title[Klein Paradox]Lessons from the Klein Paradox

# A QFT perspective (excluding Sec. 5)

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# Setup and Mode Expansion

Potential:

$$-eA_\mu = V_0\theta(x)\delta_{0\mu}, \quad e > 0, \quad V_0 > 2m$$

Dirac equation:

$$\begin{cases} (\omega - V_0\theta(x) - m)\psi_L - \partial_x\psi_R = 0 \\ (\omega - V_0\theta(x) + m)\psi_R + \partial_x\psi_L = 0 \end{cases}$$

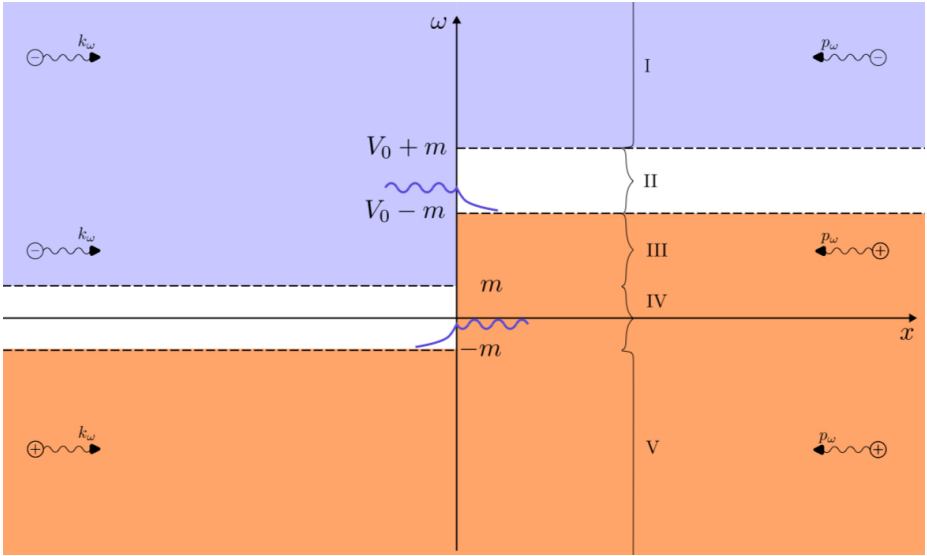
Mode definitions:

$$k_\omega = \sqrt{\omega^2 - m^2}, \quad p_\omega = \sqrt{(\omega - V_0)^2 - m^2}$$

$$\chi_\omega = \frac{k_\omega}{\omega}\frac{\omega - V_0 - m}{\omega - m}\chi(\omega)$$

$$\chi(\omega) = \begin{cases} -1, & \omega \in [m, V_0 - m] \\ 1, & \text{else} \end{cases}$$

# Figure 1: In-Modes Illustration



# Sec. 2: Static Field - Two Vacua

Mode solutions:

$$\overleftarrow{\psi}_\omega(x) = A_\omega \left\{ \theta(-x) \frac{2\chi_\omega p_\omega \chi(\omega)}{k_\omega(\chi_\omega + 1)} \begin{pmatrix} -ik_\omega \\ \omega - m \end{pmatrix} e^{-ik_\omega x} + \cdots \right\}$$

$$\overrightarrow{\psi}_\omega(x) = B_\omega \left\{ \theta(-x) \left[ \begin{pmatrix} ik_\omega \\ \omega - m \end{pmatrix} e^{ik_\omega x} + \frac{\chi_\omega - 1}{\chi_\omega + 1} \begin{pmatrix} -ik_\omega \\ \omega - m \end{pmatrix} e^{-ik_\omega x} \right] + \cdots \right\}$$

Normalization:

$$|A_\omega|^2 = \frac{1}{2p_\omega|\omega - V_0 - m|}, \quad |B_\omega|^2 = \frac{1}{2k_\omega|\omega - m|}$$

Current conservation in Klein zone:

$$j^1(x < 0) = -1 + \left( \frac{1 - \chi_\omega}{1 + \chi_\omega} \right)^2 = -4 \frac{\chi_\omega}{(1 + \chi_\omega)^2} = j^1(x > 0)$$

# Sec. 2: Static Field - Two Vacua (continued)

Field expansion (Basis 1):

$$\hat{\psi}(t,x) = \int \frac{d\omega}{2\pi} \Big\{ \theta_{\rm I}(\omega) \overrightarrow{a}_\omega \overrightarrow{\psi}_\omega(x) + [\theta_{\rm I} + \theta_{\rm II}] \overleftarrow{a}_\omega \overleftarrow{\psi}_\omega(x) + \theta_{\rm III}(\omega)[a_\omega \overrightarrow{\psi}_\omega(x) + b^\dagger_\omega \overleftarrow{\psi}_\omega(x)] + \cdots \Big\}$$

Anti-commutators:

$$\{\overleftarrow{a}_\omega, \overleftarrow{a}^\dagger_{\omega'}\} = 2\pi\delta(\omega-\omega'), \quad \{\overrightarrow{a}_\omega, \overrightarrow{a}^\dagger_{\omega'}\} = 2\pi\delta(\omega-\omega')$$

Hamiltonian:

$$\hat{H} = \int_0^\infty \frac{d\omega}{2\pi} \Big\{ \theta_{\rm I}(\omega)[\omega(\overrightarrow{a}^\dagger_\omega \overrightarrow{a}_\omega + \overleftarrow{a}^\dagger_\omega \overleftarrow{a}_\omega) + (\omega - V_0)(\overrightarrow{b}^\dagger_{\omega-V_0} \overrightarrow{b}_{\omega-V_0} + \overleftarrow{b}^\dagger_{\omega-V_0} \overleftarrow{b}_{\omega-V_0})] + \cdots \Big\}$$

# Sec. 2: Static Field - Currents

Ground state $|\mathrm{gs}\rangle$ (filled positron states in Klein zone):

$$\langle \mathrm{gs}| \overleftarrow{b}^{\dagger}_{\omega} \overleftarrow{b}_{\omega'} |\mathrm{gs}\rangle = \theta(\omega)(1-\theta_{\mathrm{I}}(\omega)-\theta_{\mathrm{II}}(\omega)) \cdot 2\pi\delta(\omega-\omega') \equiv \overleftarrow{n}^{(\omega)}_{+} \cdot 2\pi\delta(\omega-\omega')$$

General state with occupation numbers:

$$\langle \mathrm{st}| \overleftarrow{b}^{\dagger}_{\omega} \overleftarrow{b}_{\omega'} |\mathrm{st}\rangle \equiv \overleftarrow{n}^{(\omega)}_{+} \cdot 2\pi\delta(\omega-\omega'), \quad \langle \mathrm{st}| \overrightarrow{a}^{\dagger}_{\omega} \overrightarrow{a}_{\omega'} |\mathrm{st}\rangle \equiv \overrightarrow{n}^{(\omega)}_{-} \cdot 2\pi\delta(\omega-\omega')$$

Current for general state:

$$\langle \mathrm{st}|j^1|\mathrm{st}\rangle = \int_m^{V_0-m} \frac{d\omega}{2\pi} \left[1 - \overleftarrow{n}^{(\omega)}_{+} - \overrightarrow{n}^{(\omega)}_{-}\right] \frac{-4\chi_\omega}{(\chi_\omega+1)^2}$$

Two special cases:

$$\langle \mathrm{gs}|j^1|\mathrm{gs}\rangle = 0, \quad \langle 0|j^1|0\rangle = \int_m^{V_0-m} \frac{d\omega}{2\pi} \frac{-4\chi_\omega}{(\chi_\omega+1)^2}$$

# Sec. 3: Switch-on $V_0\theta(t)\theta(x)$ - Mode Matching

For $t < 0$ (free theory):

$$\overleftarrow{\psi}_0(\omega, x) = B_\omega \begin{pmatrix} -ik_\omega \\ \omega - m \end{pmatrix} e^{-ik_\omega x}, \quad \overrightarrow{\psi}_0(\omega, x) = B_\omega \begin{pmatrix} ik_\omega \\ \omega - m \end{pmatrix} e^{ik_\omega x}$$

For $t > 0$, expand over static basis:

$$\overleftarrow{\Psi}^{(+)}_\omega(t > 0, x) = \int \frac{d\omega'}{2\pi} \left[ f_{11}(\omega, \omega') \overleftarrow{\psi}_{\omega'}(x) + f_{12}(\omega, \omega') \overrightarrow{\psi}_{\omega'}(x) \right] e^{-i\omega' t}$$

Matching coefficients at $t = 0$:

$$f_{11}(\omega, \omega') = \int dx \, \overleftarrow{\psi}^\dagger_{\omega'}(x) \cdot \overleftarrow{\psi}^*_0(\omega, x)$$

$$= A_{\omega'} B^*_\omega \left\{ \frac{2\chi^*_{\omega'} p^*_{\omega'} \chi(\omega')}{k_{\omega'}(\chi^*_{\omega'} + 1)} \frac{-k_{\omega'} k_\omega + (\omega' - m)(\omega - m)}{\epsilon + i(k_{\omega'} + k_\omega)} - \cdots \right\} \theta(-\omega - m)(1 - \theta_{\rm II}(\omega'))$$

# Sec. 3: Switch-on - Pole Analysis

Key identity for poles:

$$\frac{1}{k_{\omega'} - k_\omega - i\epsilon} = \frac{k_{\omega'} + k_\omega}{(\omega' - \omega - i\epsilon \cdot \mathrm{sign}(\omega' + \omega))(\omega' + \omega - i\epsilon \cdot \mathrm{sign}(\omega' - \omega))}$$

Result for modes as $t \to \infty$:

$$\overleftarrow{\Psi}^{(+)}_\omega(t \to \infty, x) \approx -4k_{\omega+V_0}(\omega - m)\overleftarrow{\psi}_{\omega+V_0}(x)e^{-i(\omega+V_0)t} +$$

$$+\theta(-\omega - m)\left[\frac{1 - \chi(\omega + V_0)}{2} - \frac{1 + \chi(\omega + V_0)}{2}\frac{\chi^*_{\omega+V_0} - 1}{\chi^*_{\omega+V_0} + 1}\right]\overleftarrow{\psi}_{\omega+V_0}(x)e^{-i(\omega+V_0)t}$$

$$\overrightarrow{\Psi}^{(+)}_\omega(t \to \infty, x) \approx -4k_\omega(m - \omega)\overrightarrow{\psi}_\omega(x)e^{-i\omega t} + \frac{\chi_\omega - 1}{\chi_\omega + 1}\theta(-\omega - m)\overrightarrow{\psi}_\omega(x)e^{-i\omega t}$$

# Sec. 3: Switch-on - Current

Current from in-vacuum:

$$\langle \mathrm{vac}|j^1(t,x)|\mathrm{vac}\rangle\Big|_{t\to\infty} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\left[\overleftarrow{\Psi}^{(+)}_{\omega}(t,x)\gamma^0\gamma^1\overleftarrow{\Psi}^{(+)}_{\omega}(t,x) + \overrightarrow{\Psi}^{(+)}_{\omega}(t,x)\gamma^0\gamma^1\overrightarrow{\Psi}^{(+)}_{\omega}(t,x)\right]\Big|_{t\to}$$

Evaluation yields:

$$\langle \mathrm{vac}|j^1(t,x)|\mathrm{vac}\rangle\Big|_{t\to\infty} \simeq \int_{m}^{V_0-m} \frac{d\omega}{2\pi}\frac{-4\chi_\omega}{(\chi_\omega+1)^2}$$

Matches $\langle 0|j^1|0\rangle$ from static case.

# Sec. 4: General $U(x)\theta(t)$ - Mode Asymptotics

As $x \to -\infty$:

$$\psi^{(0)}_{\omega,\leftarrow}(x) \approx \theta(m - |\omega|) K_-(\omega) e^{\sqrt{m^2-\omega^2}x} \begin{pmatrix} \sqrt{m^2-\omega^2} \\ \omega - m \end{pmatrix}$$

$$\psi^{(0)}_{\omega,\rightarrow}(x) \approx \theta(m - |\omega - V_0|) \sum_{\sigma'} K_-(\sigma';\omega) e^{ik_{\omega}\sigma' x} \begin{pmatrix} ik_{\omega}\sigma' \\ \omega - m \end{pmatrix}$$

$$\psi^{(0)}_{\omega,\sigma}(x) \approx \theta(|\omega| - m)\theta(|\omega - V_0| - m) \sum_{\sigma'} \frac{N(\sigma,\sigma';\omega)}{\sqrt{2k_{\omega}|\omega - m|}} e^{ik_{\omega}\sigma' x} \begin{pmatrix} ik_{\omega}\sigma' \\ \omega - m \end{pmatrix}$$

# Sec. 4: General $U(x)\theta(t)$ - Mode Asymptotics

As $x \to +\infty$:

$$\psi^{(0)}_{\omega,\leftarrow}(x) \approx \theta(m - |\omega|) \sum_{\sigma'} K_+(\sigma';\omega) e^{ip_\omega \sigma' x} \begin{pmatrix} ip_\omega \sigma' \\ \omega - V_0 - m \end{pmatrix}$$

$$\psi^{(0)}_{\omega,\rightarrow}(x) \approx \theta(m - |\omega - V_0|) K_+(\omega) e^{-\sqrt{m^2-(\omega-V_0)^2}x} \begin{pmatrix} -\sqrt{m^2-(\omega-V_0)^2} \\ \omega - V_0 - m \end{pmatrix}$$

$$\psi^{(0)}_{\omega,\sigma}(x) \approx \theta(|\omega|-m)\theta(|\omega-V_0|-m) \sum_{\sigma',\sigma''} \frac{N(\sigma,\sigma'';\omega)}{\sqrt{2p_\omega|\omega - V_0 - m|}} S(\sigma'',\sigma';\omega) e^{ip_\omega \sigma' x} \begin{pmatrix} ip_\omega \sigma' \\ \omega - V_0 - m \end{pmatrix}$$

# Sec. 4: General $U(x)$ - S-matrix and Constraints

S-matrix structure:

$$\hat{S}(\omega) = \begin{pmatrix} \alpha(\omega) & \beta(\omega) \\ \beta^*(\omega) & \alpha^*(\omega) \end{pmatrix}$$

Current conservation:

$$|\alpha(\omega)|^2 - |\beta(\omega)|^2 = \text{sign}\left(\frac{\omega - m}{\omega - V_0 - m}\right)$$

Normalization condition from completeness:

$$\sum_{\sigma_1,\sigma_2} N^*(\sigma,\sigma_1;\omega)\left[\delta_{\sigma_1,\sigma_2} + \sum_{\sigma''} S^*(\sigma_1,\sigma'')S(\sigma_2,\sigma'')\right] N(\sigma',\sigma_2;\omega) = 2\delta_{\sigma,\sigma'}$$

Matrix form:

$$\hat{N}^\dagger(\omega)\hat{N}(\omega) = \left(\frac{1 + \hat{S}^\dagger(\omega)\hat{S}(\omega)}{2}\right)^{-1}$$

# Sec. 4: General $U(x)$ - Current

Field operator:

$$\hat{\Psi}(t,x) = \sum_{\sigma} \int_{m}^{+\infty} \left[ \Psi^{in}_{\omega,\sigma}(t,x)\hat{a}_{\omega,\sigma} + \Psi^{in}_{-\omega,\sigma}(t,x)\hat{b}^{\dagger}_{\omega,\sigma} \right] \frac{d\omega}{2\pi}$$

Current expectation:

$$\langle 0|j^1(t,x)|0\rangle\Big|_{t\to+\infty} = \sum_{\sigma} \int_{m}^{+\infty} \Psi^{in*}_{-\omega,\sigma}(t,x)\hat{\alpha}\Psi^{in}_{-\omega,\sigma}(t,x)\frac{d\omega}{2\pi}\Big|_{t\to+\infty}$$

$$\langle 0|j^1(t,x)|0\rangle\Big|_{t\to+\infty} \simeq -\int_{m}^{V_0-m} \frac{1}{|\beta(\omega)|^2}\frac{d\omega}{2\pi}$$

# Summary of Sections 1-4

**Static case:**

$$\langle \mathrm{st}|j^1|\mathrm{st}\rangle = \int_m^{V_0-m} \frac{d\omega}{2\pi}\left[1 - \overleftarrow{n}_+^{(\omega)} - \overrightarrow{n}_-^{(\omega)}\right]\frac{-4\chi_\omega}{(\chi_\omega+1)^2}$$

$$\langle 0|j^1|0\rangle = \int_m^{V_0-m} \frac{d\omega}{2\pi}\frac{-4\chi_\omega}{(\chi_\omega+1)^2}, \quad \langle \mathrm{gs}|j^1|\mathrm{gs}\rangle = 0$$

**Switch-on:**

$$\langle \mathrm{vac}|j^1|\mathrm{vac}\rangle\Big|_{t\to\infty} = \int_m^{V_0-m} \frac{d\omega}{2\pi}\frac{-4\chi_\omega}{(\chi_\omega+1)^2}$$

**General $U(x)$:**

$$\langle 0|j^1|0\rangle\Big|_{t\to\infty} = -\int_m^{V_0-m} \frac{d\omega}{2\pi}\frac{1}{|\beta(\omega)|^2}$$