Towards description of the orbifold $\mathrm{SU}(2) \times \mathrm{SU}(2) \backslash \mathrm{SU}(4) / \mathbb{T}^3$ as an entanglement space of two qubits

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Mathematical Modeling and Computational Physics (MMCP'2026)
June 22 - June 26, 2026
Joint Institute for Nuclear Research, Dubna, Russia

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Definition and discussion

The state space $\mathfrak{P}_n$ of $n-$dimensional quantum system, the qudit, consists of the positive-semidefinite (PSD), a unit trace $n \times n$ Hermitian matrices

$$\mathfrak{P}_n = \{\varrho |\, \varrho \in \mathrm{Mat}(n, \mathbb{C})\ , \varrho = \varrho^\dagger\ ,\ \mathrm{tr}\, \varrho = 1\ ,\ \ \varrho \geq 0\}\,.$$

Definition and discussion

The mathematical structure of the quantum state space $\mathfrak{P}_n$ can be considered from several points of view

- **Convex set**
- **Semialgebraic variety**
- **Topological space**
- **Measurable space**
- **Riemannian space**

The state space $\mathfrak{P}_n$ as a semialgebraic variety

The semipositivity condition $\varrho \geq 0$ of the elements of quantum state space $\mathfrak{P}_n$ restricts to certain semialgebraic variety with $\dim \mathfrak{P}_n = n^2 - 1$.

Definition and discussion

This set is not a manifold, it has singularities, where even locally it differs from $\mathbb{R}^{n^2-1}$. However, the state space pertain to a class of topological spaces which represents a

"best singular sets admitting those differential structures, the manifolds possess".

Global orbit space

The adjoint action of the special unitary group $SU(n)$ on the quantum state space $\mathfrak{P}_n$ is given by

$$(g, \varrho) \to g\varrho g^\dagger\,, \qquad g \in SU(n)\,, \quad \varrho \in \mathfrak{P}_n\,.$$

If the quantum system is isolated then these transformations are called **global unitary symmetry transformations**.
The density matrices along an unitary orbit

$$\mathcal{O}_\varrho = \{g\varrho g^\dagger, \forall g \in SU(n)\}\,,$$

represent states with the same physical characteristics.

Such a degeneracy sets the equivalence relation on the state space $\mathfrak{P}_n$ and the corresponding coset, $\mathfrak{P}_n/SU(n)$ — the *SU(n)-orbit space*, accumulates all physically relevant information about the system as a whole.

Local orbit space

Let an N-dimensional quantum system represents a union of k-subsystems each with $n_1, n_2, \ldots, n_k$ levels respectively, $N = \prod_{i=1}^{k} n_i$. For this system, according to the superposition principle, the Hilbert space reads

$$\mathcal{H}_{n_1 \times n_2 \times \cdots \times n_k} \subseteq \mathcal{H}^{n_1} \otimes \mathcal{H}^{n_2} \otimes \cdots \otimes \mathcal{H}^{n_k} .$$

As a result of the tensor product decomposition the local unitary group

$$\mathrm{G_L} = \mathrm{SU(n_1)} \times \mathrm{SU(n_2)} \times \cdots \times \mathrm{SU(n_k)} \subset \mathrm{SU}(N)$$

determines equivalent classes of states.
The non-local characteristics of composite system are encoded in the corresponding reduced structure

$$\mathcal{E}_{\mathrm{n_1 \times n_2 \times \cdots \times n_k}} = \mathfrak{P}_N / \mathrm{G_L} .$$

Local orbit space

Entanglement Space

The quotient $\mathcal{E}_{\mathrm{n}_1\times\mathrm{n}_2\times\cdots\times\mathrm{n}_\mathrm{k}}$ deserves to be especially named as the *"Entanglement Space"* because it accumulates the complete information for the non-locality characteristics of the quantum system.

From a group to cosets

Equivalence relations. Let g and $g' \in G$ be elements of the Lie group, G and $K \subset G \supset H$, then the equivalence relations

$$g \sim g' , \quad \text{if} \quad g' = g_1 g g_2^{-1} , \quad g_1 \in K , g_2 \in H$$

define the double coset $K \backslash G / H$.

Coordinatizing double coset

Starting with the coordinates on the group manifold G and restricting to an appropriate subset, to describe $K \backslash G / H$.

Task: Find a parameterization of G adapted to its subgroups structure resulting in effective description of the corresponding double coset.

Classical results. Elie Cartan, Wilhelm Killing, 1880s

Cartan decomposition.

For a real semisimple Lie algebra $\mathfrak{g}$, a Cartan decomposition is a vector space decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

This split is generated by a Cartan involution θ (an involutive automorphism where $\theta^2 = 1$):

$\mathfrak{k}$: The $+1$ eigenspace of θ, representing the compact subalgebra.

$\mathfrak{p}$: The -1 eigenspace of θ, representing an orthogonal horizontal subspace.

These subspaces obey strict commutation relations:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$

The corresponding semisimple Lie group admits the factorization

$$G = KP$$

with a maximal compact subgroup K of G assuming that its center is finite.

Classical results. Elie Cartan, Wilhelm Killing, 1880s

Generalizing the SVD (KAK Decomposition).

At the Lie group level, the Cartan decomposition leads directly to the so called KAK decomposition, which states that any element U of the connected Lie group G can be decomposed as:

$$U = K_1 A K_2 \,.$$

Where K_1 and K_2 are elements belonging to the compact subgroup K (derived from $\mathfrak{k}$).

A is an element of the Abelian subgroup A (derived from $\mathfrak{p}$).

Practical Applications

- Symmetric Spaces: It is the primary mathematical framework for classifying symmetric spaces and studying the geometry of Lie groups.
- Quantum Computing: The Cartan/KAK decomposition is extensively used in quantum compiling to factorize complex multi-qubit unitary operations (such as elements in $SU(4)$) into a product of local, easy-to-implement single-qubit gates and a non-local interaction.

Double Cosets

Generalized Cartan decomposition For the unitary group $U(n)$ the factorization

$$U = KBH$$

adapted to the double coset of the form

$$U(n_1) \times U(n_2) \times \cdots \times U(n_k) \backslash U(n) / U(m_1) \times U(m_2) \times \cdots \times U(m_r)\,,$$

with double non-commutative involutions.
Japanese Constructions Kobayashi triples

$$(U(n), U(p) \times U(q), U(m_1) \times U(m_2) \times \cdots \times U(m_r)\,.$$

An example: $N = 4 = 2 \times 2$ system, the 2-QUBIT

Global orbits of 2-qubits

The global $\mathrm{Orb}_{SU(4)}(\varrho)$ are determined by the spectrum of ϱ,

$$\varrho = W \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} W^\dagger .$$

The unitary factor W belongs to the coset

$$W \in SU(4)/\mathrm{Iso}(\varrho) ,$$

where $\mathrm{Iso}(\varrho)$ is the isotropy group of the state ϱ.

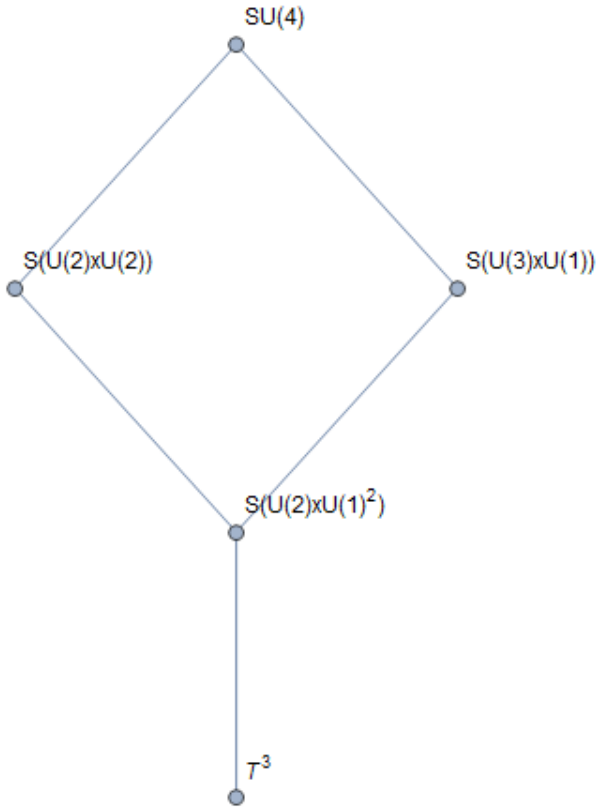
Cosets associated to the global group action

The $SU(4)$ orbits are classified according to the isotropy groups Iso, which form the hierarchy of Lie subgroups in $SU(4)$.
Hence, starting from coordinates on $SU(4)$, we intend to find the parameterization of the following cosets:

$$\frac{SU(4)}{S(U(2)\times U(2))}\,,\quad \frac{SU(4)}{S(U(3)\times U(1))}$$

and

$$\frac{SU(4)}{S(U(2)\times U(1)^2)}\,,\quad \frac{SU(4)}{T^3}$$



Local Unitary group action on $\mathrm{Orb}_{SU(4)}$

Considering the local group action $K = SU(2) \times SU(2)$ on the orbits $\mathrm{Orb}_{SU(4)}$, we arrive at the following double cosets:

$$K\backslash SU(4)/S(U(2)\times U(2))\,,\ K\backslash SU(4)/S(U(3)\times U(1))$$

and

$$K\backslash SU(4)/S(U(2)\times U(1)^2)\,,\ K\backslash SU(4)/T^3$$

Canonical charts on the Lie group

Theorem on canonical charts

If the Lie algebra $\mathfrak{g}$ of the Lie group G is decomposed in the direct sum of any number of subspaces

$$\mathfrak{g} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_m \,,$$

then the mapping

$$\Phi : \mathfrak{g} \to G, \;\; \Phi(X) = \exp a_1 \cdot \exp a_2 \cdot \ldots \cdot \exp a_m \,,$$

assuming for any $X \in \mathfrak{g}$ that $a_i \in \mathcal{A}_i \,, i = 1, \ldots, m$ are components of a vector X in decomposition (18), is a diffeomorphism from some neighborhood of $(0, 0, \ldots, 0)$ in $\mathfrak{g}$ to a neighborhood of the identity element in group G.

Canonical charts on the Lie group

The mapping $\Phi(X) = \exp a_1 \cdot \exp a_2 \cdot \ldots \cdot \exp a_m$ defines the *canonical charts* on the group manifold G.
For $m = 1$, - *canonical coordinates of the first kind*.
For $m = \dim T_e(G)$, - *canonical coordinates of the second kind*.

Aiming to describe : $SU(2) \times SU(2) \backslash SU(4) / T^3$.

We extract two subspaces in the Lie algebra $\mathfrak{su}(4)$ – its Cartan subalgebra and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ subalgebra, the corresponding complement of a vector $X \in \mathfrak{su}(4)$ affords for the coordinates of the double coset and its exponential mapping provides the sought-for parameterization of the double coset $SU(2) \times SU(2) \backslash SU(4) / T^3$.

The direct sum decomposition of $\mathfrak{su}(4)$ algebra

Proposition I: The $\mathfrak{su}(4)$ algebra admits decomposition into the following direct sum of subspaces:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{h}\,,$$

where $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{su}(4)$, $\mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and components $\mathfrak{a}$ and $\mathfrak{a}'$ are 3-dimensional Abelian subalgebras such that

$$[\mathfrak{a}'\,, \mathfrak{a}] \subseteq \mathfrak{k}\,.$$

Furthermore, the decomposition elements obey:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}\,, \quad [\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{k}\,, \quad [\mathfrak{h}, \mathfrak{a}'] \subseteq \mathfrak{k}\,, \quad [\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{a} \oplus \mathfrak{a}'\,.$$

$K\mathcal{A}T$ -decomposition of $SU(4)$

Proposition II: The decomposition of $\mathfrak{su}(4)$ defined in the Proposition I determines the canonical coordinates in the exponential map $\exp : \mathfrak{su}(4) \to \mathrm{SU}(4)$, of the following form:

$$g = K\,\mathcal{A}\,T^3\,, \qquad g \in SU(4)\,,$$

where K represents subgroup

$$K := \exp(\mathfrak{k}) = \exp(\mathfrak{su}(2)) \times \exp(\mathfrak{su}(2))\,,$$

T^3 is the maximal torus in $SU(4)$ and factor $\mathcal{A}$ associated to the sought-for double coset is product of two conjugated copies of the maximal Abelian subgroup of $SU(4)$:

$$\mathcal{A} := \exp(\mathfrak{a})\exp\left(\mathfrak{a}'\right)$$

Fano basis and adapted direct sum decomposition

The Fano basis of $\mathfrak{su}(4)$ algebra is constructed via tensor products,

$$\sigma_{\mu\nu} = \frac{1}{2i}\,\sigma_\mu \otimes \sigma_\nu\,, \qquad \sigma_\mu := (\mathbb{I}_2\,, \boldsymbol{\sigma})\,, \quad \boldsymbol{\sigma} - \text{Pauli matrices.}$$

In a single index notation, $\mathfrak{su}(4)$ algebra basis $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_{15}\}$ is

$$\boldsymbol{\lambda} := \{\sigma_{10}, \sigma_{20}, \sigma_{30}, \sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33}\}\,.$$

The sought-for decomposition $\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{h}$ reads:

$$\mathfrak{a} := \mathcal{A}_1\,, \qquad \mathfrak{a}' := \mathcal{A}_2\,, \qquad \mathfrak{k} := \mathcal{A}_3 \oplus \mathcal{A}_4\,, \qquad \mathfrak{h} := \mathcal{A}_5\,.$$

Fano basis and adapted direct sum decomposition

Here $\mathcal{A}_i = \mathrm{span}_{\mathbb{R}}(\mathbf{\Lambda}_i)$ are constructed by splitting up the basis

$$\lambda = \mathbf{\Lambda}_1 \bigcup \mathbf{\Lambda}_2 \bigcup \mathbf{\Lambda}_3 \bigcup \mathbf{\Lambda}_4 \bigcup \mathbf{\Lambda}_5$$

the five complementary subsets are

$$\begin{aligned}
\mathbf{\Lambda}_1 &= \{\lambda_1, \lambda_4, \lambda_7\}, \\
\mathbf{\Lambda}_2 &= \{\lambda_9, \lambda_{11}, \lambda_{13}\}, \\
\mathbf{\Lambda}_3 &= \{\lambda_2, \lambda_8, \lambda_{14}\}, \\
\mathbf{\Lambda}_4 &= \{\lambda_5, \lambda_{10}, \lambda_{12}\}, \\
\mathbf{\Lambda}_5 &= \{\lambda_3, \lambda_6, \lambda_{15}\}.
\end{aligned}$$

Fano basis and adapted direct sum decomposition

	$\mathfrak{a}$			$\mathfrak{a}'$			$\mathfrak{k}$					
	λ_1	λ_4	λ_7	λ_9	λ_{11}	λ_{13}	λ_2	λ_8	λ_{14}	λ_5	λ_{10}	
λ_1	0	0	0	0	$-\lambda_{14}$	λ_{10}	$-\lambda_3$	0	λ_{11}	0	$-\lambda_{13}$	
λ_4	0	0	0	λ_8	$-\lambda_{12}$	0	0	$-\lambda_9$	$-\lambda_{15}$	$-\lambda_6$	0	
λ_7	0	0	0	λ_5	0	λ_2	$-\lambda_{13}$	$-\lambda_6$	0	$-\lambda_9$	$-\lambda_3$	
λ_9	0	$-\lambda_8$	$-\lambda_5$	0	0	0	$-\lambda_{15}$	λ_4	0	λ_7	0	
λ_{11}	λ_{14}	λ_{12}	0	0	0	0	0	λ_3	λ_1	0	λ_6	
λ_{13}	$-\lambda_{10}$	0	$-\lambda_2$	0	0	0	λ_7	0	$-\lambda_6$	$-\lambda_{15}$	λ_1	
λ_2	λ_3	0	$-\lambda_{13}$	λ_{15}	0	$-\lambda_7$	0	λ_{14}	$-\lambda_8$	0	0	
λ_8	0	λ_9	λ_6	$-\lambda_4$	$-\lambda_3$	0	$-\lambda_{14}$	0	λ_2	0	0	
λ_{14}	$-\lambda_{11}$	λ_{15}	0	0	λ_1	$-\lambda_6$	λ_8	$-\lambda_2$	0	0	0	
λ_5	0	λ_6	λ_9	$-\lambda_7$	0	λ_{15}	0	0	0	0	λ_{12}	
λ_{10}	λ_{13}	0	λ_3	0	$-\lambda_6$	$-\lambda_1$	0	0	0	$-\lambda_{12}$	0	
λ_{12}	λ_{15}	$-\lambda_{11}$	0	λ_3	λ_4	0	0	0	0	λ_{10}	$-\lambda_5$	
λ_3	$-\lambda_2$	0	$-\lambda_{10}$	$-\lambda_{12}$	λ_8	0	λ_1	$-\lambda_{11}$	0	0	λ_7	
λ_6	0	$-\lambda_5$	$-\lambda_8$	0	λ_{10}	$-\lambda_{14}$	0	λ_7	λ_{13}	λ_4	$-\lambda_{11}$	
λ_{15}	$-\lambda_{12}$	$-\lambda_{14}$	0	$-\lambda_2$	0	$-\lambda_5$	λ_9	0	λ_4	λ_{13}	0	

Explicit coordinate form; $K-$factor

Local subgroup: $K=SU(2)\times SU(2)$.

The K factor in KAT-decomposition of $SU(4)$,

$$K=\exp(\mathfrak{k})\,,\quad \mathfrak{k}=\mathrm{span}_{\mathbb{R}}(\lambda_2,\lambda_8,\lambda_{14},\lambda_5,\lambda_{10},\lambda_{12})\,.$$

Noting that the standard inclusion of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(4)$ is given by $\mathrm{R}^\dagger\mathfrak{k}\mathrm{R}$ with the "magic" 4×4 unitary matrix

$$\mathrm{R}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&-i\\0&-i&-1&0\\0&-i&1&0\\1&0&0&i\end{pmatrix},$$

then $K=\mathrm{R}^\dagger\left(\exp\left(\imath\sum_i^3\varphi_i\sigma_i\right)\otimes\exp\left(\imath\sum_i^3\psi_i\sigma_i\right)\right)\mathrm{R}\,.$

Hence, two copies of $SU(2)$ coordinates, φ and ψ, parameterize K-factor in KAT-decomposition.

Explicit coordinate form; $\mathcal{A}-$factor

$\mathcal{A}$ factor is identified to $\mathbb{R}^6$ by introducing coordinates α_i and β_i:

$$\mathcal{A} = \mathcal{A}_1\mathcal{A}_2 \quad \text{with} \quad \mathcal{A}_1 = \exp(\boldsymbol{\alpha}, \boldsymbol{\Lambda}_1)\,,\ \mathcal{A}_2 = \exp(\boldsymbol{\beta}, \boldsymbol{\Lambda}_2)\,.$$

$\mathcal{A}_1$ and $\mathcal{A}_2$ are diagonalizible via the orthogonal transformations,

$$\mathcal{A}_1 = \mathcal{S}_1\,\mathrm{diag}(\boldsymbol{\alpha})\,\mathcal{S}_1^T\,, \quad \mathcal{A}_2 = \mathcal{S}_2\,\mathrm{diag}(\boldsymbol{\beta})\,\mathcal{S}_2^T\,,$$

$$\mathcal{S}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{S}_2 = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathrm{diag}(\boldsymbol{\alpha}) = ||e^{-\frac{i}{2}(\alpha_1+\alpha_2+\alpha_3)}, e^{\frac{i}{2}(\alpha_1+\alpha_2-\alpha_3)}, e^{\frac{i}{2}(\alpha_1-\alpha_2+\alpha_3)}, e^{\frac{i}{2}(-\alpha_1+\alpha_2+\alpha_3)}||$$
$$\mathrm{diag}(\boldsymbol{\beta}) = ||\,e^{-\frac{i}{2}(\beta_1+\beta_2+\beta_3)}, e^{\frac{i}{2}(\beta_1+\beta_2-\beta_3)}, e^{\frac{i}{2}(\beta_1-\beta_2+\beta_3)}, e^{\frac{i}{2}(-\beta_1+\beta_2+\beta_3)}||$$

Analyticity domain of KAT- parameters

Theorem on exponential mapping

Let G be a compact and connected matrix Lie group. The function $\exp_A$ is analytic on a bounded open neighborhood of the origin:

$$\mathcal{U} = \{A \in \mathfrak{g} \,|\, |\mathrm{Im}[\mathrm{Eigenvalue}(A)]| < \pi\}\,.$$

This determines the chart with KAT-parameters covering almost the whole SU(4). Particularly, it holds for 3-tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ each from the regular octahedron:

$$\begin{aligned}
|\alpha_1+\alpha_2+\alpha_3| &< 2\pi\,, & |\beta_1+\beta_2+\beta_3| &< 2\pi\,,\\
|\alpha_1+\alpha_2-\alpha_3| &< 2\pi\,, & |\beta_1+\beta_2-\beta_3| &< 2\pi\,,\\
|\alpha_1-\alpha_2+\alpha_3| &< 2\pi\,, & |\beta_1-\beta_2+\beta_3| &< 2\pi\,,\\
|\alpha_1-\alpha_2-\alpha_3| &< 2\pi\,, & |\beta_1-\beta_2-\beta_3| &< 2\pi\,.
\end{aligned}$$

Analyticity domain of $\mathcal{A}$-factor

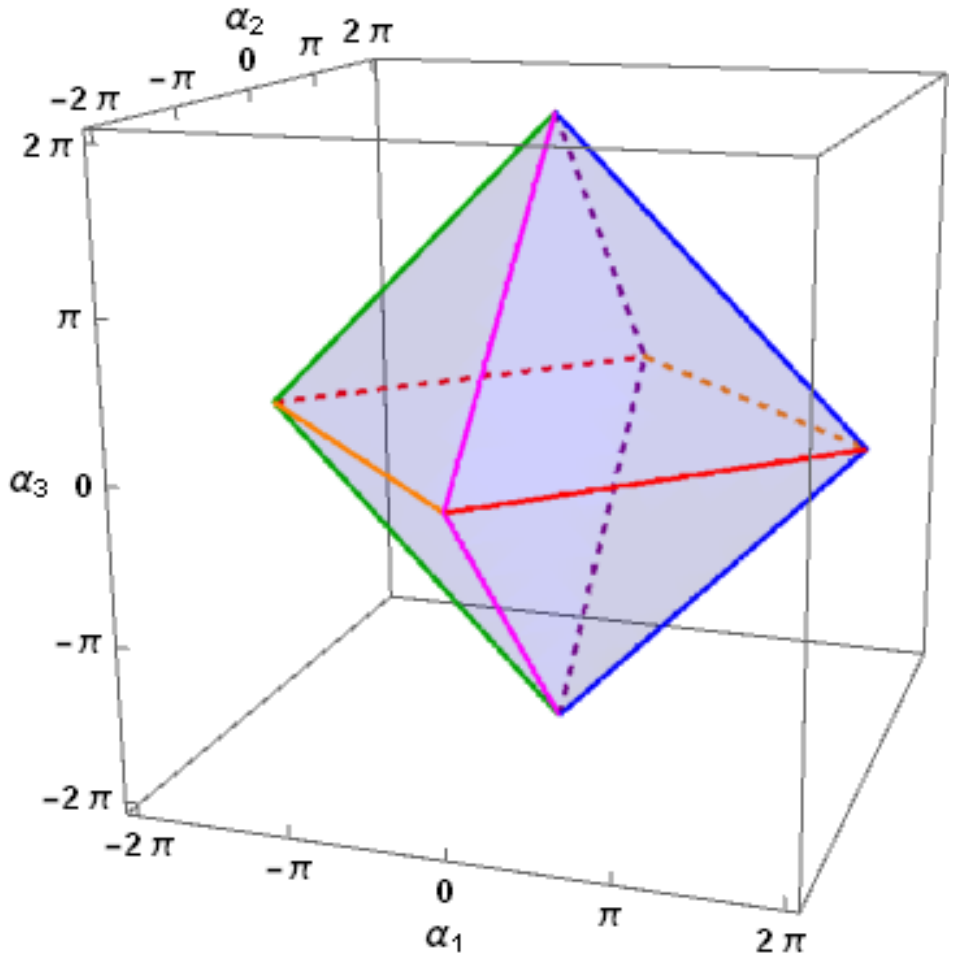


Figure 1: The regular octahedron as the domain of the analiticity of $\mathcal{A}_1$-factor in KAT-decomposition

Fundamental domain of KAT- parameters

The symmetry of KAT -decomposition

Problem: Find a subgroup $S \subset SU(4)$ such that

$$s\,\mathcal{A}_i\,s^{-1} \in \mathcal{A}_i\,, \quad \forall\, s \in S, \quad \text{and} \quad \forall\, i = 1,2,3,4,5\,.$$

Answer: The full symmetry group is O_h, which has 48 elements and is isomorphic to the direct product of the symmetric group S_4 and the cyclic group C_2 (or $S_4 \times \mathbb{Z}_2$). It includes 24 rotational symmetries and 24 additional symmetries that involve reflections.

Fundamental domain of the KAT-parameters

The fundamental domain

Hence, reduction of the analiticity domain to the fundamental one is in order. As a result, the fundamental domain of the Abelian factors $\mathcal{A}_1$ and $\mathcal{A}_2$ geometrically is given by the pair of factor spaces

$$\frac{\text{Regular octahedron}}{O_h = S_4 \times \mathbb{Z}_2}$$

The double coset $SU(2) \times SU(2) \backslash SU(4) / T^3$ as a 6D Orbifold

Let $\mathfrak{P}_4$ be the state space of two qubits. Taking into account a physical argumentation, states $\varrho \in \mathfrak{P}_4$ are classified by virtue of the equivalence relations imposed by the (K, H)-action of two subgroups $K, H \subseteq SU(4)$:

$$\varrho_1 \simeq \varrho_2 \quad \text{if} \quad \varrho_2 = k \varrho_1 h^{-1}, \quad k \in K,\, h \in H\,.$$

If $K = H = \mathrm{SU}(4)$, then all maximal rank states with the same spectrum are mapped to a point of the orbit space of the adjoint action of the $\mathrm{SU}(4)$ group

$$\mathfrak{P}_4 / \mathrm{SU}(4)\,.$$

The double coset $SU(2) \times SU(2) \backslash SU(4) / T^3$ as a 6D Orbifold

Considering a generic density matrix of maximal rank with non-degenerate eigenvalues we fix the unitary matrix U as an element of the coset

$$U \in \mathrm{SU}(4)/\mathrm{T}^3 \,.$$

For the parameterization of a generic 6-dimensional $SU(2) \times SU(2)$-orbit we use a recently proposed factorized representation of $\mathrm{SU}(4)$ group

$$\mathrm{SU}(4) = \mathrm{K}\mathcal{A}\mathrm{T}^3 \,,$$

where $\mathrm{K} = \mathrm{SU}(2) \times \mathrm{SU}(2)$, T^3 is the maximal 3-torus subgroup, and the $\mathcal{A}$-factor denotes the product of two exponents, $\mathcal{A} = \exp\{\mathfrak{a}\} \exp\{\mathfrak{a}'\}$ with two 3-dimensional Abelian subalgebras $\mathfrak{a}$ and $\mathfrak{a}'$ representing subspaces in the following direct sum decomposition of $\mathfrak{su}(4)$

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{t} \,.$$

The double coset SU(2) × SU(2)\SU(4)/T^3 as a 6D Orbifold

Let the density matrix of two qubits has the Singular Value Decomposition (SVD):

$$\varrho = \frac{1}{4}\left[I_2 \otimes I_2 + \sum_{i=1}^{3} a_i \sigma_i \otimes I_2 + \sum_{i=1}^{3} b_i I_2 \otimes \sigma_i + \sum_{i,j=1}^{3} c_{ij} \sigma_i \otimes \sigma_j\right] =$$

$$= U\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\, U^{\dagger}\,,$$

where the diagonal factor is expanded over the basis elements of the Cartan subalgebra of $\mathfrak{su}(4)$ algebra as follows:

$$\lambda_1 = \frac{1}{4}(1 + x + y + z)\,, \quad \lambda_2 = \frac{1}{4}(1 + x - y - z)\,,$$

$$\lambda_3 = \frac{1}{4}(1 - x + y - z)\,, \quad \lambda_4 = \frac{1}{4}(1 - x - y + z)\,.$$

The double coset SU(2) × SU(2)\SU(4)/T^3 as a 6D Orbifold

Furthermore, on orbits of the principal stratum, which is realized as the interior of 3D simplex Δ_3 of the ordered eigenvalues of the density matrix,

$$1 > \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0,$$

one can consider the double action with the pair of elements from $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $H = \mathrm{T}^3$ – the maximal Abelian torus.
One can be convinced that according to the double coset parameterization for $SU(4)$ any non-degenerate rank-4 state is $SU(2) \times SU(2)$-equivalent to the following 9-parameter representative matrix $\varrho_{d=6}$:

$$\varrho_{d=6} = \mathcal{A}\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\,\mathcal{A}^\dagger\,.$$

Using the parameterization for a generic state of two qubits we prove the following

A Result

Proposition *In the vicinity of a state ϱ consisting of maximal rank states with a non-degenerate spectrum the entanglement space admits the representation:*

$$\mathcal{E}_{2\times 2} = \mathbb{B}_6 \times \Delta_3 \,.$$

Here the 9D entanglement space has a 6D component in the form of the double coset of the SU(4) group $\mathbb{B}_6 = \mathrm{SU}(2) \times \mathrm{SU}(2)\backslash\mathrm{SU}(4)/\mathrm{T}^3$ and the factor Δ_3 denotes the interior of the ordered 3-simplex of eigenvalues of ϱ .

The double coset $SU(2) \times SU(2) \backslash SU(4)/T^3$ as a 6D Orbifold

One of the basic objects of our study is the double coset/biquotient

$$\mathbb{B}_6 = \mathrm{SU}(2) \times \mathrm{SU}(2) \backslash \mathrm{SU}(4)/\mathbb{T}^3$$

This biquotient represents a *stratified manifold*.
Consider a compact Lie group with two subgroups $K, H \subset G$ and the following action

$$(k, h) \cdot g = kgh^{-1}, \text{for} k \in K, h \in H$$

For the biquotient $K \backslash G/H$ to be a smooth manifold, the action of $K \times H$ on G must be free. The condition for this action to be free is that the stabilizer of any element is trivial. Formally, for every $g \in G$, the following condition must hold:

$$kgh^{-1} = g \longrightarrow k = h = e$$

where e is the identity element of G.

The double coset SU(2) × SU(2)\SU(4)/T^3 as a 6D Orbifold

This can be rewritten to state that for all g, the intersection of K and the conjugate of H by g must be trivial:

$$K \cap gHg^{-1} = \{e\}$$

In terms of the ranks of the groups, there is a fundamental constraint derived from the dimension of the stabilizers.
For the action to be free (or even locally free/orbifold), the following must hold:

$$\mathrm{rank}(K) + \mathrm{rank}(H) \leq \mathrm{rank}(G)$$

The double coset SU(2) × SU(2)\SU(4)/T^3 as a 6D Orbifold

For our case, the manifold condition is violated and biquotient $\mathbb{B}_6$ has a stratified form. To describe the stratified structure of the biquotient $\mathbb{B}_6$ in detail, we must classify the stabilizer subgroups of the action $K \times T$ on G. In a biquotient, the stabilizer of an element $g \in G$ is defined as:

$$\mathrm{Stab}(g) = \{(k,t) \in K \times T \mid kgt^{-1} = g\} \simeq K \cap gTg^{-1}\,.$$

The double coset $SU(2) \times SU(2) \backslash SU(4)/T^3$ as a 6D Orbifold

The strata are the sets of points whose stabilizers are conjugate in the transformation group.

Table 3: Metric and Curvature Properties of the 6D Biquotient Strata

Stratum	**Dim**	**Metric Tensor** g_{ij}	**Sectional Curvature** K
Principal	6	$g_{ij} > 0$ (Non-degenerate)	Strictly Positive ($K > c > 0$)
Facets	5	$g_{ii} \to 0$ for 1 direction	K jumps (Reflection boundar
Cells	3	3-rank deficiency	High Curvature/Concentratic
Edges	1	5-rank deficiency	Singular/Focal Set
Vertices	0	$g_{ij} = 0$ (Total collapse)	Divergent/Ill-defined

Discussion

Future plans

- Studying the cases of the non-generic global orbits $\mathrm{Orb}_{SU(4)}$;
- Further investigation of the entanglement space of two qubits $\mathfrak{P}_4/SU(2) \times SU(2)$ in terms of the two octahedron coordinates and simplex of the density matrix eigenvalues;
- Construction of the Wigner function of 2-qubits in terms of the introduced parameterization of $SU(4)$.

Thank you for your attention!