Horndeski gravity: Some cosmological and astrophysical aspects

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Based on

- P.E. Kashargin, S.V. Sushkov, Anti-de Sitter neutron stars in the theory of gravity with nonminimal derivative coupling, arxiv:2205.08949 (2022)
- R. Galeev, R.K. Muharlyamov,A.A. Starobinsky, S.V. Sushkov, M.S. Volkov, Anisotropic cosmological models in Horndeski gravity, Phys.Rev.D 103 (2021) 104015
- R.V. Korolev, F.S.N. Lobo, S.V. Sushkov, General constraints on Horndeski wormhole throats, Physical Review D, 2020, 101(12), 124057
- R.V. Korolev, F.S.N. Lobo, S.V. Sushkov, Kinetic gravity braiding wormhole geometries, Phys. Rev. D 102, 104016 (2020)
- A.A. Starobinsky, S.V. Sushkov, M.S. Volkov, Anisotropy screening in Horndeski cosmologies, Physical Review D, 2020, 101(6), 064039
- Matsumoto J. General dynamical properties of cosmological models with nonminimal kinetic coupling / J. Matsumoto, S. V. Sushkov // Journal of Cosmology and Astroparticle Physics. - 2018. - 01. - 040
- Sushkov S.V. (2017) Horndeski Wormholes. In: Lobo F. (eds) Wormholes, Warp Drives and Energy Conditions. Fundamental Theories of Physics, vol 189. Springer, Cham

Based on... (continuation)

- R.V. Korolev and S.V. Sushkov, Exact wormhole solutions with nonminimal kinetic coupling, Physical Review D 90, 124025 (2014)
- Abzalov R. Cosmological perturbations in the theory of gravity with nonminimal kinetic coupling / R. Abzalov, S.V. Sushkov // TSPU Bulletin. - 2014. - 12(153) - P.13-16
- M.A. Skugoreva, S.V. Sushkov, A.V. Toporensky, Cosmology with nonminimal kinetic coupling and a power-law potential, Physical Review D 88, 083539 (2013)
- S.V. Sushkov, Realistic cosmological scenario with nonminimal kinetic coupling, Physical Review D 85, 123520 (2012)
- S.V. Sushkov, R. Korolev, Scalar wormholes with nonminimal derivative coupling, Classical Quantum Gravity 29, 085008 (2012)
- E.N. Saridakis, S.V. Sushkov, Quintessence and phantom cosmology with nonminimal derivative coupling, Physical Review D 81, 083510 (2010)
- S.V. Sushkov, Exact cosmological solutions with nonminimal derivative coupling, Physical Review D 80 (2009) 103505

Motivation

- GR has successfully been exploited for a long time to describe celestial motion in Solar system, a bending of light rays, gravitational waves, the universe expansion (ΛCDM model)
- GR is unable to solve the number already existing problems and appearing new ones
  - cosmological and black hole singularities
  - dark energy (accelerating expansion of the Universe)
  - initial inflation
  - large scale structure of the universe
  - dark matter evidence
  - cosmological constant problem
  - etc. . .
- These amazing discoveries have set new serious challenges before theoretical cosmology faced the necessity of radical *modification* or *extension* of General Relativity

Plan of the talk

- Scalar-tensor theories of gravity
  - Generalized Brans-Dicke theories
  - Horndeski theories; Subclasses
  - Extended scalar-tensor theories (DHOST)
- Theory of gravity with nonminimal derivative coupling
- Cosmological models with nonminimal derivative coupling
  - No potential
  - Cosmological constant
  - Power-law potential
  - The screening Horndeski cosmologies
  - Anisotropic cosmologies
- Perturbations
- Compact objects: black holes, wormholes, neutron stars
  - "Strange" neutron stars

Scalar-tensor theories

$$S = \int d^4x\sqrt{-g}\left[F(\phi)R - Z(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U(\phi)\right] + S_m\left[\psi_m, g_{\mu\nu}\right]$$

- generalizations of the Brans-Dicke theories
- the scalar field is
  - minimally coupled with ordinary matter (physical or Jordan frame)
  - non-minimally coupled with the scalar curvature by the term $F(\phi)R$

Notice: Non-minimal coupling of the scalar field with the scalar curvature is provided by the terms $F(\phi)R$

Horndeski theory

In 1974, *Gregory Walter Horndeski* derived the action of the most general scalar-tensor theories with second-order equations of motion
[G.Horndeski, *Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space*, IJTP **10**, 363 (1974)]

Horndeski Lagrangian:

$$L_{\rm H} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)$$

$$\mathcal{L}_2 = G_2(\phi, X)\ ,$$
$$\mathcal{L}_3 = G_3(\phi, X)\,\Box\phi\ ,$$
$$\mathcal{L}_4 = G_4(\phi, X)R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\ ,$$
$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})\,,$$

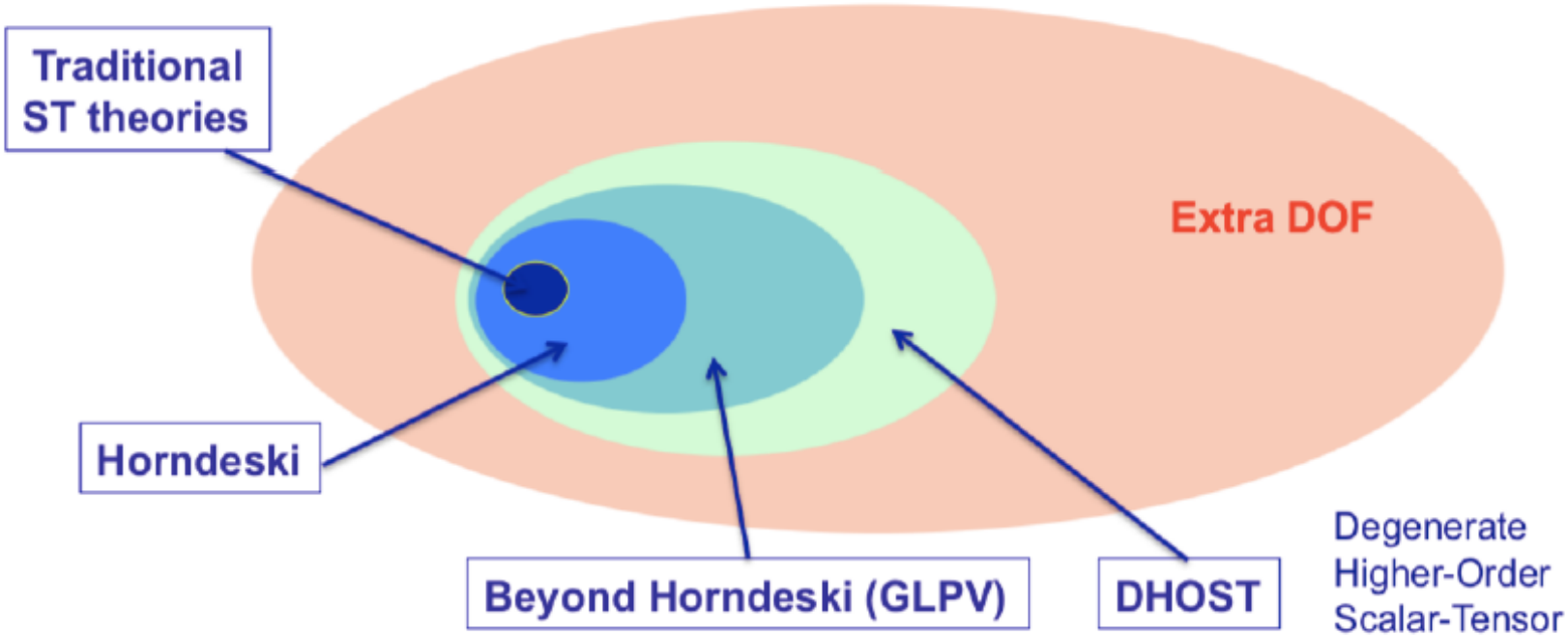
where $\phi_\mu = \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, $X = -\frac{1}{2}(\nabla\phi)^2$,
and $G_a(\phi, X)$ are four arbitrary functions,

Subclasses of the Horndeski theory

$$\mathcal{L}_H = \mathcal{L}\{G_2, G_3, G_4, G_5\}$$

- Hilbert-Einstein action (GR):
  $G_4(\phi, X) = \frac{1}{2}M_{Pl}^2 \quad \rightarrow \quad \mathcal{L}_H \sim \frac{1}{2}M_{Pl}^2 R$
- Nonminimal coupling: $G_4(\phi, X) = f(\phi) \quad \rightarrow \quad \mathcal{L}_H \sim f(\phi)R$
- GR with a scalar field: $G_2(\phi, X) = \epsilon X - V(\phi)$
- k-essence: $G_2 = K(\phi, X)$
- Kinetic gravity braiding (KGB):
  $G_3 = B(\phi, X) \quad \rightarrow \quad \mathcal{L}_H \sim B(\phi, X)\Box\phi$
- Nonminimal kinetic coupling:
  $G_5(\phi, X) = \eta\phi \quad \rightarrow \quad \mathcal{L}_H \sim \eta G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$
- Fab Four, Gallileons, etc.

Extended scalar-tensor theories



Landscape of scalar-tensor theories

D. Langlois, Dark energy and modified gravity
in degenerate higher-order scalar-tensor (DHOST) theories: A review
Int. J. Mod. Phys. D 28 (2019), no. 05 1942006

DHOST theories

$$S = \int d^4x \sqrt{-g} \Big[F_{(2)}(\phi, X) R + P(\phi, X) + Q(\phi, X) \Box\phi$$

$$+ F_{(3)}(\phi, X)\, G_{\mu\nu}\phi^{\mu\nu} + \sum_{a=1}^{5} A_a(\phi, X) L_a^{(2)} + \sum_{a=1}^{10} B_a(\phi, X) L_a^{(3)} \Big]$$

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \qquad L_2^{(2)} = (\Box\phi)^2\,, \qquad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu\,,$$
$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \qquad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\,.$$

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho\,,$$
$$L_4^{(3)} = (\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,,$$
$$L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,,$$
$$L_9^{(3)} = \Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,.$$

Non-minimal coupling of the scalar field with curvature

Notice: There are only two qualitatively different terms describing non-minimal coupling of the scalar field with curvature: $M(\phi, X)R$ and $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

- $M(\phi, X)R$ — Brans-Dicke-like theories
- $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ — theories with non-minimal derivative coupling

Theory with nonminimal derivative coupling

Action:

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2\,R-(\varepsilon\,g_{\mu\nu}+\eta\,G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi-2V(\phi)\right]+S_{\rm m}$$

Field equations:

$$M_{\rm Pl}^2 G_{\mu\nu}=T^{(\phi)}_{\mu\nu}+\eta\,\Theta_{\mu\nu}+T^{(\rm m)}_{\mu\nu}$$

$$[\epsilon g^{\mu\nu}+\eta G^{\mu\nu}]\nabla_\mu\nabla_\nu\phi=V'_\phi$$

$$T^{(\phi)}_{\mu\nu}=\epsilon\left[\nabla_\mu\phi\nabla_\nu\phi-\tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right]-g_{\mu\nu}V(\phi),$$

$$\Theta_{\mu\nu}=-\tfrac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi\,R+2\nabla_\alpha\phi\,\nabla_{(\mu}\phi R^\alpha_{\nu)}-\tfrac{1}{2}(\nabla\phi)^2G_{\mu\nu}+\nabla^\alpha\phi\,\nabla^\beta\phi\,R_{\mu\alpha\nu\beta}$$
$$+\nabla_\mu\nabla^\alpha\phi\,\nabla_\nu\nabla_\alpha\phi-\nabla_\mu\nabla_\nu\phi\,\Box\phi+g_{\mu\nu}\big[-\tfrac{1}{2}\nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\nabla_\beta\phi+\tfrac{1}{2}(\Box\phi)^2$$
$$-\nabla_\alpha\phi\,\nabla_\beta\phi\,R^{\alpha\beta}\big]$$

$$T^{(\rm m)}_{\mu\nu}=(\rho+p)U_\mu U_\mu+pg_{\mu\nu}$$

Notice: *The field equations are of second order!*

Cosmological models in theories of gravity with nonminimal derivative coupling

Cosmological models: General formulas

Ansatz:

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2,$$

$$\phi = \phi(t)$$

$a(t)$ *cosmological factor*, $H = \dot{a}/a$ *Hubble parameter*

Field equations:

$$3M_{\rm Pl}^2 H^2 = \frac{1}{2}\dot{\phi}^2 \left(\epsilon - 9\eta H^2\right) + V(\phi),$$

$$M_{\rm Pl}^2(2\dot{H} + 3H^2) = -\frac{1}{2}\dot{\phi}^2 \left[\epsilon + \eta\left(2\dot{H} + 3H^2 + 4H\ddot{\phi}\dot{\phi}^{-1}\right)\right] + V(\phi),$$

$$\frac{d}{dt}\left[(\epsilon - 3\eta H^2)a^3\dot{\phi}\right] = -a^3\frac{dV(\phi)}{d\phi}$$

Cosmological models: General formulas

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$$\frac{d}{dt}\left[(\epsilon - 3\eta H^2)a^3\dot{\phi}\right] = -a^3\frac{dV(\phi)}{d\phi}$$

$$V(\phi) \equiv const \quad \Longrightarrow \quad \dot{\phi} = \frac{Q}{a^3(\epsilon - 3\eta H^2)} \qquad Q \text{ is a scalar charge}$$

Cosmological models: I. No potential $V(\phi) \equiv 0$

Trivial model without kinetic coupling, i.e. $\eta = 0$

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[M_{\rm Pl}^2 R - (\nabla\phi)^2 \right]$$

Cosmological models: I. No potential $V(\phi) \equiv 0$

Trivial model without kinetic coupling, i.e. $\eta = 0$

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2 R - (\nabla\phi)^2\right]$$

Solution:

$$a_0(t) = t^{1/3}; \quad \phi_0(t) = \frac{1}{2\sqrt{3\pi}}\ln t$$

$$ds_0^2 = -dt^2 + t^{2/3}d\mathbf{x}^2$$

$t = 0$ *is an initial singularity*

Cosmological models: II. No potential $V(\phi) \equiv 0$

Model without free kinetic term, i.e. $\epsilon = 0$

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[M_{\rm Pl}^2 R - \eta G^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right]$$

Cosmological models: II. No potential $V(\phi) \equiv 0$

Model without free kinetic term, i.e. $\epsilon = 0$

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2 R - \eta G^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right]$$

Solution:

$$a(t) = t^{2/3}; \quad \phi(t) = \frac{t}{2\sqrt{3\pi|\eta|}}, \quad \eta < 0$$

$$ds_0^2 = -dt^2 + t^{4/3}d\mathbf{x}^2$$

$t = 0$ *is an initial singularity*

Notice: *The parameter η is non-pertubative!*
There is no limit $\eta \to 0$!

Cosmological models: III. No potential $V(\phi) \equiv 0$

Model for an ordinary scalar field ($\epsilon = 1$) with nonminimal kinetic coupling $\eta \neq 0$

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[M_{\rm Pl}^2 R - (g^{\mu\nu} + \eta G^{\mu\nu}) \phi_{,\mu} \phi_{,\nu} \right]$$

Model for an ordinary scalar field ($\epsilon = 1$) with nonminimal kinetic coupling $\eta \neq 0$

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2 R - (g^{\mu\nu} + \eta G^{\mu\nu})\phi_{,\mu}\phi_{,\nu}\right]$$

Asymptotic for $t \to \infty$:

$$a(t) \sim a_0(t) = t^{1/3}; \quad \phi(t) \sim \phi_0(t) = \frac{1}{2\sqrt{3\pi}}\ln t$$

Notice: *At large times the model with $\eta \neq 0$ has the same behavior like that with $\eta = 0$*

Cosmological models: III. No potential $V(\phi) \equiv 0$

Asymptotics for early times

The case $\eta < 0$:

$$a_{t\to 0} \approx t^{2/3}; \quad \phi_{t\to 0} \approx \frac{t}{2\sqrt{3\pi|\eta|}}$$

$$ds^2_{t\to 0} = -dt^2 + t^{4/3} d\mathbf{x}^2$$

$t = 0$ *is an initial singularity*

The case $\eta > 0$:

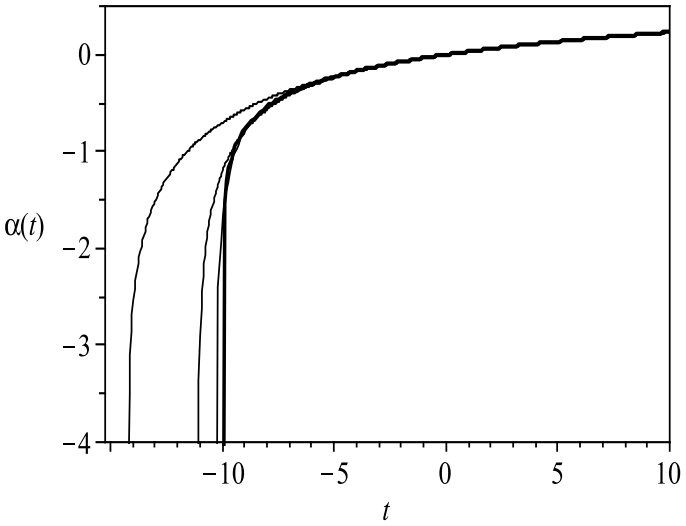
$$a_{t\to -\infty} \approx e^{H_\eta t}; \quad \phi_{t\to -\infty} \approx Ce^{-t/\sqrt{\eta}}$$

$$ds^2_{t\to -\infty} = -dt^2 + e^{2H_\eta t} d\mathbf{x}^2$$

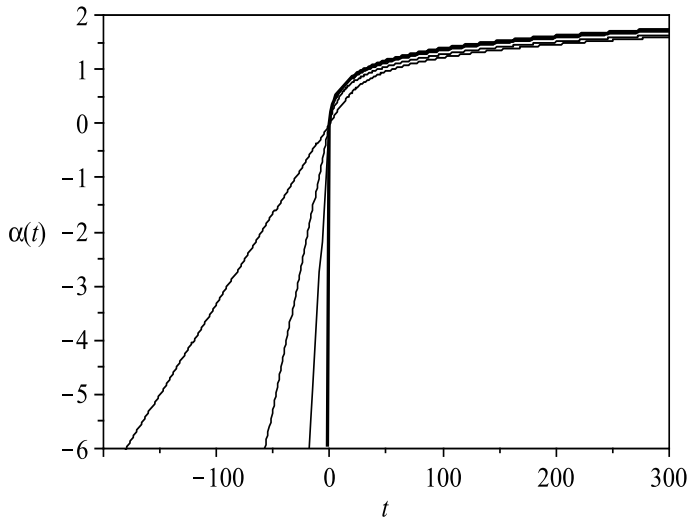
de Sitter asymptotic with $H_\eta = 1/\sqrt{9\eta}$

Cosmological models: III. No potential $V(\phi) \equiv 0$

Plots of $\alpha = \ln a$ in case $\eta \neq 0$, $\epsilon = 1$, $V = 0$.



(a) $\eta < 0$;
$\eta = 0; -1; -10; -100$



(b) $\eta > 0$;
$\eta = 0; 1; 10; 100$

De Sitter asymptotics: $\alpha(t) = \dfrac{t}{\sqrt{9\eta}} \quad \Rightarrow \quad H = \dfrac{1}{\sqrt{9\eta}}$

Notice: *In the model with nonmnimal kinetic coupling one get de Sitter phase without any potential!*

Models with the constant potential $V(\phi) = M_{\rm Pl}^2\Lambda = const$

$$S = \int d^4x\sqrt{-g}\left[M_{\rm Pl}^2(R - 2\Lambda) - [\epsilon g^{\mu\nu} + \eta G^{\mu\nu}]\phi_{,\mu}\phi_{,\nu}\right]$$

Models with the constant potential $V(\phi) = M_{\rm Pl}^2\Lambda = const$

$$S = \int d^4x\sqrt{-g}\left[M_{\rm Pl}^2(R - 2\Lambda) - [\epsilon g^{\mu\nu} + \eta G^{\mu\nu}]\phi_{,\mu}\phi_{,\nu}\right]$$

There are two exact de Sitter solutions:

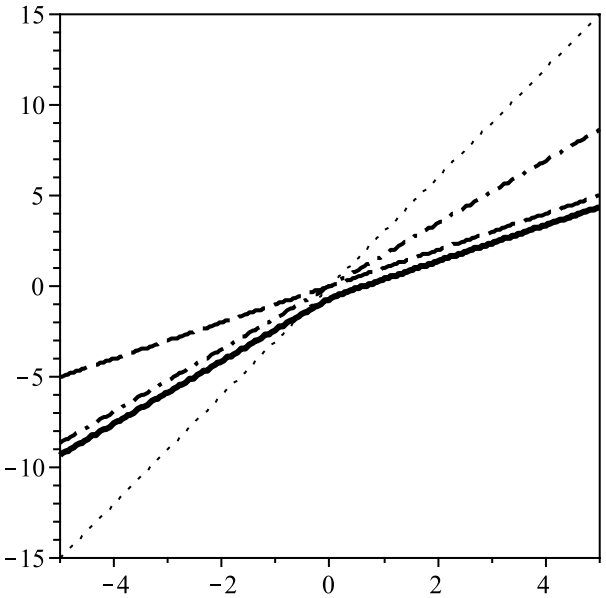
I. $\alpha(t) = H_\Lambda t, \quad \phi(t) = \phi_0 = const,$

II. $\alpha(t) = \dfrac{t}{\sqrt{3|\eta|}}, \quad \phi(t) = M_{\rm Pl}\left|\dfrac{3\eta H_\Lambda^2 - 1}{\eta}\right|^{1/2} t,$

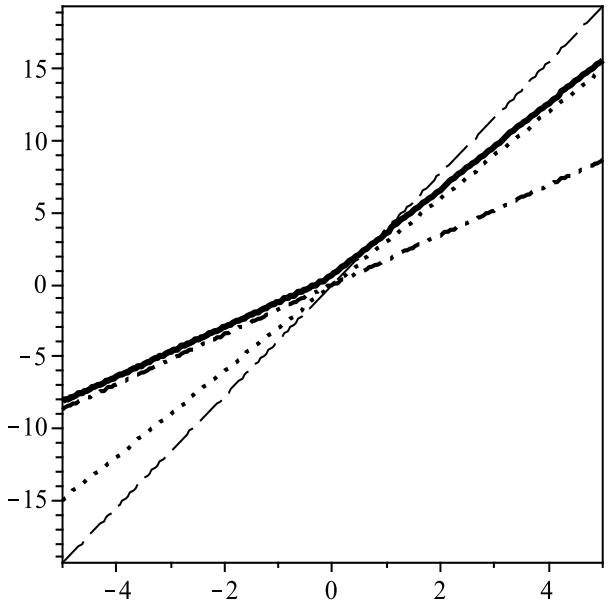
$$H_\Lambda = \sqrt{\Lambda/3}$$

Cosmological models: IV. Cosmological constant

Plots of $\alpha(t)$ in case $\eta > 0$, $\epsilon = 1$, $V = M_{\rm Pl}^2 \Lambda$



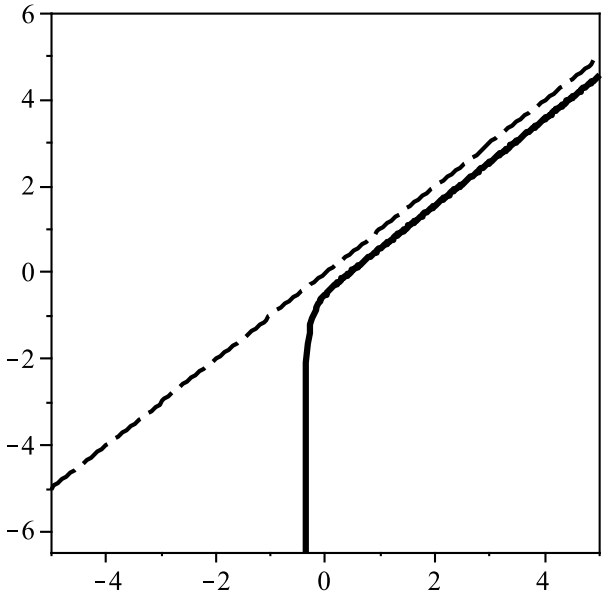
(a) $H_\Lambda^2 < \dot{\alpha}^2 < 1/9\eta$



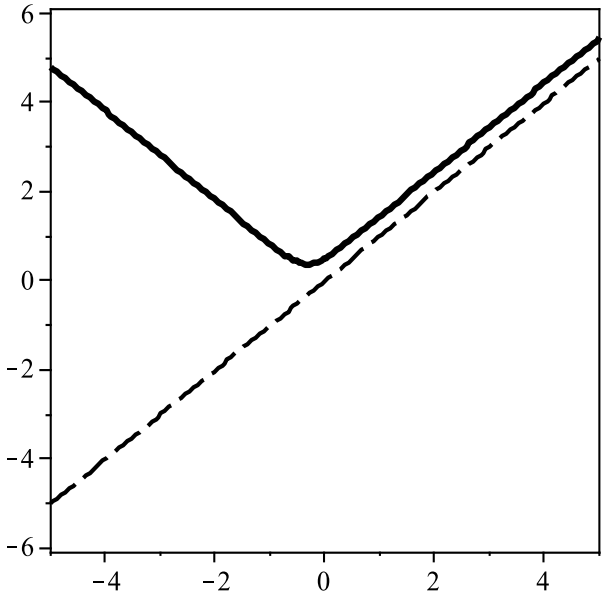
(b) $1/9\eta < \dot{\alpha}^2 < 1/3\eta < H_\Lambda^2$

De Sitter asymptotics:
$\alpha_1(t) = H_\Lambda t$ *(dashed),*
$\alpha_2(t) = t/\sqrt{9\eta}$ *(dash-dotted),*
$\alpha_3(t) = t/\sqrt{3\eta}$ *(dotted).*

Plots of $\alpha(t)$ in cases $\eta > 0$, $\epsilon = -1$ and $\eta < 0$, $\epsilon = 1$



(a) $\eta < 0$, $\epsilon = 1$



(b) $\eta > 0$, $\epsilon = -1$

De Sitter asymptotic:
$\alpha_1(t) = H_\Lambda t$ *(dashed).*

Role of potential

$$S = \int d^4x\sqrt{-g}\left\{M_{\rm Pl}^2 R - [g^{\mu\nu} + \eta G^{\mu\nu}]\phi_{,\mu}\phi_{,\nu} - 2V(\phi)\right\}$$

What a role does a potential play in cosmological models with the nonminimal derivative coupling?

Power-law potential $V(\phi) = V_0\phi^N$

Skugoreva, Sushkov, Toporensky, PRD 88, 083539 (2013)

Models with the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$

Primary (early-time) "kinetic" inflation:

$$H_{t\to-\infty} \approx \frac{1}{\sqrt{9\eta}}(1 + \tfrac{1}{2}\eta m^2)$$

Late-time cosmological scenarios:

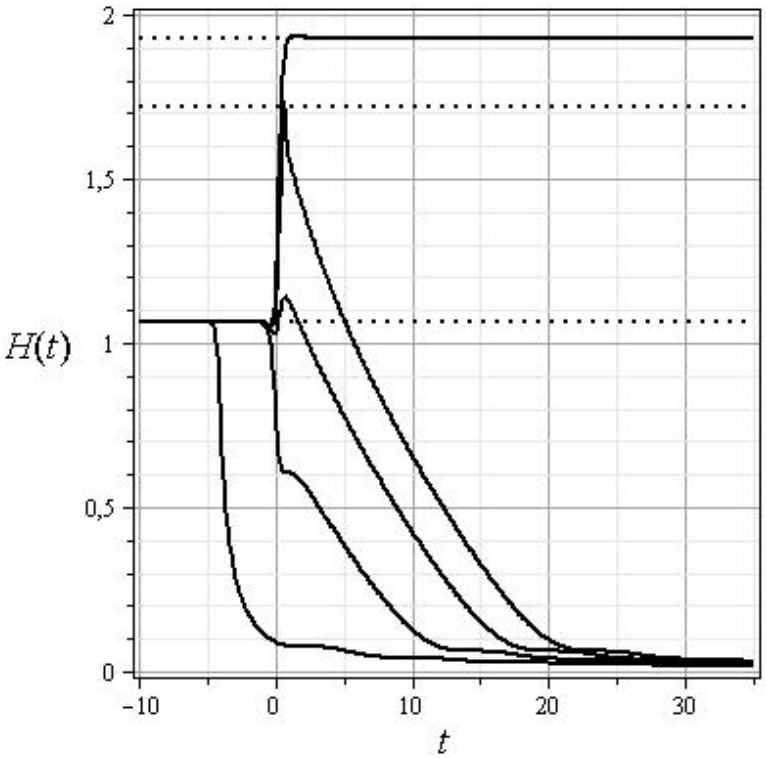
Oscillatory asymptotic or "graceful" exit from inflation

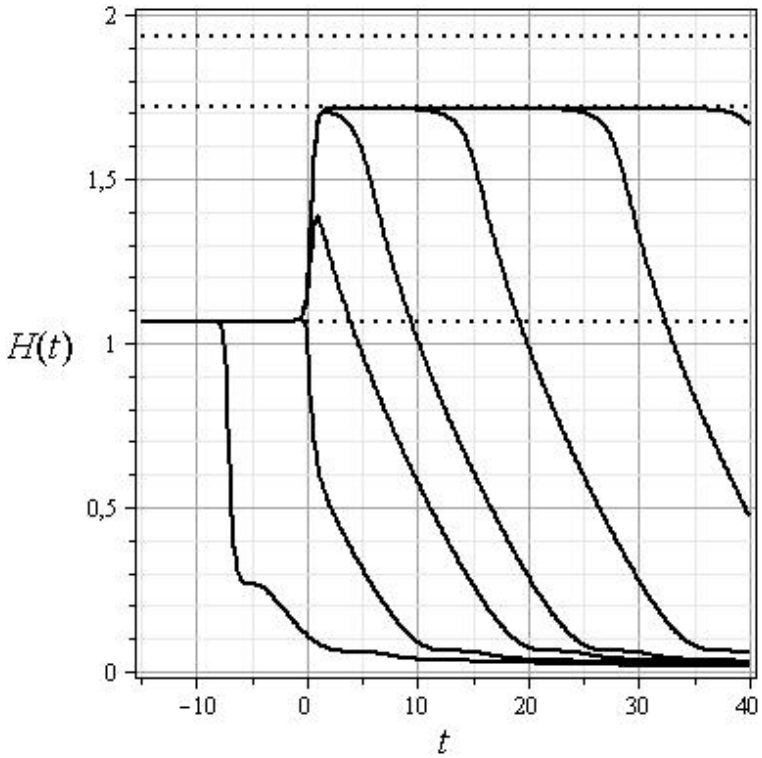
$$H_{t\to\infty} \approx \frac{2}{3t}\left[1 - \frac{\sin 2mt}{2mt}\right]$$

quasi-de Sitter asymptotic or secondary inflation

$$H_{t\to\infty} \approx \frac{1}{\sqrt{3\eta}}\left(1 \pm \sqrt{\tfrac{1}{6}\eta m^2}\right)$$

Cosmological models: Power-law potential





Initial conditions
$\phi_0 = \dot{\phi}_0$

Initial conditions
$\phi_0 = -\dot{\phi}_0$

De Sitter asymptotics: $H_{t\to-\infty} \approx 1/\sqrt{9\eta}(1+\frac{1}{2}\eta m^2)$,

$$H_{t\to\infty} \approx 1/\sqrt{3\eta}\left(1 \pm \sqrt{\tfrac{1}{6}\eta m^2}\right).$$

Screening properties of Horndeski model:

Starobinsky, Sushkov, Volkov, JCAP, 2015

The FLRW ansatz for the metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right],$$

$a(t)$ *cosmological factor*, $H = \dot{a}/a$ *Hubble parameter*

Gravitational equations:

$$-3M_{\rm Pl}^2\left(H^2 + \frac{K}{a^2}\right) + \frac{1}{2}\varepsilon\,\psi^2 - \frac{3}{2}\eta\,\psi^2\left(3H^2 + \frac{K}{a^2}\right) + \Lambda + \rho = 0,$$

$$-M_{\rm Pl}^2\left(2\dot{H} + 3H^2 + \frac{K}{a^2}\right) - \frac{1}{2}\varepsilon\,\psi^2 - \eta\,\psi^2\left(\dot{H} + \frac{3}{2}H^2 - \frac{K}{a^2} + 2H\frac{\dot{\psi}}{\psi}\right) + \Lambda - p = 0.$$

The scalar field equation:

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\left(3\eta\left(H^2 + \frac{K}{a^2}\right) - \varepsilon\right)\psi\right) = 0,$$

where $\psi = \dot{\phi}$, and $\phi = \phi(t)$ is a homogeneous scalar field

Isotropic and homogeneous cosmological models

Ansatz: $V \equiv 0$ (no potential), $\varepsilon = +1$ (ordinary scalar)
$\phi = \phi(t)$, $T^{(m)}_{\mu\nu} = diag(\rho(t), p(t), p(t), p(t))$, and
the FLRW metric

$$ds^2 = -dt^2 + \mathrm{a}^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

$k = 0, \pm 1$, $\mathrm{a}(t)$ *cosmological factor*, $H(t) = \dot{\mathrm{a}}(t)/\mathrm{a}(t)$ *Hubble parameter*

Gravitational equations:

$$3\left(H^2 + \frac{k}{\mathrm{a}^2}\right) = \Lambda + 8\pi\rho + 4\pi\psi^2\left(1 - 9\eta\left(H^2 + \frac{k}{3\mathrm{a}^2}\right)\right),$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = \Lambda - 8\pi p - 4\pi\psi^2\left[1 + 2\eta\left(\dot{H} + \frac{3}{2}H^2 - \frac{k}{a^2} + 2H\frac{\dot{\psi}}{\psi}\right)\right]$$

The scalar field equations:

$$\mathrm{a}^3\psi\left(1 - 3\eta\left(H^2 + \frac{k}{\mathrm{a}^2}\right)\right) = Q = const$$

where $\psi = \dot{\phi}$

Modified Friedmann equation (Master equation). I

Material content is a mixture of radiation and non-relativistic component:

$$\rho = \rho_m + \rho_r = \rho_{m0}\left(\frac{\mathrm{a}_0}{\mathrm{a}}\right)^3 + \rho_{r0}\left(\frac{\mathrm{a}_0}{\mathrm{a}}\right)^4$$

Introducing the dimensionless scales factor a, Hubble parameter h, and coupling parameter ζ:

$$a = \frac{\mathrm{a}}{\mathrm{a}_0}, \quad h = \frac{H}{H_0}, \quad \zeta = \eta H_0^2,$$

and the dimensionless density parameters:

$$\Omega_0 = \frac{\Lambda}{3H_0^2}, \quad \Omega_2 = \frac{k}{\mathrm{a}_0^2 H_0^2}, \quad \Omega_3 = \frac{\rho_{m0}}{\rho_{cr}}, \quad \Omega_4 = \frac{\rho_{r0}}{\rho_{cr}}, \quad \Omega_6 = \frac{4\pi Q^2}{3\mathrm{a}_0^6 H_0^2},$$

where $\rho_{cr} = 3H_0^2/8\pi$ is the critical density, one has

Modified Friedmann equation

$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

Modified Friedmann equation (Master equation). II

Modified Friedmann equation

$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

- Assuming $\Lambda \geq 0$, one has $\Omega_0 \geq 0$
- $\Omega_2 = k/\mathrm{a}_0^2 H_0^2$, hence
  $\Omega_2 = 0$, $\Omega_2 < 0$, $\Omega_2 > 0$ if $k = 0, -1, +1$, respectively
- $\zeta = \eta H_0^2 = \pm\left(\ell/\ell_H\right)^2$, where $\ell_H = 1/H_0$, hence
  ζ is proportional to the square of ratio of two characteristic scales, hence $\zeta \ll 1$???
- In case $\Omega_6 = 0$ (no scalar with non-minimal derivative coupling) one has the standard master equation of ΛCDM cosmological model
- In case $\Omega_6 \neq 0$ but $\zeta = 0$ (no non-minimal derivative coupling) one has a cosmological model with an ordinary scalar field

Modified Friedmann equation (Master equation). III

Denoting $y = h^2$ one can rewrite the master equation as a cubic in y algebraic equation

$$c_3 y^3 + c_2(a) y^2 + c_1(a) y + c_0(a) = 0$$

with the coefficients

$$c_3 = 9\zeta^2$$

$$c_2 = -6\zeta\left(1 - \frac{3\zeta\Omega_2}{a^2}\right) - 9\zeta^2\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right),$$

$$c_1 = \left(1 - \frac{3\zeta\Omega_2}{a^2}\right)^2 + 6\zeta\left(1 - \frac{3\zeta\Omega_2}{a^2}\right)\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right) + \frac{9\zeta\Omega_6}{a^6},$$

$$c_0 = -\left(1 - \frac{3\zeta\Omega_2}{a^2}\right)^2\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right) - \left(1 - \frac{3\zeta\Omega_2}{a^2}\right)\frac{\Omega_6}{a^6}.$$

Notice: Roots $h = h(a)$ of the cubic polynomial (30) define a global cosmological behavior as follows

$$\int_{a=1}^{a} \frac{d\tilde{a}}{\tilde{a}h(\tilde{a})} = H_0(t - t_0).$$

Turning points and bounces in the Universe evolution

A turning point in the Universe evolution may occur at a moment $t = t_*$, when the scale factor $\mathrm{a}(t)$ reaches its extremal, either maximal or minimal value, $a(t_*) = a_*$. Correspondingly, $y(a_*) = h^2(a_*) = 0$.

The polynomial $P(a, y) = c_3 y^3 + c_2(a) y^2 + c_1(a) y + c_0(a)$ has a root $y(a_*) = 0$ if and only if $c_0(a_*) = 0$,and hence we obtain two separate algebraic conditions for a_*:

$$\left(1 - \frac{3\zeta\Omega_2}{a_*^2}\right)\left(\Omega_0 - \frac{\Omega_2}{a_*^2} + \frac{\Omega_3}{a_*^3} + \frac{\Omega_4}{a_*^4}\right) + \frac{\Omega_6}{a_*^6} = 0, \tag{5}$$

$$\left(1 - \frac{3\zeta\Omega_2}{a_*^2}\right) = 0. \tag{6}$$

NOTICE: The conditions (1) and (2) have NO solutions in case $\Omega_2 \leq 0$. Therefore, in cosmological models with negative or zero spatial curvature there are no turning points.

Turning points and bounces: $\Omega_2 > 0$ (positive spatial curvature)

Condition 1: $$\boxed{\left(1-\frac{3\zeta\Omega_2}{a_*^2}\right)\left(\Omega_0-\frac{\Omega_2}{a_*^2}+\frac{\Omega_3}{a_*^3}+\frac{\Omega_4}{a_*^4}\right)+\frac{\Omega_6}{a_*^6}=0}$$

In the simplest case: $\Omega_0=\Omega_3=\Omega_4=0, \zeta=0$, one has

$$a_*^2=\sqrt{\Omega_6/\Omega_2}=\sqrt{(1+\Omega_2)/\Omega_2}.$$

Supposing $\Omega_2 \ll 1$, we get $a_*^2=a_{max}^2 \approx 1/\Omega_2^{1/2} \gg 1$

Therefore, the Universe's expansion is stopped when the scale factor achieves its maximal value a_{max} and then replaced by contraction.

This is a turning point!

Thus, a root (if exists) of the Condition 1 gives a maximal value $a_*=a_{max}(\Omega_0,\Omega_2,\Omega_3,\Omega_4,\zeta)$ which does generally depend on *all* parameters of the model.

Turning points and bounces: $\Omega_2 > 0$ (positive spatial curvature)

Condition 2: $\boxed{1 - \dfrac{3\zeta\Omega_2}{a_*^2} = 0} \quad \longrightarrow \quad a_*^2 = 3\zeta\Omega_2$

Since $\Omega_2 \ll 1$ and $\zeta \ll 1$, we get $a_*^2 = a_{min}^2 \ll 1$

Therefore, the Universe's contraction is stopped when the scale factor achieves its minimal value $a_{min} = (3\zeta\Omega_2)^{1/2}$.

NOTICE:

- The value $a_{min} = (3\zeta\Omega_2)^{1/2}$ depends ONLY on the product $\zeta\Omega_2$, and does NOT depend on Ω_0, Ω_3, Ω_4!
- Following [a], we may say that the cosmological constant and material substance are *screened* at the early stage and makes no contribution to the universe evolution.

[a] A. A. Starobinsky, S. V. Sushkov, and M. S. Volkov, *The screening Horndeski cosmologies*, *JCAP* **1606** (2016), no. 06 007

Bounce solution

Let us consider an asymptotic behavior of h near $a = a_* \equiv (3\zeta\Omega_2)^{1/2}$:

$$9\zeta h^2 \approx 2\frac{\Delta a}{a_*} - 3\left(\frac{\Delta a}{a_*}\right)^2 + 4\left(\frac{\Delta a}{a_*}\right)^3 + \ldots,$$

where $\Delta a = a - a_*$.
Integrating, we obtain

$$a(\tau) = a_{min}\left(1 + \frac{\Delta\tau^2}{18\zeta}\right) + O(\Delta\tau^4),$$

$$h(\tau) = \frac{\Delta\tau}{9\zeta} + O(\Delta\tau^3),$$

where $a_{min} = a_*$, $\Delta\tau = \tau - \tau_*$, and τ_* is a constant of integration.

Evidently: $a(\tau) \to a_{min}$ and $h(\tau) \to 0$ as $\Delta\tau \to 0$, i.e. $\tau \to \tau_*$.

NOTICE: The spacetime geometry is regular when approaching to the "bounce" a_{min}!

Bounce solution

Is the point $a_*^2 = a_{min}^2 = 3\zeta\Omega_2$ a bounce?

Scalar field equation:

$$\phi' = \frac{q}{a^3\left(1 - 3\zeta\left(h^2 + \frac{\Omega_2}{a^2}\right)\right)}$$

Asymptotics:

$$\phi' \approx \frac{27\zeta q}{2a_{min}^3\Delta\tau^2}.$$

Thus, one has $\phi' \propto 1/\Delta\tau^2 \to \infty$ as $\Delta\tau \to 0$, i.e. $\tau \to \tau_*$.

NOTICE: One has a singular behavior of the scalar field when approaching to the "bounce" a_{min}.

A 'singular' bounce!

Screening properties of Horndeski model

The first integral of the scalar field equation:

$$a^3\left(3\eta\left(H^2+\frac{K}{a^2}\right)-\varepsilon\right)\psi=Q,$$

where Q is the Noether charge associated with the shift symmetry $\phi\to\phi+\phi_0$.

Let $Q=0$. One finds in this case two different solutions:

GR branch: $\psi=0 \quad\Longrightarrow\quad H^2+\dfrac{K}{a^2}=\dfrac{\Lambda+\rho}{3M_{\rm Pl}^2}$

Screening branch: $H^2+\dfrac{K}{a^2}=\dfrac{\varepsilon}{3\eta} \quad\Longrightarrow\quad \psi^2=\dfrac{\eta\,(\Lambda+\rho)-\varepsilon\,M_{\rm Pl}^2}{\eta\,(\varepsilon-3\eta\,K/a^2)}$

NOTICE: The role of the cosmological constant in the screening solution is played by $\varepsilon/3\eta$ while the Λ-term is screened and makes no contribution to the universe acceleration.

Note also that the matter density ρ is screened in the same sense.

Cosmological scenarios

The case $\zeta \neq 0$ and $\Omega_3 = \Omega_4 = 0$ (no matter)

Master equation:

$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

The early time universe evolution (the limit $a \to 0$)

Asymptotics:

$$h^2 = -\frac{\Omega_2}{3a^2} + \left(\frac{1}{9\zeta} - \frac{8\zeta\Omega_2^3}{27\Omega_6}\right) + O(a^2)$$

- First two major terms in the asymptotic (37) do not contain the cosmological constant Ω_0!
- Following [1], we may say that the cosmological constant is *screened* at the early stage and makes no contribution to the universe evolution.

[1] A. A. Starobinsky, S. V. Sushkov, and M. S. Volkov, *The screening Horndeski cosmologies*, *JCAP* **1606** (2016), no. 06 007

Cosmological scenarios
The case $\zeta \neq 0$ and $\Omega_3 = \Omega_4 = 0$ (no matter)

Zero spatial curvature ($k = 0$, $\Omega_2 = 0$):

$$h^2 = \frac{1}{9\zeta} + O(a^6)$$

- **Therefore** at early cosmological times one has an *eternal* ($t \to -\infty$) inflation with the quasi-De Sitter behavior of the scale factor: $a(t) \propto e^{H_\eta t}$, where $H_\eta = 1/\sqrt{9\eta}$.
- **Notice:** that the primary inflationary epoch is only driven by non-minimal derivative or *kinetic* coupling between the scalar field and curvature without introducing any fine-tuned potential, and so one can call this epoch as a *kinetic* inflation.

Cosmological scenarios

The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Negative spatial curvature ($k = -1$, $\Omega_2 < 0$):

$$h^2 = \frac{|\Omega_2|}{3a^2} + \left(\frac{1}{9\zeta} + \frac{8\zeta|\Omega_2|^3}{27\Omega_6} \right) + O(a^2).$$

- The Hubble parameter h has a *singular* behavior at $a \to 0$, so that $h^2 \approx |\Omega_2|/3a^2 \to \infty$
- As a increases, the first term in the asymptotic (**??**) decreases and becomes negligible with respect to the second one. As the scale factor a grows further, the behavior of Hubble parameter is determined by the second term in (**??**), so that $h^2 \approx h_{dS}^2 = \frac{1}{9\zeta} + \frac{8\zeta|\Omega_2|^3}{27\Omega_6}$ and $a(t) \propto e^{h_{dS}(H_0 t)}$. This stage can be called as a *quasi-de Sitter era* with the de Sitter parameter h_{dS}.

Cosmological scenarios

The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Positive spatial curvature ($k = +1$, $\Omega_2 > 0$):

$$h^2 = -\frac{\Omega_2}{3a^2} + \left(\frac{1}{9\zeta} - \frac{8\zeta\Omega_2^3}{27\Omega_6}\right) + O(a^2).$$

- There exists some small minimal value of $a = a_{min}$,

$$a_{min}^2 = 3\zeta\Omega_2\,,$$

such that the value of h^2 becomes to be zero!!!

- A moment t_B when the Hubble parameter h, or $\dot{a}$, equals to zero is a turning point in the universe evolution, or a *bounce*, when the stage of contraction is changing to expansion one.
- The minimal size of the universe can be estimated as follows

$$\mathrm{a}_{min} = \sqrt{3}\,\ell,$$

where ℓ is the characteristic scale of nonminimal derivative coupling.

Cosmological scenarios
The case $\zeta \neq 0$ and $\Omega_3 = \Omega_4 = 0$

Master equation:

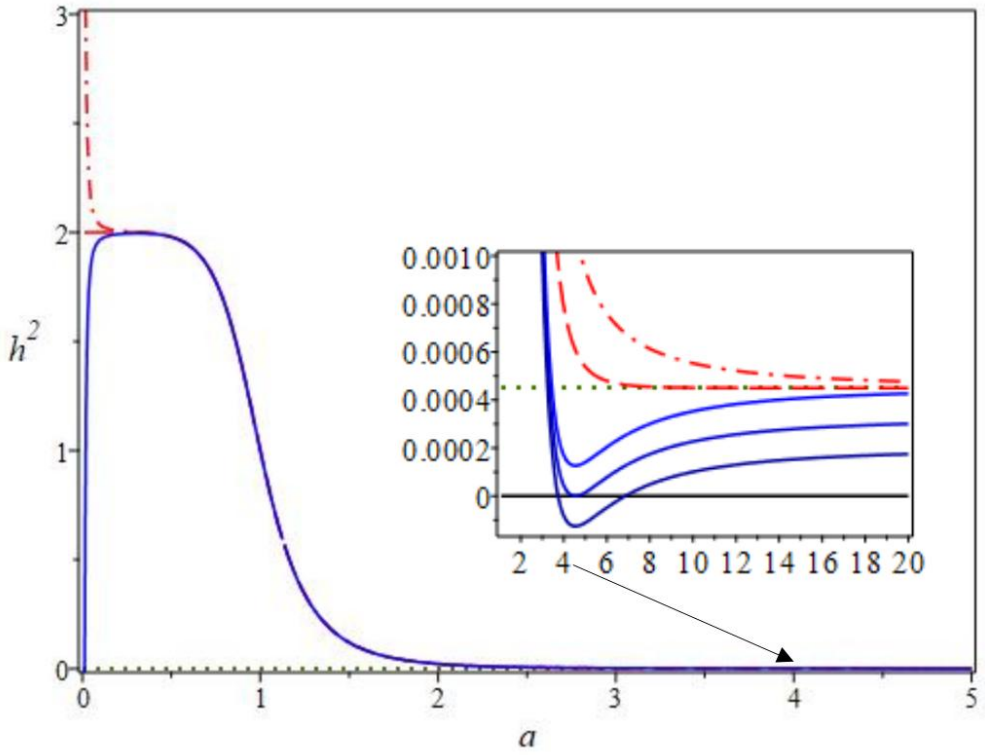
$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

The late time universe evolution (the limit $a \to \infty$)

- In the case $\Omega_2 \leq 0$, at the late stage of evolution the universe enters a *secondary inflation epoch* with $h^2 = \Omega_0$, i.e. $H = H_\Lambda = \sqrt{\Lambda/3}$.
- In the case $\Omega_2 > 0$, the squared Hubble parameter has an extremal value h^2_{extr} such that $d(h^2)/da = 0$. In case $h^2_{extr} > 0$ one has the inflationary asymptotic $h^2 = \Omega_0$. In case $h^2_{extr} \leq 0$, there is a turning point in the universe evolution, when the expansion stage is changing to contraction one.
- In the last case one has a *cyclic scenario* of the universe evolution.

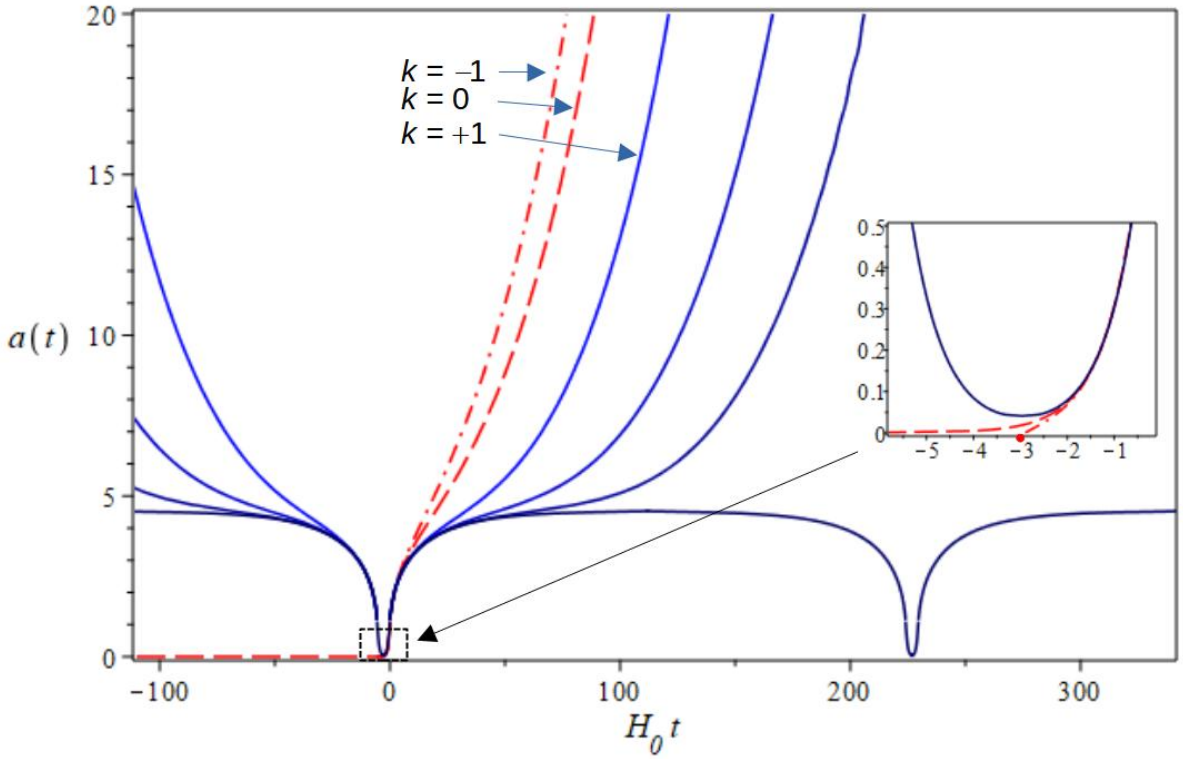
Examples of numerical solutions. I.
The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Plots of h^2 versus a



Examples of numerical solutions. II.
The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Plots of a versus t



Role of anisotropy

Notice: Small anisotropy of the universe observed today could be catastrophically large on early stages of the universe evolution. Therefore the results obtained for isotropic cosmological models may not be valid!

Anisotropic cosmologies: Bianchi I model. I

The Bianchi I metric

$$ds^2 = -dt^2 + a_1^2\,dx_1^2 + a_2^2\,dx_2^2 + a_3^2\,dx_3^2\,,$$

where $a_i = a_i(t)$ and $\phi = \phi(t)$

Let us use **the standard parametrization**:

$$a_1 = a e^{\beta_+ + \sqrt{3}\beta_-}, \quad a_2 = a e^{\beta_+ - \sqrt{3}\beta_-}, \quad a_3 = a e^{-2\beta_+}$$

$\sigma^2 = \dot\beta_+^2 + \dot\beta_-^2$ is the *anisotropy parameter*, and $H = \dot a/a$

Field equations:

$$3M_{\rm Pl}^2(H^2 - \sigma^2) = \tfrac{1}{2}\left(1 - 9\eta\left(H^2 - \sigma^2\right)\right)\dot\phi^2 + \Lambda,$$

$$\frac{d}{dt}\left[a^3\dot\beta_\pm(2M_{\rm Pl}^2 + \eta\dot\phi^2)\right] = 0,$$

$$\frac{d}{dt}\left[a^3\left(3\eta\left(H^2 - \sigma^2\right) - 1\right)\dot\phi\right] = 0.$$

Anisotropic cosmologies: Bianchi I model. II

Anisotropy parameter:

$$\sigma^2 = \frac{C^2}{a^6(2M_{\rm Pl}^2 + \eta\dot{\phi}^2)^2}$$

Asymptotic behavior of anisotopy:

As expected, at late times anisotropy is *damping* in the usual way

$$a \to \infty \qquad \Longrightarrow \qquad \sigma^2 \sim a^{-6} \to 0$$

Suprisingly, unlike GR, anisotropy is *screened* at early times!

$$a \to 0, \;\; \dot{\phi}^2 \sim a^{-6} \qquad \Longrightarrow \qquad \sigma^2 \sim a^6 \to 0$$

Therefore, contrary to what one would normally expect, the early state of the Universe in the theory cannot be anisotropic!

Anisotropic cosmologies: Bianchi IX model. I

The Bianchi IX metric

$$ds^2 = -dt^2 + \tfrac{1}{4}a_1^2\,\omega_1 \otimes \omega_1 + \tfrac{1}{4}a_2^2\,\omega_2 \otimes \omega_2 + \tfrac{1}{4}a_3^2\,\omega_3 \otimes \omega_3\,,$$

where ω_a are 1-forms, $d\omega_a = \varepsilon_{abc}\,\omega_b \wedge \omega_c$

Parameterization: $a_1 = ae^{\beta_+ + \sqrt{3}\beta_-}$, $a_2 = ae^{\beta_+ - \sqrt{3}\beta_-}$, $a_3 = ae^{-2\beta_+}$

$a^3 = a_1 a_2 a_3$ — a volume;
$H = \dot{a}/a$ — an 'average' Hubble parameter
$\beta_\pm$ parameterize deviation from isotropy
$\sigma^2 = \dot{\beta}_+^2 + \dot{\beta}_-^2$ — an anisotropy parameter
$\mathcal{H}^2 = H^2 - \sigma^2$ where $\mathcal{H}$ is an 'anisotropic' Hubble parameter

The effective spacial curvature $\mathcal{K}$: if $\beta_\pm = 0$, then $\mathcal{K} = 1$

$$\mathcal{K} = -\frac{1}{3}e^{-8\beta_+}\left(4e^{6\beta_+}\cosh^2(\sqrt{3}\beta_-) - 1\right)\left(4e^{6\beta_+}\sinh^2(\sqrt{3}\beta_-) - 1\right)$$

Field equations:

$$3M_{\rm Pl}^2\left(\mathcal{H}^2+\frac{\mathcal{K}}{a^2}\right)=\frac{1}{2}\dot\phi^2+\Lambda-\frac{9}{2}\eta\,\dot\phi^2\left(\mathcal{H}^2+\frac{\mathcal{K}}{3a^2}\right), \quad (7)$$

$$\frac{1}{a^2}\frac{d}{dt}\left[(2M_{\rm Pl}^2+\eta\dot\phi^2)a\dot a\right]=(2M_{\rm Pl}^2+\eta\dot\phi^2)(\tfrac{1}{2}\mathcal{H}^2-\sigma^2)$$

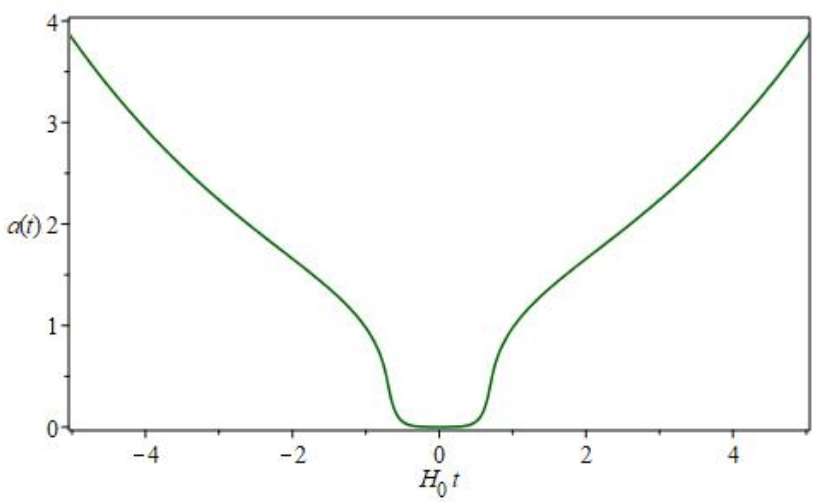
$$-(M_{\rm Pl}^2-\tfrac{1}{2}\eta\dot\phi^2)\frac{\mathcal{K}}{a^2}-\frac{1}{2}\dot\phi^2+\Lambda, \quad (8)$$

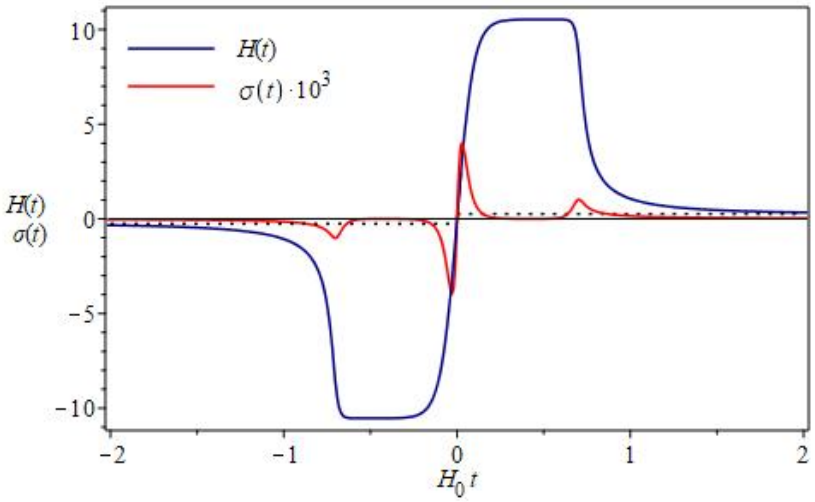
$$\frac{d}{dt}\left[a^3\dot\beta_\pm(2M_{\rm Pl}^2+\eta\dot\phi^2)\right]=a(M_{\rm Pl}^2-\tfrac{1}{2}\eta\dot\phi^2)\frac{\partial\mathcal{K}}{\partial\beta_\pm}, \quad (9)$$

$$\frac{d}{dt}\left[a^3\dot\phi\left(1-3\eta\left(\mathcal{H}^2+\frac{\mathcal{K}}{a^2}\right)\right)\right]=0. \quad (10)$$

Anisotropic cosmologies: Bianchi IX model. III

Numerical solution:





Notice: Contrary to the Belinskii-Khalatnikov-Lifshits mechanism of oscillatory approaching to the singularity, the anisotropy tends to zero at the moment of the bounce!

Concluding remarks

- We have explored bounce scenarios in the framework of homogeneous and isotropic cosmological models with arbitrary spatial curvature in the theory of gravity with non-minimal derivative coupling.
- In general, the model depends on five independent dimensionless parameters: the coupling parameter ζ, and density parameters Ω_0, Ω_2, Ω_3, Ω_4.
- A bounce cosmological scenario is most general for the homogeneous and isotropic cosmological model with positive spatial curvature ($\Omega_2 > 0$, $k = +1$).
- Near the bounce one has $a(\tau) \approx a_{min}\left(1 + \Delta\tau^2/18\zeta\right) \to a_{min}$ and $h(\tau) \approx \Delta\tau/9\zeta \to 0$ as $\Delta\tau \to 0$. Therefore, the spacetime geometry is regular when approaching to the bounce.
- However, the scalar field diverges near the bounce as follows: $\phi' \propto 1/\Delta\tau^2 \to \infty$ as $\Delta\tau \to 0$.
- Therefore, we can term this scenario as a **'singular' bounce**.

Conclusions (Continuation. . .)

- The cosmological constant Λ (or Ω_0) turns out to be *screened* at early times and makes no contribution to the universe evolution
- Depending on model parameters, there are three qualitatively different initial state of the universe: an *eternal kinetic inflation*, an *initial singularity*, and a *singular bounce*.
- For *all* types of spatial geometry, we found that the universe goes inevitably through the *primary quasi-de Sitter* (inflationary) epoch with the de Sitter parameter $h^2_{dS} = \frac{1}{9\zeta}$.
- For $k = 0$ this epoch lasts eternally to the past, when $t \to -\infty$. When $k = -1$ or $+1$, the primary inflationary epoch starts soon after a birth of the universe from an initial singularity, or after a bounce, respectively.
- The mechanism of primary or *kinetic* inflation is provided by non-minimal derivative coupling and needs *NO* fine-tuned potential.

Conclusions (Continuation. . .)

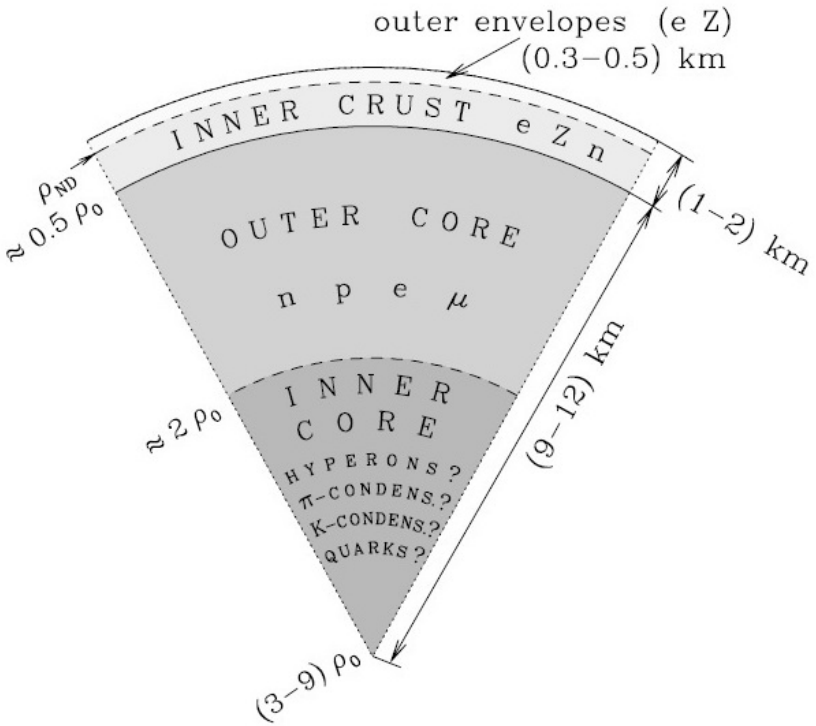
- In the course of cosmological evolution the domination of η-terms is canceled, and this leads to a *change* of cosmological epochs.
- The late-time universe evolution depends both on k and Λ. In the case $k = 0$ (zero spatial curvature), or $k = -1$ (negative spatial curvature), at late times the universe enters an epoch of *accelerated expansion* or a secondary inflationary epoch with $H = H_\Lambda = \sqrt{\Lambda/3}$. In case $k = +1$ (positive spatial curvature), there is a *turning point* in the universe evolution, when the expansion stage is changing to contraction one.
- Depending on model parameters, there are *cyclic scenarios* of the universe evolution with the non-singular bounce at a minimal value of the scale factor, and a turning point at the maximal one.
- Contrary to what one would normally expect, anisotropy is *dumped* at early stages of the universe evolution!

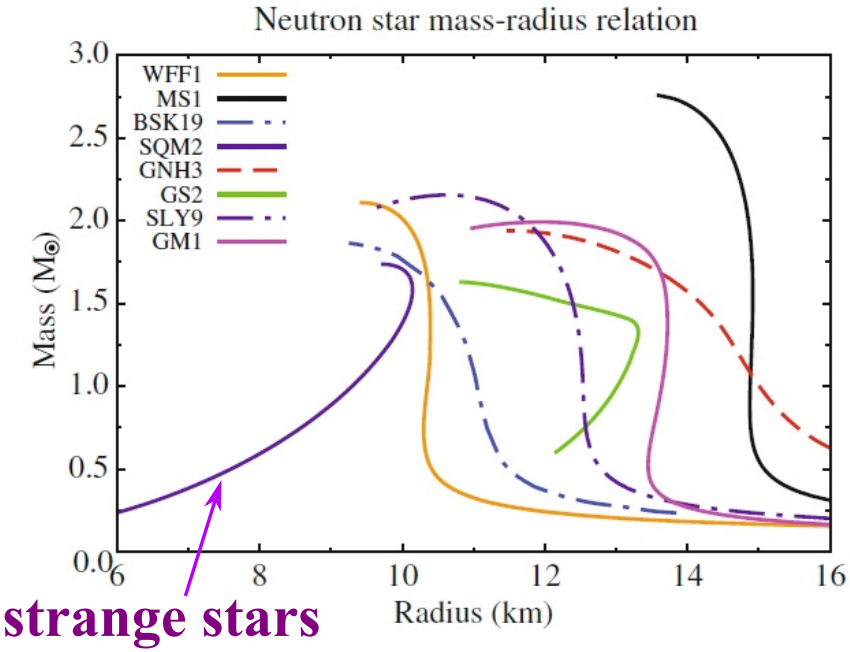
Compact objects: Black holes, wormholes, neutron stars

- P.E. Kashargin, S.V. Sushkov, Anti-de Sitter neutron stars in the theory of gravity with nonminimal derivative coupling, arxiv:2205.08949 (2022)
- R.V. Korolev, F.S.N. Lobo, S.V. Sushkov, General constraints on Horndeski wormhole throats, Physical Review D, 2020, 101(12), 124057
- R.V. Korolev, F.S.N. Lobo, S.V. Sushkov, Kinetic gravity braiding wormhole geometries, Phys. Rev. D 102, 104016 (2020)
- Sushkov S.V. (2017) Horndeski Wormholes. In: Lobo F. (eds) Wormholes, Warp Drives and Energy Conditions. Fundamental Theories of Physics, vol 189. Springer, Cham
- R.V. Korolev and S.V. Sushkov, Exact wormhole solutions with nonminimal kinetic coupling, Physical Review D 90, 124025 (2014)
- S.V. Sushkov, R. Korolev, Scalar wormholes with nonminimal derivative coupling, Classical Quantum Gravity 29, 085008 (2012)

Neutron stars

Schematic structure of a neutron star[1] **Neutron star mass-radius relation**[2]





The available data suggests that the most neutron stars have masses close to 1.3 – 1.4 M_{sun} and characteristic star radii is 10 – 13 km, but lower and higher masses exist and this ranges can be extended[3].

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[1] Neutron Stars: Equation of State and Structure, P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, Astrophysics and Space Science Library, (Springer, New York, 2007).

[2] The Physics and Astrophysics of Neutron Stars, ed L. Rezzolla et al. Springer, Cham **457** (2018).

Neutron star configuration

Action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\varkappa}(R - 2\Lambda_0) - \frac{1}{2}\left(\varepsilon_1 g^{\mu\nu} + \varepsilon_2 \ell^2 G^{\mu\nu}\right)\phi_\mu\phi_\nu\right] + S^{(m)}$$

where $\varepsilon_i = \pm 1$, ℓ is a characteristic length, Λ_0 is a bare cosmological constant.

Static spherically symmetric configuration:

$$ds^2 = -A(r)c^2dt^2 + \frac{dr^2}{B(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$\phi = \phi(r)$$

Gravitational field equations:

$$\frac{1}{\varkappa}\left(-\frac{B'}{r}+\frac{1-B}{r^2}\right)=\epsilon+\frac{1}{\varkappa}\Lambda_0+\frac{1}{2}\varepsilon_1 B\psi^2-\varepsilon_2\ell^2\frac{B\psi^2}{2r^2}\left(1+B+3rB'+4rB\frac{\psi'}{\psi}\right),$$

$$\frac{1}{\varkappa}\left(\frac{BA'}{rA}-\frac{1-B}{r^2}\right)=p-\frac{1}{\varkappa}\Lambda_0+\frac{1}{2}\varepsilon_1 B\psi^2-\varepsilon_2\ell^2\frac{B\psi^2}{2r^2}\left(1-3B-3rB\frac{A'}{A}\right),$$

where a prime means a derivative with the respect of r, and $\psi=\phi'$.

Equation of the scalar field:

$$\left[\varepsilon_1 r^2-\varepsilon_2\ell^2\left(1-B-rB\frac{A'}{A}\right)\right]\psi\sqrt{AB}=Q,$$

where Q is a "scalar charge"

Equation of motion of the perfect fluid:

$$\frac{A'}{A} = -\frac{2p'}{\epsilon + p}.$$

Equation of state

$$p = K\rho_0^{\Gamma}, \quad \epsilon = \rho_0 c^2 + \frac{p}{\Gamma - 1},$$

where ρ_0 is a baryonic mass density, $\Gamma = 2$ is the adiabatic index and $K = 1.79 \times 10^5$ (CGS) is the polytropic constant [1-3].

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1. R.F. Tooper, Adiabatic Fluid Spheres in General Relativity. The Astrophysical Journal **142**, 1541-1562 (1965).

2. J.M. Lattimer, The Nuclear Equation of State and Neutron Star Masses, Annu. Rev. Nucl. Part. Sci. **62**, pp.485-515 (2012).

3. A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D 92, 044050 (2015).

Dimensionless values:

$\xi = \Lambda_0 \ell^2$, $x = \dfrac{r}{\ell}$, $\mathcal{E} = \varkappa \ell^2 \epsilon$, $\mathcal{P} = \varkappa \ell^2 p$, $\Psi^2 = \varkappa \ell^2 \psi^2$.

Normal system of field equations

$$\frac{dB}{dx} = -\frac{1}{\Delta}\Big[\Big((1+\varepsilon\xi)x^4 + (\varepsilon - 5\xi)x^2 + 2\Big)B + \Big((1-\varepsilon x^2)\mathcal{E} + 2(3-\varepsilon x^2)\mathcal{P}\Big)x^2 B$$
$$- (1-\varepsilon x^2)^2\left(2 - (\varepsilon+\xi)x^2 - x^2\mathcal{E}\right)\Big],$$

$$\frac{dA}{dx} = \frac{A(1-B-\varepsilon x^2)}{xB}, \qquad \frac{d\mathcal{P}}{dx} = -\frac{(\mathcal{E}+\mathcal{P})(1-B-\varepsilon x^2)}{2xB},$$

$$\Psi^2 = -\frac{x^2(\varepsilon - \xi + \mathcal{P})}{\varepsilon_2 B\,(1-\varepsilon x^2)}, \qquad \mathcal{E} = \varkappa\ell^2 c^2\left(\frac{\mathcal{P}}{\varkappa\ell^2 K}\right)^{1/\Gamma} + \frac{\mathcal{P}}{\Gamma - 1},$$

where $\Delta = x(1-\varepsilon x^2)\big(2 - (\varepsilon+\xi)x^2 + x^2\mathcal{P}\big)$,

$\varepsilon = \varepsilon_1/\varepsilon_2 = \pm 1$, and ξ is a free parameter.

Boundary conditions for $A(r)$, $B(r)$, $\Psi(r)$ and $\mathcal{P}(r)$:

$$\begin{aligned} p(0) = K\rho_{0c}^{\Gamma}, \quad & p'(0) = 0, \\ B(0) = 1, \quad & B'(0) = 0, \\ A'(0) = 0, \quad & A(R) = A_{out}(R), \\ \psi(0) = 0, \quad & \end{aligned}$$

where ρ_{0c} – central barionic mass density, $A_{out}(r)$ is the external vacuum solution, R is the radius of the star.

Radius R of the star:

$$\mathcal{P}(R) = 0.$$

Consequence of the boundary conditions:

$$Q = 0 \quad \Rightarrow \quad J^{\mu} = 0 \quad \Rightarrow \quad \text{no hair solution.}$$

External vacuum solution

Vacuum solution[1] in the case $\rho = 0$ and $p = 0$:

$$B(x) = \frac{(x^2+1)^2}{\left((1-\xi)x^2+2\right)^2} F(x),$$
$$A(x) = 3C_2 F(x),$$
$$\Psi^2(x) = \frac{x^2(1+\xi)}{\varepsilon_2 B (1+x^2)},$$

where C_1 and C_2 are constants of integration and

$$F(x) = (1-\xi)(3+\xi) + \frac{1}{x}\left((1+\xi)^2 \arctan x + C_1\right) + \frac{x^2}{3}(1-\xi)^2.$$

The case $\xi = -1$ (anti-de Sitter-Schwarzschild black hole):

$$A(r) = B(r) = 1 - \frac{r_g}{r} + \frac{|\Lambda_{AdS}|}{3} r^2, \quad \Psi^2(r) = 0.$$

[1] M. Rinaldi, Phys. Rev. D **86**, 084048 (2012).

External vacuum solution

Asymptotics at $x \to \infty$:

$$B(x)=\frac{x^2}{3}+\frac{7+\xi}{3(1-\xi)}+\frac{C_1+\frac{1}{2}(1+\xi)^2\pi}{(1-\xi)^2}\,\frac{1}{x}+\mathcal{O}(x^{-2}),$$

$$A(x)=3C_2(1-\xi)(3+\xi)\left[1+\frac{1-\xi}{3(3+\xi)}x^2+\frac{C_1+\frac{\pi}{2}(1+\xi)^2}{(3+\xi)(1-\xi)}\,\frac{1}{x}\right]+\mathcal{O}(x^{-2}),$$

$$\Psi^2(x)=\frac{3(1+\xi)}{\varepsilon_2 x^2}+\mathcal{O}(x^{-4}).$$

Conditions:

- t is the time of a distant observer $\Rightarrow$ $C_2=[3(1-\xi)(3+\xi)]^{-1}$;
- Metric signature $\Rightarrow$ $C_2>0$ $\Rightarrow$ $-3<\xi<1$.

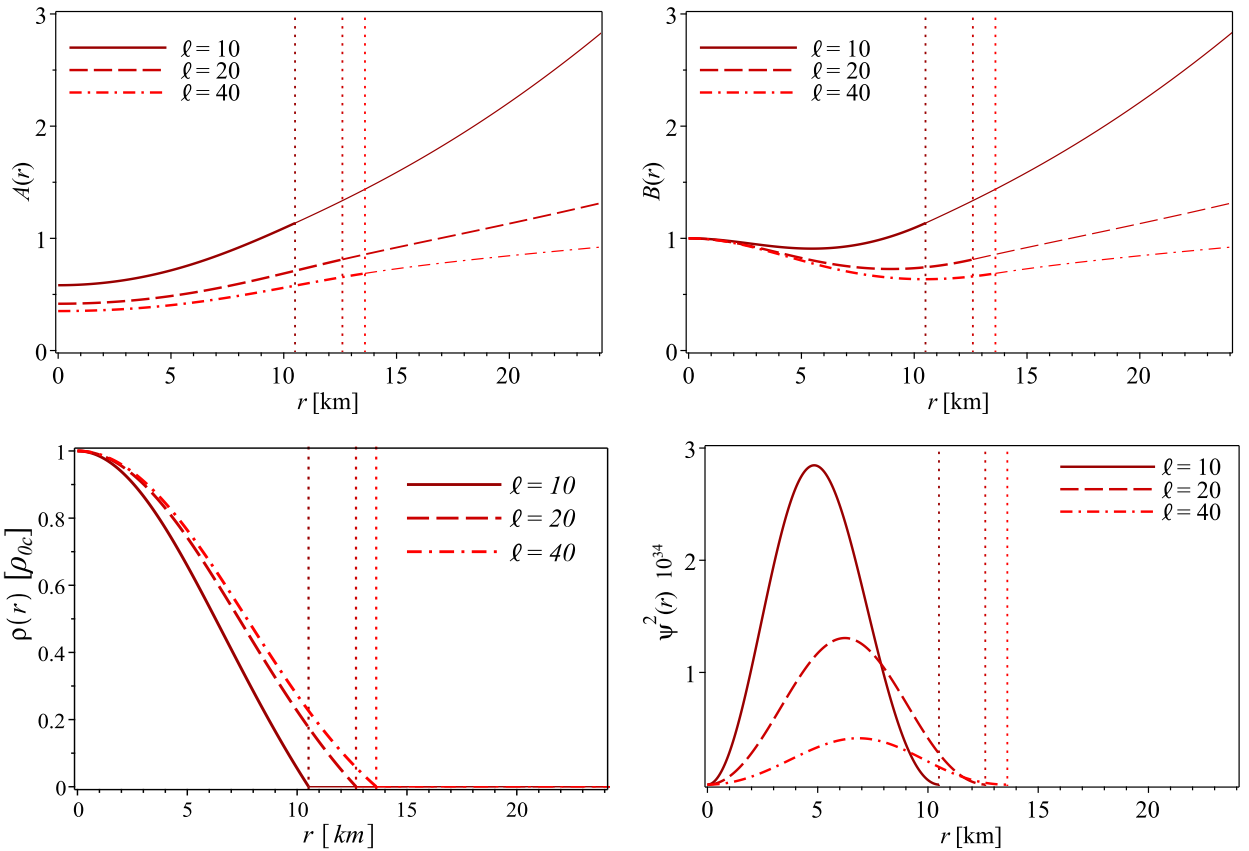
Anti-de Sitter-Schwarzschild asymptotics:

$$A(r)\approx 1-\frac{r_g}{r}+\frac{|\Lambda_{AdS}|}{3}\,r^2,$$

$$r_g=\frac{2GM}{c^2}=-\ell\,\frac{C_1+\frac{\pi}{2}(1+\xi)^2}{(3+\xi)(1-\xi)},\quad \Lambda_{AdS}=-\frac{1-\xi}{3+\xi}\,\frac{1}{\ell^2},$$

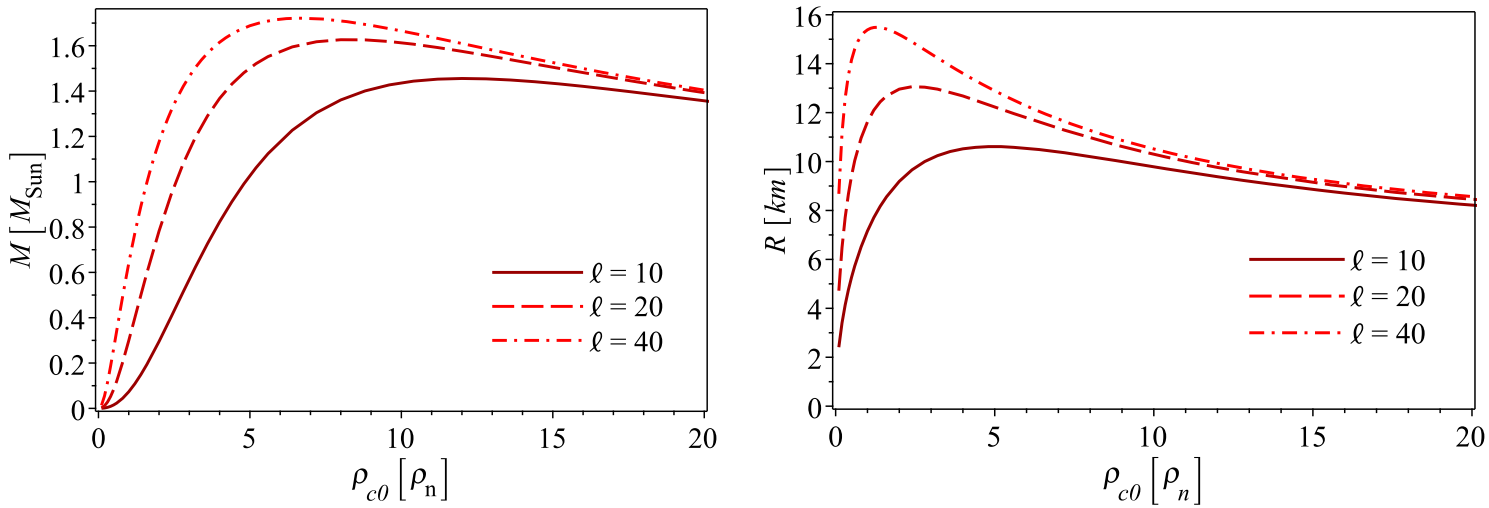
where r_g is the Schwarzschild radius, M is the Schwarzschild mass and Λ_{AdS} is the effective negative cosmological constant, $\xi\equiv\Lambda_0\ell^2$.

Numerical results: $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$



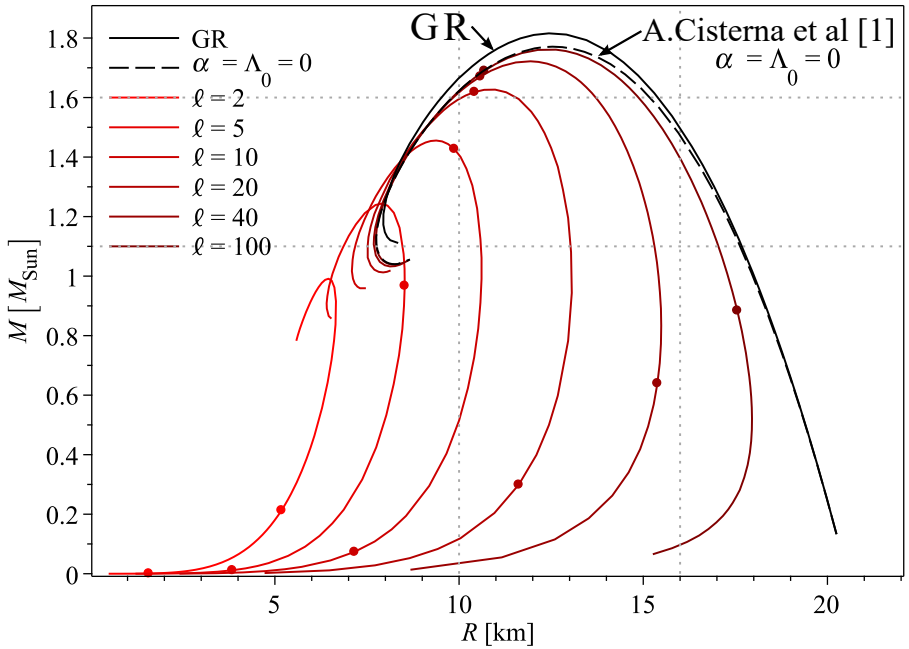
Graphs of the functions $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$ in the case $\xi = -1$ are shown for three values of the nonminimal derivative coupling parameter $\ell = 10, 20, 40\ km$ and the central baryonic mass density $\rho_{0c} = 10^{15}\ g/cm^3$. Vertical dotted lines mark the boundary of star ($\ell = 10, 20, 40$ from left to right).

Numerical results: Mass M and radius R versus the central baryonic mass density ρ_{0c}



The dependence of the mass M (left panel) and the radius R (right panel) on the central baryonic mass density ρ_{0c} in the case $\xi = -1$ is shown for three values of the nonminimal derivative coupling parameter $\ell = 10, 20, 40$ km. Values of ρ_{0c} are given in term of the nuclear density $\rho_n = 2.5 \times 10^{14}$ g/cm^3.

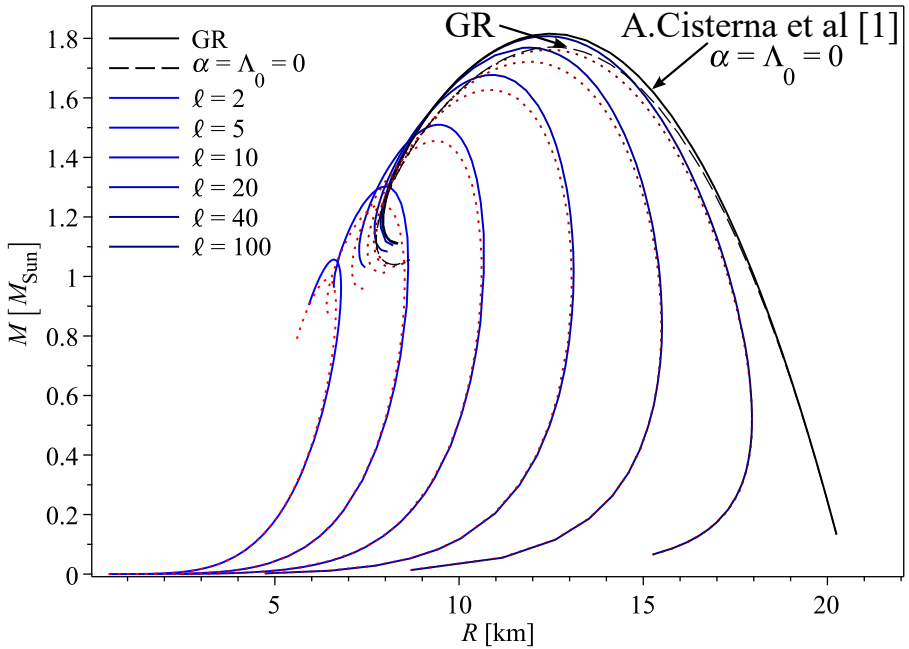
Numerical results: Mass-radius diagram



Mass-radius diagram. $\xi = -1$, $\Lambda_{AdS} = -\ell^{-2}$. Solid red curves correspond to different values of the nonminimal derivative coupling parameter $\ell = 2, 5, 10, 20, 40, 100$ km (from left to right). All graphs are built in the range of values of the central baryonic mass density ρ_{0c} from $0.1\rho_n$ to $100\rho_n$ (from bottom to up), where $\rho_n = 2.5 \times 10^{14}\ g/cm^3$ is the nuclear density. Small solid circles on the curves mark the values $\rho_{0c} = \rho_n$ and $\rho_{0c} = 10\rho_n$.

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1. A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D **92**, 044050 (2015).



Mass-radius diagram in the theory

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\varkappa}(R + 2|\Lambda_{AdS}|)\right] + S^{(m)},$$

where $\Lambda_{AdS} = -1/\ell^2$ is a negative cosmological constant.

- In the case $\xi = -1$ the external vacuum solution has a form of the Schwarzschild-anti de Sitter black hole. We have the anti-de Sitter-Schwarzschild asymptotics outside the star in the case $\xi \in (-3, 1)$.
- In the case $\xi = -1$ the mass-radius diagram has an essential property, namely, radius decreases monotonically with decreasing mass. The specific "strange" relation between mass and radius corresponds to the so-called bare strange stars, and in our case it is forming due to the effective negative cosmological constant Λ_{Ads}.
- In the case $\ell \gg 1\,km$ the mass-radius diagram approaches the results without kinetic term of scalar field in the Lagrangian obtained earlier by A.Cisterna et al.

Concluding remarks

- Particular subclasses of Horndeski theory, such as the model with nonminimal derivative coupling, provide new interesting features of cosmological and astrophysical models:
  - *Essentially new* inflationary mechanism which does not need any fine-tuned potential
  - *Screening properties*, when cosmological anisotropy is screened at early times
  - Forming of *"strange" neutron stars*, when the star's radius decreases monotonically with decreasing mass

THANKS FOR YOUR ATTENTION!