On a Generic 2-Qubit Entangler

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About What my Talk is ?

Notion of the quantum entanglers in quantum computing:

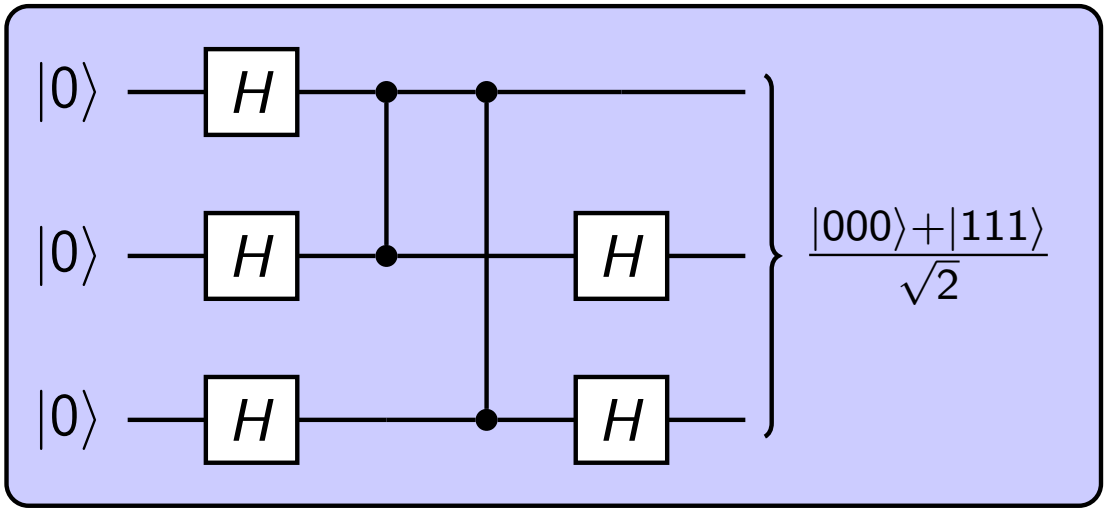
> *Entanglers are two-qubit or multi-qubit quantum logic gates capable of transforming independent, unentangled qubit states into inseparable, entangled states.*

I am aiming to specify one aspect of the above notion:

- How one can describe a variety of an admissible entanglers ?

Generating 2-Qubit Entangled States

It is well known how to generate specific entangled states, e.g.



Bell State Generation

However for certain problems, say Variational Quantum Eigensolver (VQE), we need to manipulate states with a set of parameters that are optimized in a way providing maximal entanglement capacity.

We need to implement gates with a certain entanglement power on a quantum computer.

What Google AI thinks about Entanglers

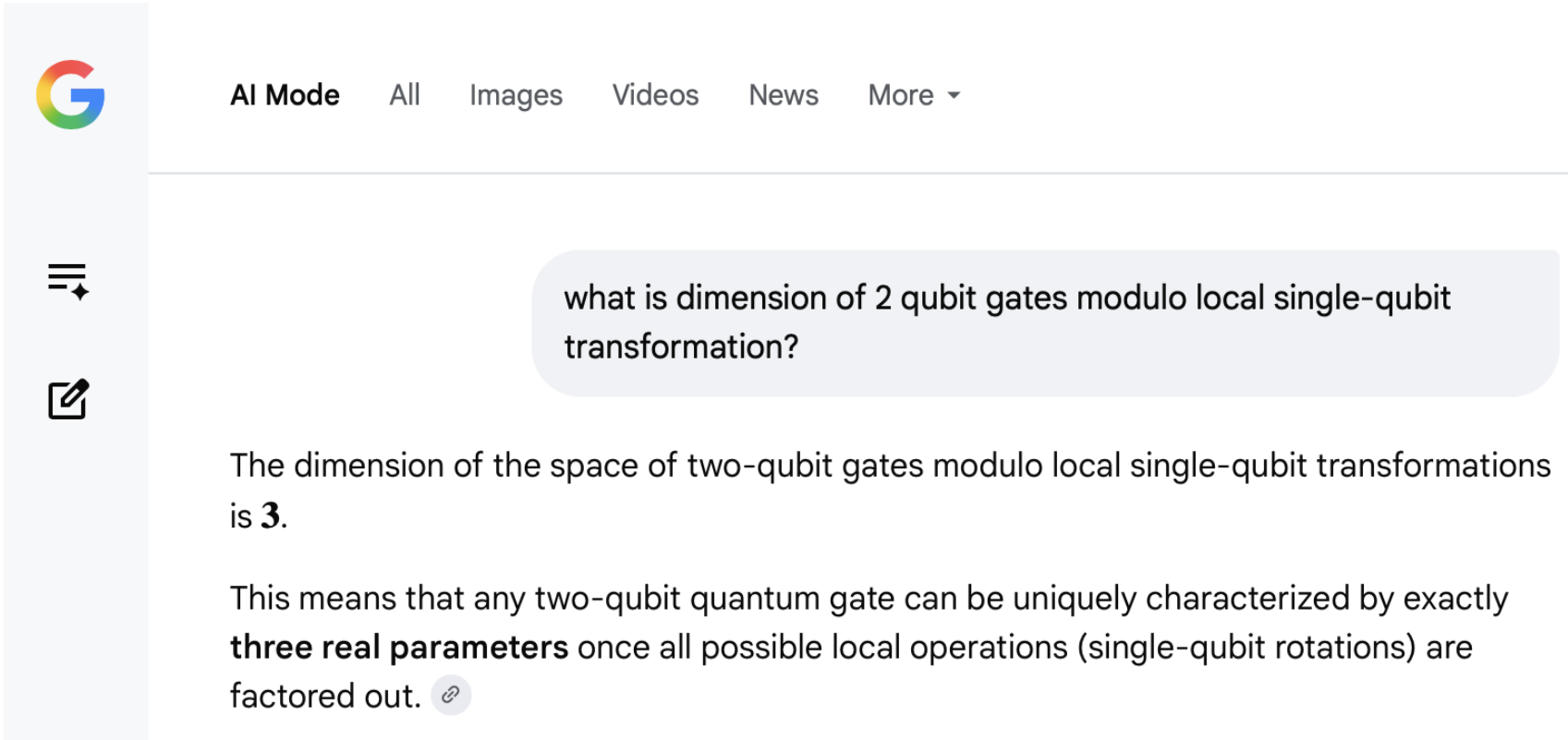


Figure 1: Google AI answer about the dimension of entanglers of 2 qubits

Why it is Interesting in Quantum Computing

We are living in the era of Quantum Advantage (2025-p.d.), when a prevailed tendency in quantum computing is towards a solving of problems inspired and motivated by practical needs.

A basic working strategy includes:

- Identify a quantum resource;
- Develop efficient classical-quantum algorithms;
- Implement on classical and quantum computer.

The Task

RESOURCE: ENTANGLEMENT

Accumulation and Production in Quantum Circuits.

- Entanglement of States;
- Entanglement Production by Gates.

Three Parts of the Tackled Problem

I. A Unified view on the Entanglement of States & Entanglers;

II. Math behind the Entanglement;

III. Implementing resources in quantum computing.

Discussion on this Conference

II. Parameterization of the unitary group adapted to the Hilbert space decomposition in tensor product corresponding to a composite system.

Report: *D.Mladenov*

III. Construction of a generic 6-parametric entangler using the adaptive decomposition of the SU(4) group;

Report: *A.Bogolubskaya, Y.Palii, I.Rogojin, and K.Verkhovtseva*

Quantum Resources

A core of the Entanglement in Composite Systems:

Superposition & Tensor Product of States

- Phenomenon of Entanglement of States – specific correlations between parts of a quantum composite system that have no analogue in its classical counterpart;
- Generalized Statistical Description – relaxing Kolmogorov's probability axioms exploiting the quasiprobability distributions.

Entanglement of States:

Let $\mathfrak{P}_N$ be the state space of an N-dimensional bipartite system composed of two subsystems: $\mathcal{H}_{A\times B} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$.
Two subsets of $\mathfrak{P}_N$:

- the family of separable mixed states $\mathfrak{S}_N \subset \mathfrak{P}_N$ given by a convex combination of tensor products of the states of subsystems ϱ_A^k and ϱ_B^k

$$\mathfrak{S}_N : \quad \{\, \varrho_A^k \in \mathfrak{P}_{N_A},\ \varrho_B^k \in \mathfrak{P}_{N_B} \,|\, \mathrm{conv}(\varrho_A^k \otimes \varrho_B^k) \,\}\,.$$

- its complement – the set of entangled mixed states.

Symmetry and Composition

Symmetry of an elementary system:

- Global Unitary SU(N)-symmetry due to the underlining Hilbert space of N-level system $\mathcal{H}_N = \mathbb{C}^N$;

Symmetry of a composite system:

- Local Unitary symmetry due to the tensor product structure of Hilbert space of a binary AB system:

$$\mathcal{H}_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B \,.$$

The binary composite system possesses the local symmetry with respect to the Local Unitary Transformations:

$$G_L = SU(N_A) \times SU(N_B) \subset SU(N) \,.$$

Role of Two Groups

Two groups, $SU(N)$ and G_L prescribe physical properties of states;

- Invariants under G_L describe only local properties of subsystems;
- Invariants under SU(N) group describe the global characteristics of system as structureless unit but impact on the non-local properties as well.

A basic concept:

The transformation that affect on non-local characteristics of a composite system are given by SU(N) transformations modulo the local ones.

Stratification of $\mathfrak{P}_N$ by Global Unitary Group Action

The adjoint $SU(N)$-action on $\mathfrak{P}_N$ sets equivalence relations between its elements and defines its stratification:

$$\mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]} \,.$$

where

$$\mathfrak{P}_{[H_\alpha]} := \{x \in \mathfrak{P}_N |\ H_x \text{ is conjugate to } H_\alpha\} \,.$$

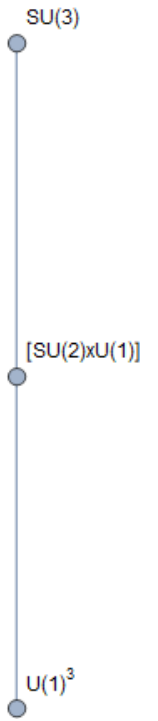
Definition: A subgroup $H_x \subset SU(N)$ is the isotropy group of a point $x \in \mathfrak{P}_N$,

$$H_x = \{g \in SU(N) \mid g \cdot x = x\} \,,$$

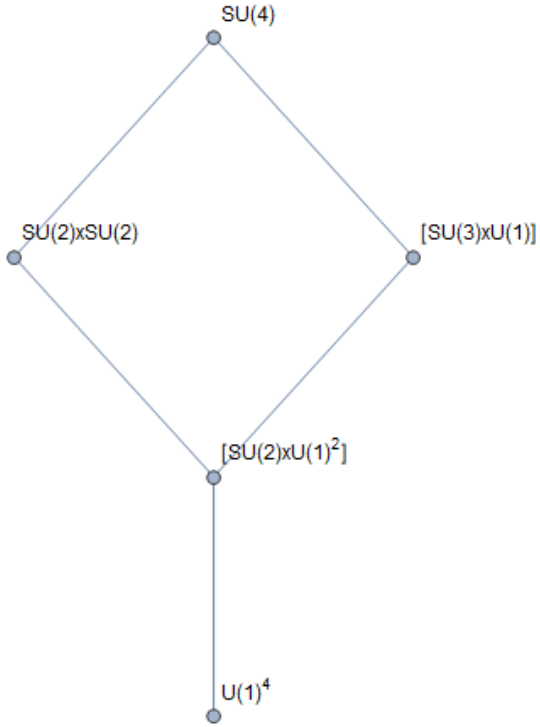
and points $x \,, y \in \mathfrak{P}_N$ are said to be of the same type if their stabilizers H_x and H_y are conjugate subgroups of $SU(N)$ group.

The Isotropy Group Classification

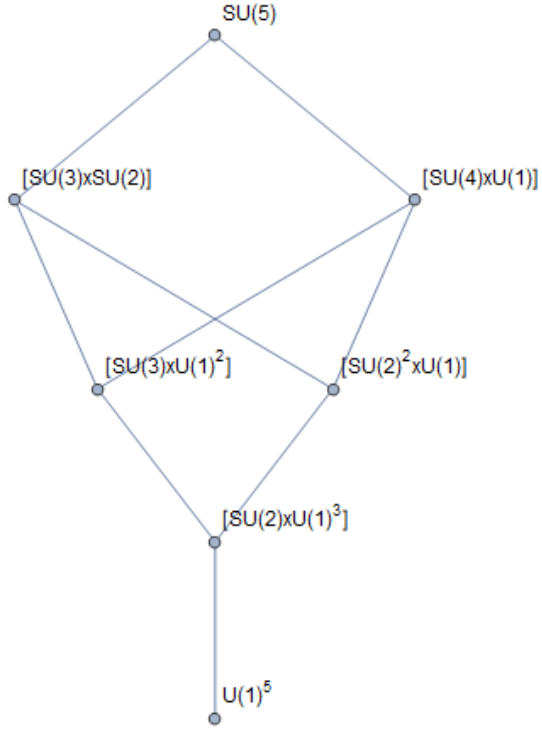
The isotropy groups H_α of N-level system with respect to SU(N) group action are classified by the Hasse diagrams.



$N = 3$



$N = 4$



$N = 5$

Entanglement Space

The subgroup $G_L \in SU(N)$ acts independently on the subsystems density matrices,

$$\varrho'_{N_\alpha} = g^\dagger \varrho_{N_\alpha}\, g\,, \qquad g \in SU(N_\alpha),\ \alpha = A, B\,.$$

preserving their local characteristics. Whole information about a non-locality is accumulated in the following orbit space:

$$\mathcal{E}_{N_A \times N_B} = \mathfrak{P}_N / SU(N_A) \times SU(N_B)\,.$$

The space $\mathcal{E}_{N_A \times N_B}$ is termed as **Entanglement Space**.

N.B. $\quad \dim(\mathcal{E}_{N_A \times N_B}) = N_A^2 N_B^2 - N_A^2 - N_B^2 + 1$

Local Structure of the Entanglement Space

A local structure of bipartite entanglement space in a vicinity of state with the isotropy group H_α

$$\begin{aligned}\mathcal{E}_{N_A\times N_B} &= \bigcup_{\text{orbit type}\,[\mathrm{H}_\alpha]} \mathcal{E}_{N_A\times N_B}[H_\alpha] \\ &= \bigcup_{\text{orbit type}\,[\mathrm{H}_\alpha]} \Delta_N^{(\alpha)} \times G_L\backslash G/H_\alpha\,.\end{aligned}$$

$\Delta_N^{(\alpha)}$ - subset of eigenvalues simplex Δ_{N-1} with a given degeneracy.

Double coset as Space of Entanglers

Definition of entanglers:

For a binary composite $N = N_A N_B$-dimensional quantum system one can introduce a family of entanglers defined as a set of unitary transformations forming the double coset

$$G_L \backslash SU(N) / H_\alpha \, .$$

Entanglers are classified according to the isotropy groups H_α with respect the $SU(N)$ group conjugation.

Local structure of 2-qubit entanglement space

Consider 2-qubit entanglement space ($N = 4, N_A = N_B = 2$)

$$\mathcal{E} = \mathfrak{P}_4 / SU(2) \times SU(2)\,.$$

In the vicinity of a state ϱ consisting of maximal rank states with a non-degenerate spectrum, the entanglement space is

$$\mathcal{E}_{2\times 2}[\mathrm{T}^3] = \mathrm{Int}\Delta_3 \times \mathbb{B},$$

where T^3 - maximal torus of $SU(4)$, the factor $\mathrm{Int}\Delta_3$ - an interior of 3-simplex of ordered eigenvalues, and $\mathbb{B}$ is the 6D double coset:

$$\mathbb{B} = SU(2) \times SU(2) \backslash SU(4) / T^3\,.$$

The double coset $\mathbb{B}$ is an arena where a generic entanglers act

Math Behind the Problem

Parametric 6D Entanglers

Towards a Unique Description

A unified landscape of the entanglement characteristics of states and their associated gates.

The steps we are performing in this direction:

- Elaborating the concept of a generic entangler of 2-qubits, as a 6-parameter SU(4) gate capable of generating any state within the existing nine entanglement classes.
- Implementing the generic entangler in parameterised quantum circuits.

SU(4) Group Decomposition

Proposition: The $\mathfrak{su}(4)$ algebra admits decomposition into the direct sum:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{k}' \,,$$

where $\mathfrak{a}$ and $\mathfrak{a}'$ are Abelian subalgebras with the algebra

$$[\mathfrak{a}' \,, \mathfrak{a}] \subset \mathfrak{k} \,,$$

$\mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2) \,, \quad \mathfrak{k}' := \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \,,$ such that:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \,, \qquad [\mathfrak{k}', \mathfrak{k}'] \subset \mathfrak{k}' \,, \qquad [\mathfrak{k}, \mathfrak{k}'] \subset \mathfrak{a} \oplus \mathfrak{a}' \,,$$

The exponential map $\exp : \mathfrak{su}(4) \to SU(4)$ results in corresponding Cartan-type coordinates in vicinity of the identity,

$$g := K \exp(\mathfrak{a}) \exp\left(\mathfrak{a}'\right) T^3 \,, \qquad K \in SU(2) \times SU(2) \,,$$

where T^3 is the maximal torus in $SU(4)$.

Explicit form of the Entangler $\mathcal{A}$

Entangler $\mathcal{A} = \exp(\mathfrak{a}) \exp(\mathfrak{a}')$ is parametrized by six real parameters α_i and β_i which are eigenphases of each factor

$$\mathcal{A}_1 = \mathcal{S}_1 \, \mathrm{diag}(\boldsymbol{\alpha}) \, \mathcal{S}_1^T \,, \quad \mathcal{A}_2 = \mathcal{S}_2 \, \mathrm{diag}(\boldsymbol{\beta}) \, \mathcal{S}_2^T \,,$$

where the diagonalizing orthogonal transformations read

$$\mathcal{S}_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \; \mathcal{S}_2 = \frac{1}{2} \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathrm{diag}(\boldsymbol{\alpha}) = || e^{-\frac{i}{2}(\alpha_1+\alpha_2+\alpha_3)}, e^{\frac{i}{2}(\alpha_1+\alpha_2-\alpha_3)}, e^{\frac{i}{2}(\alpha_1-\alpha_2+\alpha_3)}, e^{\frac{i}{2}(-\alpha_1+\alpha_2+\alpha_3)} ||$$

$$\mathrm{diag}(\boldsymbol{\beta}) = || \, e^{-\frac{i}{2}(\beta_1+\beta_2+\beta_3)}, e^{\frac{i}{2}(\beta_1+\beta_2-\beta_3)}, e^{\frac{i}{2}(\beta_1-\beta_2+\beta_3)}, e^{\frac{i}{2}(-\beta_1+\beta_2+\beta_3)} ||$$

Analyticity domain of $\mathcal{A}_1$-factor

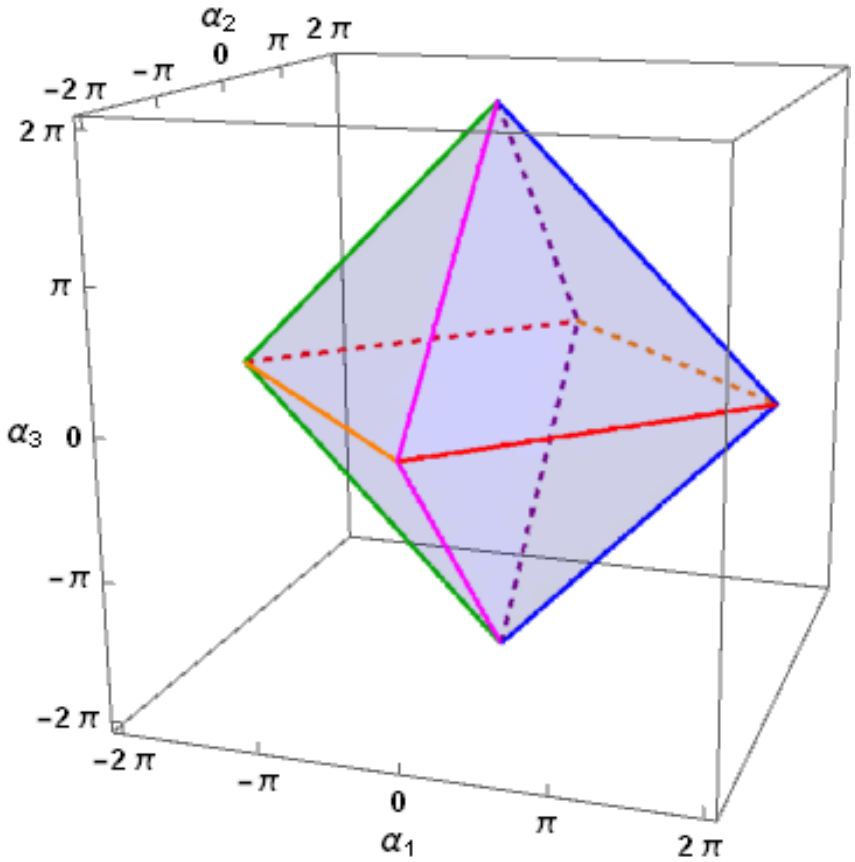


Figure 2: The regular octahedron as the domain of the analiticity of $\mathcal{A}_1$-factor in KAT-decomposition

Instead of Conclusion

Consider ensemble of a random SU(4) gates with the measure

$$\mathrm{d}\mu_{\mathrm{SU}(4)} \propto \sqrt{\det \mathrm{g}_{\mathrm{SU}(4)}} \prod_{i=1}^{15} \mathrm{d}x_i = \mathrm{d}\mu_K \mathrm{d}\mu_{T^3} \mathrm{d}\mu_{\mathcal{A}} \,.$$

Computing the joint non-local probability density function $P(\vec{\alpha}, \vec{\beta})$ of random entanglers using the Haar metric $\mathrm{g}_{\mathrm{SU}(4)}$ in KAT coordinates gives,

$$P(\vec{\alpha}, \vec{\beta}) \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta} \propto \left| \vec{V}_1 \cdot \left(\vec{V}_2 \times \vec{V}_3 \right) \right| \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta}\,,$$

where $(C_i = \cos\alpha_i\,, S_i = \sin\alpha_i,\ \ i = 1, 2, 3)$:

$$\vec{V}_1 = \frac{1}{2}\left\{(C_1 + C_3)\cos(\beta_1)\cos(\beta_2),\ \ S_3 \sin(\beta_2)\cos(\beta_3),\ \ -S_1 \sin(\beta_1)\cos(\beta_3)\right\}$$

$$\vec{V}_2 = \frac{1}{2}\left\{S_3 \sin(\beta_2)\cos(\beta_1),\ \ (C_2 + C_3)\cos(\beta_2)\cos(\beta_3),\ \ -S_2 \sin(\beta_3)\cos(\beta_1)\right\}$$

$$\vec{V}_3 = \frac{1}{2}\left\{-S_1 \sin(\beta_1)\cos(\beta_2),\ \ -S_2 \sin(\beta_3)\cos(\beta_2),\ \ (C_1 + C_2)\cos(\beta_1)\cos(\beta_3)\right\}$$

Visualization of the non-local factor of the Haar measure

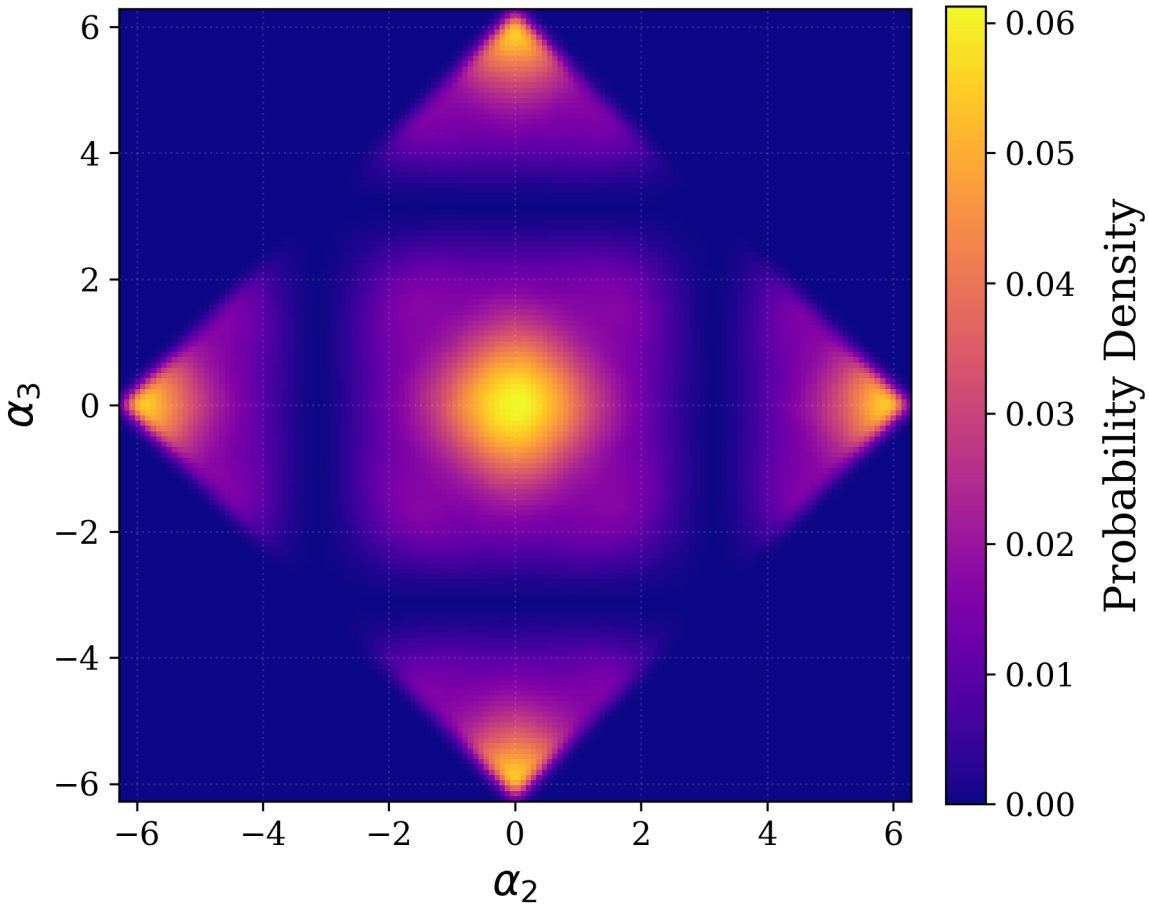


Figure 3: 2D slice of non-local PDF over (α_2, α_3) plane.

Visualization of the non-local factor of the Haar measure

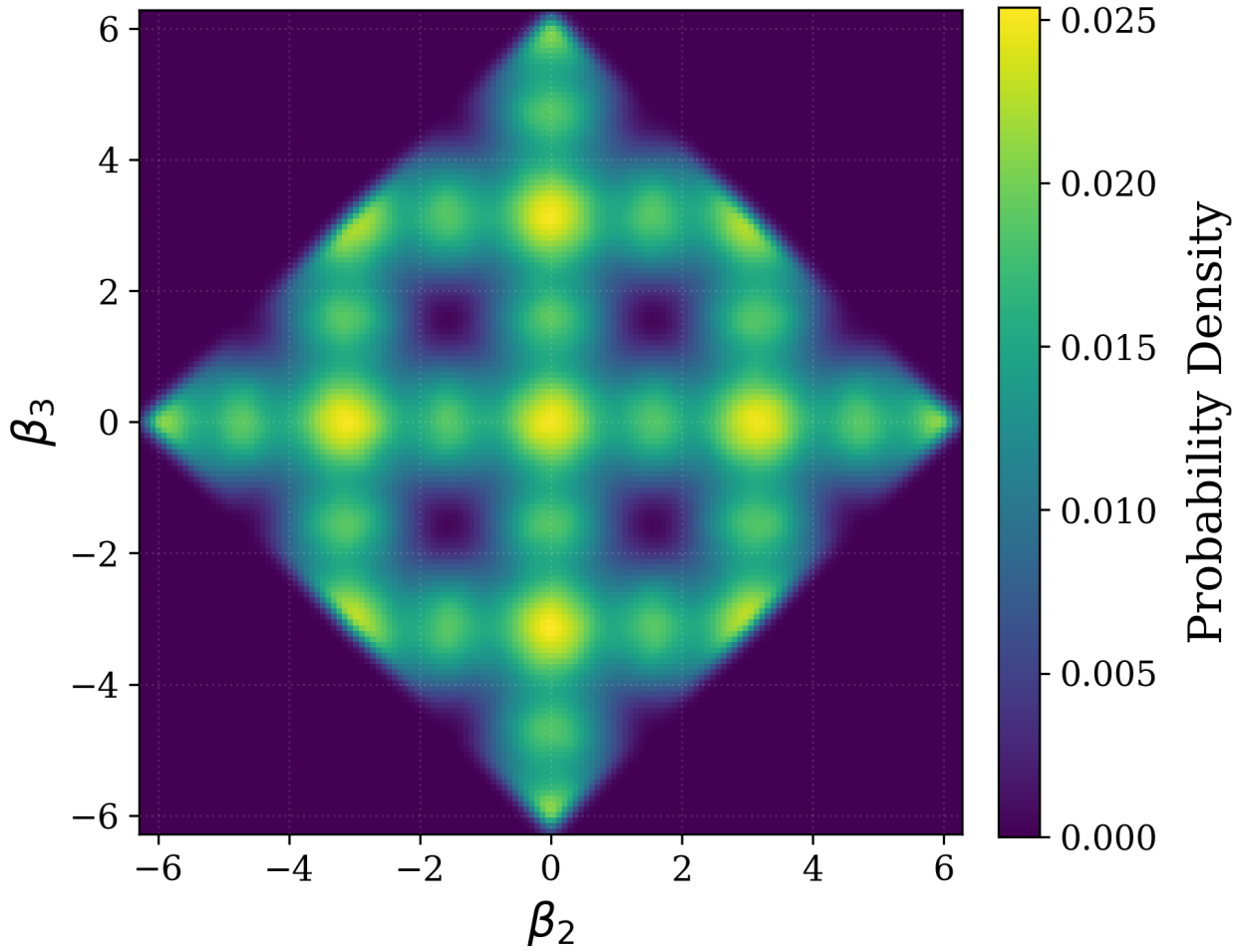


Figure 4: 2D slice of non-local PDF over (α_2, α_3) plane.

Visualization of PDF on two octahedon boundaries

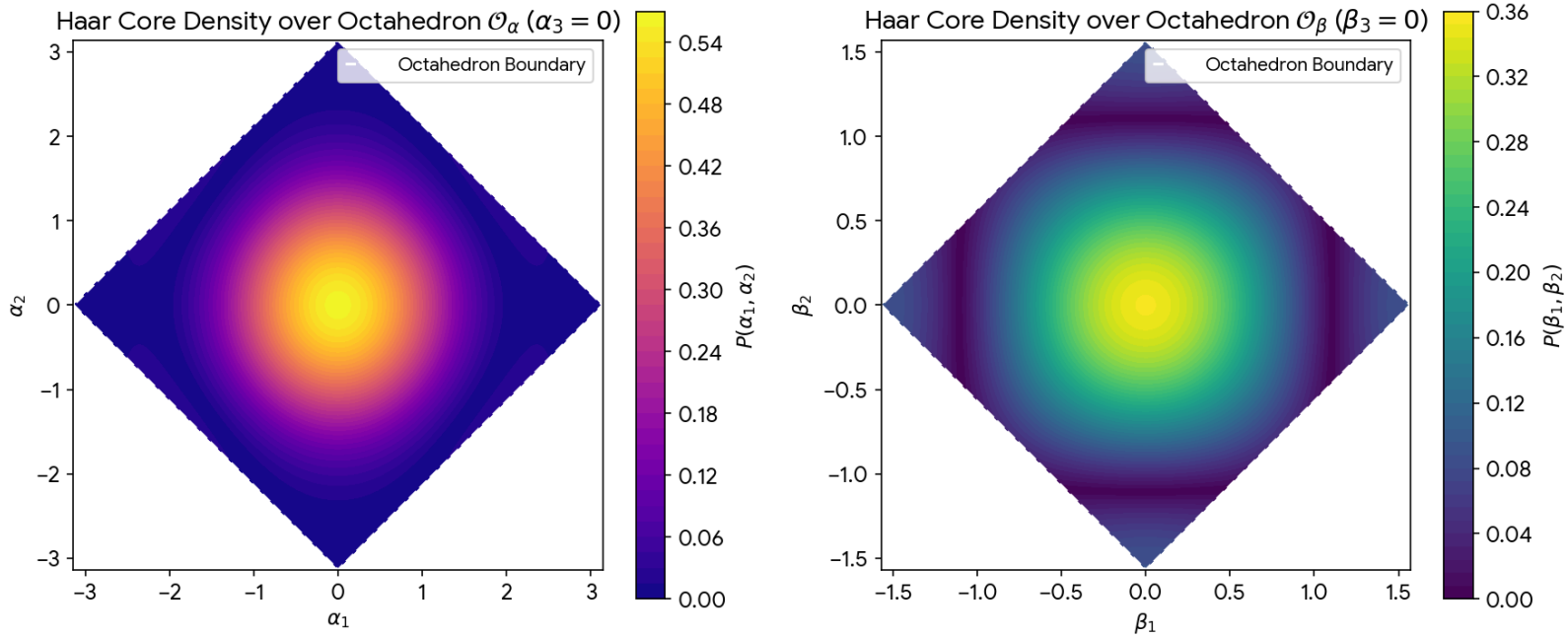


Figure 5: Marginal distributions for $\alpha_3 = 0$ and $\beta_3 = 0$.

Implementing 6D Entangler on Quantum Coprocessors

Summarising:

Information about all non-local 2 qubit correlations is encoded in the 6D entangler

$$\mathcal{A} = \exp(\mathfrak{a}) \exp(\mathfrak{a}') \,,$$

Acting on states it creates and destroys entanglement resource.

An ongoing work:

Implementation of the entangler $\mathcal{A}$ in parameterized quantum circuits on the SnowDrop4Q and SnowDrop8Q coprocessors developed by the Bauman Moscow State Technical University and the All-Russian Research Institute of Automation.

Thank you for your attention!