlin subalgebra

The result of Lemma 2 can also be obtained by methods that are used to construct an involutive family of integrals of the full Toda system (Deift,... 1986). Let us first consider a statement about the analogue of the commutative Gelfand-Tsetlin subalgebra for anti-diagonal minors.

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Let's also consider the upper-right $k \times k$ corner of the operator $L - \lambda E$ and its determinant

$$\Delta_k^+(\lambda) = \Delta_k^+(L - \lambda E) = det((L - \lambda E)^{(k)}).$$

Theorem

Coefficients $\Delta_k^+(\lambda)$ lie in $Z(S(\mathfrak{gl}_n^{(k)}))$ and therefore collectively form a commutative family with respect to the standard Lie-Poisson-Kirillov-Kostant bracket.

Gelfand-Tsetlin subalgebra

The result of Lemma 2 can also be obtained by methods that are used to construct an involutive family of integrals of the full Toda system (Deift,... 1986). Let us first consider a statement about the analogue of the commutative Gelfand-Tsetlin subalgebra for anti-diagonal minors.
Consider a family of nested Lie subalgebras

$$\mathfrak{gl}_n^{(1)} \subset \mathfrak{gl}_n^{(2)} \subset \ldots \subset \mathfrak{gl}_n^{(n-1)} \subset \mathfrak{gl}_n^{(n)} = \mathfrak{gl}_n,$$

in which $\mathfrak{gl}_n^{(k)}$ is a subalgebra formed by the upper-right corner of the Lax operator, that is, by generators e_{ij} with

$$k \geq i, \quad j \geq n - k + 1.$$

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Remark

There is a stronger statement that these functions are invariant with respect to the upper triangular nilpotent subalgebra or the corresponding Lie subgroup.

The Theorem and the Remark above has two curious consequences:

Lemma

Δ_k^+ for $k < n/2$ belong to the center of $S(\mathfrak{n}_+)$.

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Proof.

First of all, we note that for such values of the integer k, the function $\Delta_k^+(\lambda)$ does not depend on λ. The statement itself is a consequence of the fact that Δ_k^+ lie in the centralizer $S(\mathfrak{n}_+)$ and, moreover, for $k < n/2$ they themselves are elements of this algebra. □

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We can find the center of the field of fractions $\mathcal{F}S(\mathfrak{b}_+)$

Lemma

Let us denote by Δ_k^0 and Δ_k^1 the coefficients in the expansion of $\Delta_k^+(\lambda)$ for $k > n/2$ on the parameter λ:

$$\Delta_k^+(\lambda) = \lambda^{2k-n}\Delta_k^0 + \lambda^{2k-n-1}\Delta_k^1 + \dots$$

Then

$$\Delta_k^1/\Delta_k^0 \in Z(\mathcal{F}S(\mathfrak{b}_+)).$$

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Remark

The expression Δ_k^1/Δ_k^0 exactly coincides with the function Δ_k^*, defined previously.

Concluding remarks

- The characteristic property of our family of compatible quadratic Poisson brackets is the log-canonicity of brackets between anti-diagonal minors and generators of the Poisson algebra. It is very important but cannot yet be considered well understood.

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- The characteristic property of our family of compatible quadratic Poisson brackets is the log-canonicity of brackets between anti-diagonal minors and generators of the Poisson algebra. It is very important but cannot yet be considered well understood.
- This family is a generalization for the Poisson-Lie groups, we hope to find its geometric interpretations in terms of some factorization problems, Manin triples and pairs etc.
- In the framework of possible future research, we are particularly interested in the quantization of these results, concerning the quantization of a compatible families of quadratic brackets, their linear incarnations, the centers of both brackets, and commutative families of functions.

Thank you!