Black hole entanglement entropy

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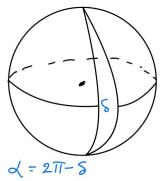
1. Result
2. Replica trick
3. Entanglement entropy
4. Entanglement entropy from local thermodynamics
5. BTZ

Result

Standard definition of entanglement entropy (Replica Trick):

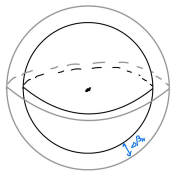
$$S_A^{\text{off shell}} = \overset{R=\bar{R}+\delta(\Sigma)}{\text{}} = (\beta\partial_\beta - 1)\log Z\Big|_{\beta=\beta_H} = S_q - \xi Q \sim A,$$

$$S_q = -Tr\hat{\rho}\log\hat{\rho} \quad Q \sim \int d\Sigma\langle\phi^2\rangle$$

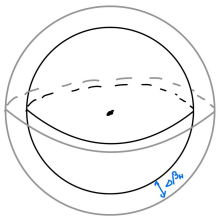
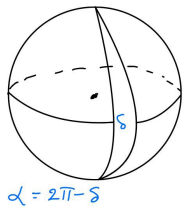
Problems:

- Statistical interpretation of the "contact" term Q
- Puzzle of renormalization quantum corrections due to the self-interaction
- $\log Z \sim ... + (\beta - \beta_H)^2 \int dV\delta(\Sigma)^2$

We develop another method (local thermodynamics $\langle T_\nu^\mu \rangle = \{-\rho, p_r, ...\}$) :

$$S_V^{\text{on shell}} = \text{} = \int^{T_H} \frac{dT_H}{T_H} \int d^{d-1}x\sqrt{|g|}\left(\frac{\partial\rho}{\partial T_H} + (p+\rho)\frac{1}{\upsilon}\frac{\partial\upsilon}{\partial T_H}\right)$$

where $\upsilon = r^{d-2}\sqrt{|\det(g_{ij})|}$ - measure of the volume element on a constant time slice

$$\text{} = S_V^{\text{on shell}} = S_A^{\text{off shell}} = \text{}$$

Motivation

The main motivation for studying entanglement entropy has always been, and still is, the attempt to interpret the Bekenstein-Hawking entropy as an entropy of entanglement.

My little motivation: I don't like the replica trick

Bekenstein-Hawking entropy

First law:

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \Omega \delta J + \Phi \delta Q \tag{1}$$

Temperature:

$$T_H = \frac{\kappa}{2\pi} \tag{2}$$

Entropy:

$$S_{BH} = \frac{A}{4} \tag{3}$$

Generalized second law:

$$\delta S_{BH} + \delta S_{ent} \geq 0 \tag{4}$$

which states that the sum of the Bekenstein-Hawking entropy of a black hole and the entropy of surrounding matter does not decrease in physical processes.

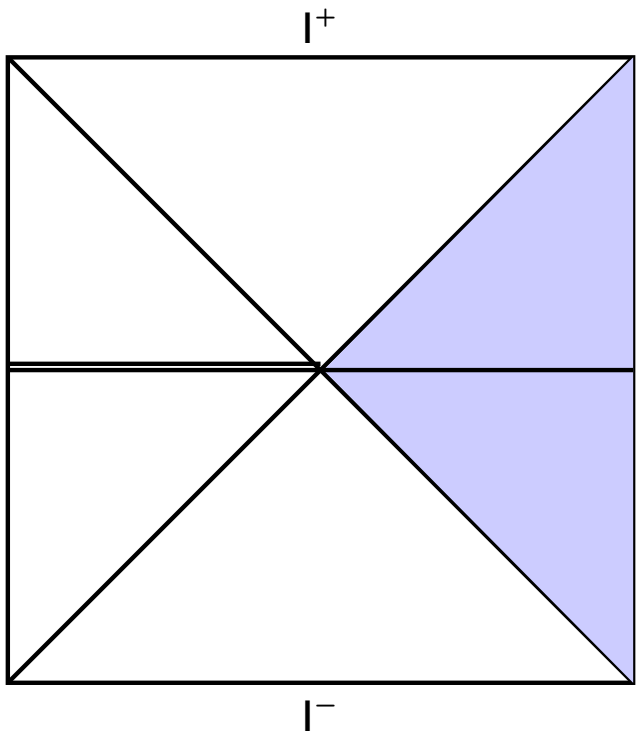
$$S_{ent} = -Tr\hat{\rho}\log\hat{\rho} \tag{5}$$

where:

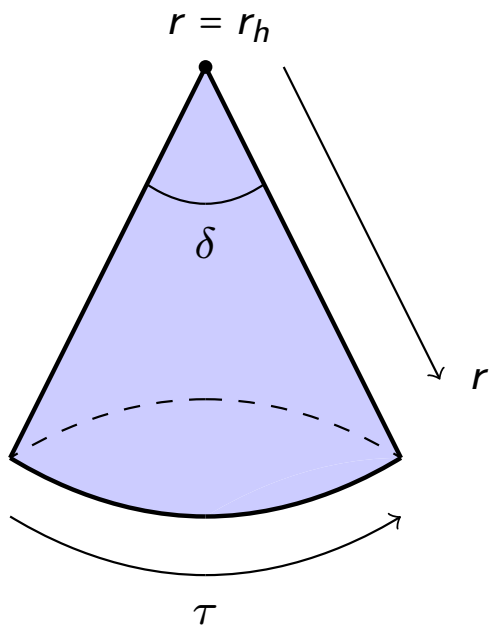
$$\hat{\rho} = \int D\phi_L \langle\phi_L|0\rangle\langle 0|\phi_L\rangle = e^{-\beta_H \hat{H}} \tag{6}$$

As an example:

$$|0\rangle = \prod_{\omega}\sum_{n_\omega} e^{-\pi/H\omega n_\omega}|n_\omega, L\rangle \otimes |n_\omega, R\rangle \quad \rightarrow \quad \hat{\rho}_R = \mathrm{Tr}_L|BD\rangle\langle BD| = e^{-\beta_H H_R}. \tag{7}$$



$$Z_\alpha = Tr\hat{\rho}^\alpha = \int_{\phi(0)=\phi(\tau=2\pi\alpha)} D\phi e^{-W} \tag{8}$$



The entanglement entropy is defined as follows:

$$S_{ent} = -(\alpha\partial_\alpha - 1)\log Z_\alpha\Big|_{\alpha=1} = -Tr\hat{\rho}\log\hat{\rho} \tag{9}$$

BH entropy

The Riemannian tensor contains delta-like surface contributions near the horizon:

$$\lim_{\alpha\to 1} R^{M_\alpha}_{\mu\nu\rho\sigma} = R^{M}_{\mu\nu\rho\sigma} + 2\pi(1-\alpha)\left(n^1_\mu n^2_\rho - n^2_\mu n^1_\rho\right)\left(n^1_\nu n^2_\sigma - n^2_\nu n^1_\sigma\right)\delta(\Sigma) + O\left((1-\alpha)^2\right), \tag{10}$$

where $n^{1,2}$ are two orthonormal vectors orthogonal to the horizon surface Σ. The classical action in the limit $\alpha \to 1$ is given by:

$$\begin{aligned} W[\alpha] = \\ = -2\pi\alpha\frac{1}{16\pi}\int d^{d-1}x\sqrt{g}\left(R^M - 2\Lambda\right) - \\ -(1-\alpha)\frac{1}{4}\int d^{d-1}x\sqrt{g}\delta(\Sigma) + O\left((1-\alpha)^2\right). \end{aligned} \tag{11}$$

Applying the replica formula, we find that only the surface term contributes to the entropy:

$$S_A^{\text{off shell}} = -(\alpha\partial_\alpha - 1)W[\alpha]\Big|_{\alpha=1} = \frac{A}{4}. \tag{12}$$

Entanglement entropy

The Euclidean path integral for the non-minimally coupled scalar field is:

$$Z^{div}_{matter} = \int D\phi \; e^{-\frac{1}{2}\int_{M_\alpha} d^4x\sqrt{g}\;\phi\left(-\Box+m^2+\xi R^{M_\alpha}\right)\phi}. \tag{13}$$

The Ricci scalar contains a delta-like singularity on the horizon surface:

$$R^{M_\alpha} = R^M + 4\pi(1-\alpha)\delta(\Sigma) + O\big((1-\alpha)^2\big). \tag{14}$$

We can expand the terms containing the delta function up to the first order in $(1-\alpha)$:

$$\log Z^{div}_{matter} = \log \bar{Z}^{div}_{matter} - 2\pi\xi(1-\alpha)\int d\Sigma \langle\phi^2\rangle^{div}_{\bar{Z}}, \tag{15}$$

$$S_A^{\text{off shell}} = \bar{S} - \xi\beta_H\langle\hat{Q}\rangle. \tag{16}$$

To calculate $\bar{Z}^{div}_{matter}$ and $\langle\phi^2\rangle^{div}_{\bar{Z}}$ we can use the heat kernel expansions:

$$\log \bar{Z}^{div}_{matter} = \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}e^{-sm^2}\int_{M_\alpha} d^4x\sqrt{g}\ \bar{K}_{M_\alpha}(s,x,x), \tag{17}$$

and

$$\langle\phi^2\rangle^{div}_{\bar{Z}} = \int_{\epsilon^2}^{\infty} ds e^{-sm^2}\bar{K}_{M_\alpha}(s,x,x). \tag{18}$$

The heat kernel on a manifold with a conical singularity is given by:

$$\bar{K}_{M_\alpha}(s,x,x) = \frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_n \bar{a}_n(x)s^n, \tag{19}$$

with the coefficients expressed as:

$$\bar{a}_n(x) = a_n^{st}(x) + (1-\alpha)\,a_n^{\alpha}(x)\delta(\Sigma) + O\left((1-\alpha)^2\right), \tag{20}$$

If we know the heat kernel for $M_{\alpha=1}$:

$$K_{M_\alpha}(s,x,y,\tau) = K_{M_1}(s,x,y,\tau) + \frac{1}{4\pi\alpha}\int_\Gamma dw \cot\left(\frac{w}{2\alpha}\right)K_{M_1}(s,x,y,\tau+\omega) \tag{21}$$

Thus, it can be seen that up to the zeroth and first orders in $(1-\alpha)$ all divergences are completely removed by the standard renormalization of the gravitational coupling constants:

$$W_{ren} = W_{gr} + W_{matter} = \tag{22}$$

$$= \alpha \int_M d^4x\sqrt{g}\left(-\frac{1}{16\pi G_{ren}}\left(R^M - 2\Lambda_{ren}\right) + c_1^{ren} R^M R^M + c_2^{ren} R^M_{\mu\nu} R^{M,\mu\nu} + c_3^{ren} R^M_{\mu\nu\rho\sigma} R^{M,\mu\nu\rho\sigma}\right) +$$

$$+4\pi(1-\alpha)\int d\Sigma\left(-\frac{1}{16\pi G_{ren}} + 2c_1^{ren} R^M + c_2^{ren} R^M_{\mu\nu} n_i^\mu n_i^\nu + 2c_3^{ren} R^M_{\mu\nu\rho\sigma} n_i^\mu n_i^\rho n_j^\nu n_j^\sigma\right) +$$

$$+O\left((1-\alpha)^2\right) + \text{finite terms.}$$

The renormalized constants are given by:

$$\frac{1}{G_{ren}} = \frac{1}{G_B} + \frac{(6\xi-1)}{12\pi\epsilon^2} + \frac{m^2(6\xi-1)}{6\pi}\log(\epsilon m), \tag{23}$$

$$c_1^{ren} = c_1^B - \frac{(1-6\xi)^2\log(em)}{1152\pi^2},$$

$$c_2^{ren} = c_2^B + \frac{1}{2880\pi^2}\log(\epsilon m),$$

$$c_3^{ren} = c_3^B - \frac{1}{2880\pi^2}\log(\epsilon m),$$

$$\frac{\Lambda_{ren}}{G_{ren}} = \frac{\Lambda_B}{G_B} - \frac{1}{8e^4\pi} + \frac{m^2}{4e^2\pi} + \frac{m^4\log(em)}{4\pi}.$$

Thus, it is sufficient to utilize the standard renormalization procedure up to the second order in $(1-\alpha)^2$.

Contact terms

$$\log Z_{matter} = \log \bar{Z}_{matter} - 2\pi\xi(1-\alpha)\int d\Sigma \langle \phi^2 \rangle_{\bar{Z}}, \tag{24}$$

using that $\alpha = \beta/\beta_H$ we can show that:

$$-\partial_\beta \log Z_{matter} = \langle E \rangle = \langle H \rangle - \xi \langle Q \rangle \tag{25}$$

as a result:

$$S_A^{\text{off shell}} = \bar{S} - \xi\beta_H \langle \hat{Q} \rangle. \tag{26}$$

Here, $\bar{S}$ is the standard definition of the entropy:

$$\bar{S} = \beta_H \langle \hat{H} \rangle - \beta_H F = Tr\,\hat{\rho} \log \hat{\rho} \tag{27}$$

Edge modes

In the 90s, Kabat showed that for gauge theory:

$$S_{photon} = 2\pi A \int_{\epsilon^2}^{\infty} ds \frac{1}{(4\pi s)^{d/2}} \left(\frac{1}{6}(d-2) - 1 \right) \tag{28}$$

The same terms arise for any integer spin
Dowker showed that in de Sitter space the logarithmic term for the entropy for spin $0, 1/2, 1$ is given by:

$$S \sim (\text{Conf. Anom.}) \log \epsilon \tag{29}$$

But for spin 1:

$$S \sim (\text{not Conf. Anom.}) \log \epsilon \tag{30}$$

Local thermodynamics

Let's use the first and second laws of thermodynamics for local quantities

$$T_{\text{loc}}\delta s = \delta(\rho\upsilon) + p\,\delta\upsilon, \tag{31}$$

where s - entropy density, $T_{\text{loc}} = T/\sqrt{|g_{00}|}$ - Tolman temperature, $\upsilon = r^{d-2}\sqrt{|\det(g_{ij})|}$ - measure of the volume element on a constant time slice, $\sqrt{|g|} = \upsilon\sqrt{|g_{00}|}$. Then

$$\delta s = \frac{p+\rho}{T}\sqrt{|g|}\frac{1}{\upsilon}\frac{\partial\upsilon}{\partial T}\delta T + \sqrt{|g|}\frac{\partial\rho}{\partial T}\frac{\delta T}{T} \tag{32}$$

The variation of total entropy:

$$\delta S = \int d^{d-1}x\,\delta s \tag{33}$$

As a result:

$$S_V^{\text{on shell}} = \int^{T_H}\frac{dT_H}{T_H}\int d^{d-1}x\sqrt{|g|}\left(\frac{\partial\rho}{\partial T_H} + (p+\rho)\frac{1}{\upsilon}\frac{\partial\upsilon}{\partial T_H}\right) \tag{34}$$

dS example

In de Sitter spacetime:

$$\langle T_{\mu\nu} \rangle \sim H^4 g_{\mu\nu} \quad \text{and} \quad T = \frac{H}{2\pi} \tag{35}$$

Energy density:

$$\rho = -\frac{\langle T^{\mu}_{\mu} \rangle}{4} \tag{36}$$

$$S_V = -\int \frac{dT_H}{T_H} \int_{r=\frac{1}{2\pi T_H}} d^{d-1}x\sqrt{g}\, \frac{\partial \rho_{\text{vac}}(T_H)}{\partial T_H} = \tag{37}$$

$$= \int d^4x\sqrt{g}\langle T^{\mu}_{\mu} \rangle \log(H/\epsilon)$$

BTZ

The BTZ black hole metric is given by:

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right) dt^2 + \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2 d\phi^2. \tag{38}$$

The scalar curvature is constant:

$$R = -6\frac{1}{l^2} \tag{39}$$

BH temperature:

$$T_H = \frac{M}{2\pi} \tag{40}$$

SET

$$\langle T_\mu^\nu \rangle = F(r)\mathrm{diag}(1,1,-2) + G(r)\mathrm{diag}(1,0,-1) \tag{41}$$

Not of the hydrodynamical type:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{42}$$

where $u_\mu = (\sqrt{|g_{00}|}, 0, 0, 0)$ is the four-velocity.

- For $M = 0$ we have negative energy density $\sim$ Casimir
- Non-thermal type $T_\mu^\nu \not\sim T_H^3$
- Vanishes in the limit $M \to \infty$
- Green's function satisfies the KMS thermality condition

BTZ

For a conformally coupled scalar field theory, the replica trick gives:

$$K^{AdS} \to K^{BTZ}_{\alpha=1} \to K^{BTZ}_{\alpha} \to W_\alpha \to W^{ren}_\alpha \to S_A : \tag{43}$$

$$W_\alpha = \sum_{n=1}^{\infty} \frac{1}{4n} \frac{\sinh \frac{An}{\alpha}}{\sinh An} \frac{e^{-\frac{An}{2}}}{\sinh^2 \frac{An}{2\alpha}} + (1-\alpha) A \frac{1}{2\sqrt{2}\pi} \left(\frac{1}{\sqrt{\cosh An - 1}} - \frac{1}{\sqrt{\cosh An + 1}} \right) \tag{44}$$

This coincides with the local thermodynamics entropy:

$$K^{AdS} \to K^{BTZ}_{\alpha=1} \to \langle T^\nu_\mu \rangle \to \langle T^\nu_\mu \rangle_{ren} \to S_V \tag{45}$$

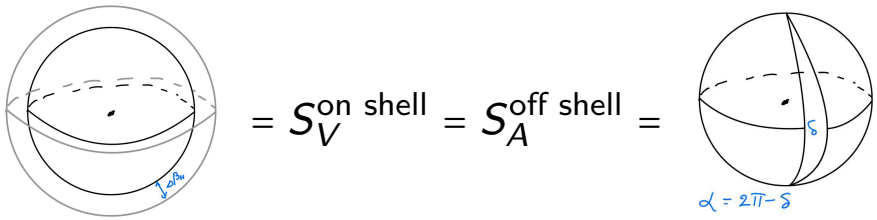
$$\langle T^\nu_\mu \rangle = F(r, T_H)\mathrm{diag}(1,1,-2) + G(r, T_H)\mathrm{diag}(1,0,-1) \tag{46}$$

$$g_{00} = -g^{11} = \left(\frac{r^2}{l^2} - M \right) = \left(\frac{r^2}{l^2} - 2\pi T_H \right) \tag{47}$$

$$S_V^{\text{on shell}} = \int^{T_H} \frac{dT_H}{T_H} \int d^{d-1}x \sqrt{|g|} \left(\frac{\partial \rho}{\partial T_H} + (p+\rho) \frac{1}{v} \frac{\partial v}{\partial T_H} \right) \tag{48}$$

Conclusion

We show that:



- Explicit volume dependence $S \sim \int dV \ldots$
- Solve puzzle (???) of renormalization quantum corrections due to the self-interaction
- Using local thermodynamics we can compute heat capacity without the problem of $\log Z \sim \ldots + (\beta - \beta_H)^2 \int dV \delta(\Sigma)^2$