Propagation of Two-Photon Zernike States in Atmospheric Turbulence

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The Scope

In broad strokes this presentation is about

The *robustness* of a specific *state* of *structured quantum light* while *propagating* through a *turbulent quantum channel*.

The expected results are going to be of the following nature

- Mathematical: New (to the best of our knowledge) identities and properties of Zernike functions
- Computational: Reduction of an 8-dimensional integral to a rapidly converging algebraic form
- Physical: Hierarchic properties of the photon leakage to neghbouring modes

Context of Zernike Polynomials

Historically

Zernike polynomials were introduced and are still widely used in optics in the context of aberrations of optical systems.

The remarkable properties of ZP as seen from the optician perspective

- Orthogonality on a unit disk (matching the circular exit pupil of an imaging system);
- Non vanishing nature over the boundary of the disk;
- Axial symmetry;
- The most common light aberrations are described by the lowest order polynomials;
- etc.

Characteristic equation

Consider the following diferential operator

$$\partial^2 + \alpha(\mathsf{r}\cdot\partial)^2 + \beta\,\mathsf{r}\cdot\partial, \qquad \mathsf{r} := (\mathsf{x}^1, \mathsf{x}^2), \quad \partial := (\partial_1, \partial_2).$$

It turns out that it is Hermitian

with respect to the standard invariant measure of integration over the unit disk, for the values of parmeters

$$\alpha = -1 \text{ and } \beta = -2$$

ZP's are

exactly the eigen-functions of this operator.

Moreover

Independent of the values of the parameters, its eigenvalues are real. This being a consequence of its pseudo-Hermicity.

Zernike Polynomials

For $n, m \in \mathbb{Z}, 2 \mid (n - |m|) \geqslant 0$ Zernike polynomials are defined as

$$Z_n^m(\boldsymbol{\rho}) = \begin{cases} \sqrt{n+1}\, R_n^{|m|}(\rho) e^{im\theta}, & 0 \leqslant \rho \leqslant 1, \\ 0 & \rho > 1, \end{cases}, \quad \text{where}$$

$$R_n^{|m|}(\rho) = \rho^{|m|} P^{(0,|m|)}_{\frac{n-|m|}{2}}(2\rho^2 - 1) = \sum_{k=0}^{\frac{n-|m|}{2}} \frac{(-1)^k (n-k)!}{k!\left(\frac{n+m}{2} - k\right)!\left(\frac{n-m}{2} - k\right)!} \rho^{n-2k}.$$

The corresponding Fourier transforms have a simple representation in terms of Bessel function

$$\begin{aligned} \widetilde{Z}_n^m(q) &= \int_0^1 d\rho\, \rho \int_0^{2\pi} d\theta\, Z_n^m(\rho, \theta) e^{2\pi i \rho q \cos(\theta - \phi)} \\ &= 2\pi i^n \sqrt{n+1} \frac{J_{n+1}(2\pi q)}{2\pi q} e^{im\phi}, \end{aligned}$$

Orthogonality and completeness relations

Zernike

$$\int \mathrm{d}^2\mathsf{s}\, \mathsf{Z}_\mathsf{n}^\mathsf{m}(\mathsf{s})\, \mathsf{Z}_{\mathsf{n}'}^{\mathsf{m}'*}(\mathsf{s}) = \pi\, \delta_{\mathsf{n}\mathsf{n}'}\, \delta_{\mathsf{m}\mathsf{m}'},$$

$$\sum_{\mathsf{n},\mathsf{m}} \mathsf{Z}_\mathsf{n}^\mathsf{m}(\mathsf{s}_1)\, \mathsf{Z}_\mathsf{n}^{\mathsf{m}*}(\mathsf{s}) = \pi\, \delta_\mathsf{D}(\mathsf{s}_1 - \mathsf{s}).$$

Fourier Zernike

$$\int \mathrm{d}^2\mathsf{q}\, \widetilde{\mathsf{Z}}_\mathsf{n}^\mathsf{m}(\mathsf{q})\, \widetilde{\mathsf{Z}}_{\mathsf{n}'}^{\mathsf{m}'*}(\mathsf{q}) = \pi\, \delta_{\mathsf{n}\mathsf{n}'}\, \delta_{\mathsf{m}\mathsf{m}'},$$

$$\sum_{\mathsf{n},\mathsf{m}} \widetilde{\mathsf{Z}}_\mathsf{n}^\mathsf{m}(\mathsf{q}_1)\, \widetilde{\mathsf{Z}}_\mathsf{n}^{\mathsf{m}*}(\mathsf{q}) = \pi\, \frac{\mathsf{J}_1(2\pi|\mathsf{q}-\mathsf{q}_1|)}{|\mathsf{q}-\mathsf{q}_1|} = \pi\, \widetilde{\mathsf{Z}}_0^0(\mathsf{q}-\mathsf{q}_1).$$

Where

$\delta_\mathsf{D}(\mathbf{s}_1 - \mathbf{s}) = \mathbb{1}_\mathsf{D}(\mathsf{s}_1)\mathbb{1}_\mathsf{D}(\mathsf{s}_2)\, \delta(\mathbf{s}_1 - \mathbf{s}_2)$ is the delta function defined as the kernel of the projection onto the functions supported on a subset of the unit disk D, while $\widetilde{\mathsf{Z}}_0^0(\mathsf{q}-\mathsf{q}_1)$ acts as the integral kernel for the projector onto the subspace of $\mathcal{F}\left\{\mathsf{L}^2_\mathsf{D}\right\} \subset \mathsf{L}^2_{\mathbb{R}^2}$.

Some relations on Zernike polynomials

The mode coupling in the forthcoming Zernike basis are governed by two complementary sets of coefficients

A real-space linearization coefficients

$$A^{m_1 m_2 M}_{n_1 n_2 N} = \frac{1}{\pi}\int d^2 s\, Z^{m_1}_{n_1}(\mathbf{s})\, Z^{m_2}_{n_2}(\mathbf{s})\, Z^{M*}_{N}(\mathbf{s}),$$

$$Z^{m_1}_{n_1}(\mathbf{s})\, Z^{m_2}_{n_2}(\mathbf{s}) = \sum_{NM} A^{m_1 m_2 M}_{n_1 n_2 N}\; Z^{M}_{N}(\mathbf{s}).$$

Γ Fourier space linearization coefficients

$$\Gamma^{mm'm''}_{nn'n''} = \frac{1}{\pi}\int d^2 q \tilde{Z}^{m}_{n}(q)\tilde{Z}^{m'}_{n'}(q)\tilde{Z}^{m''*}_{n''}(2q).$$

$$\tilde{Z}^{m}_{n}(q)\tilde{Z}^{m'}_{n'}(q) = 4\sum_{n''m''}\Gamma^{mm'm''}_{nn'n''}\tilde{Z}^{m''}_{n''}(2q),$$

Some new identities dictating the following addition rules

$$\sum_{\substack{n_1,m_1\\ n_2,m_2}} A^{m_1 m_2 M}_{n_1 n_2 N}\; Z^{m_1}_{n_1}(s_1)\, Z^{m_2}_{n_2}(s_2) = \pi\, Z^{M}_{N}(s_1)\,\delta_D(s_1 - s_2),$$

$$\sum_{\substack{n_1,m_1\\ n_2,m_2}} A^{m_1 m_2 M}_{n_1 n_2 N}\; \tilde{Z}^{m_1}_{n_1}(q_1)\, \tilde{Z}^{m_2}_{n_2}(q_2) = \pi\, \tilde{Z}^{M}_{N}(q_1 + q_2).$$

Some relations on Zernike polynomials

And complimentary

$$\sum_{\substack{n_1,m_1\\ n_2,m_2}} \Gamma^{m_1 m_2 M}_{n_1 n_2 N} \tilde{Z}^{m_1 *}_{n_1}(q_1) \tilde{Z}^{m_2 *}_{n_2}(q_2) = \pi \int d^2 q\, \tilde{Z}^{M *}_{N}(2q)\, \tilde{Z}^{0}_{0}(q - q_1)\, \tilde{Z}^{0}_{0}(q - q_2),$$

$$\sum_{\substack{n_1,m_1\\ n_2,m_2}} \Gamma^{m_1 m_2 M}_{n_1 n_2 N} Z^{m_1 *}_{n_1}(s_1) Z^{m_2 *}_{n_2}(s_2) = \frac{\pi}{4} Z^{M *}_{N}\left(\frac{s_1 + s_2}{2}\right) 1_D(s_1) 1_D(s_2).$$

An important identity coupling the A and Γ coefficients is

$$\sum_{\substack{n_1,m_1\\ n_2,m_2}} \Gamma^{m_1 m_2 M *}_{n_1 n_2 N} A^{m_1 m_2 M_1}_{n_1 n_2 N_1} = \frac{\pi}{4} \delta_{N N_1} \delta_{M M_1},$$

and similarly under permutations of $((n_1, m_1), (n_2, m_2)$, and $(N, M))$

Zernike modes and statement of the problem

Quantum $|z_n^m\rangle = (z_n^m)^\dagger|0\rangle$ Zernike modes are defined by

$$(z_n^m)^\dagger = \frac{1}{\sqrt{\pi}} \int d^2 q \tilde{Z}_n^{m*}(q) a^\dagger(q), \quad [z_n^m, z_{n'}^{m'\dagger}] = \delta_{mm'}\delta_{nn'}$$

.

It turns out if the non-linear crystal is pumped by such a mode after SPDC two photons exit with a joint state

$$|\psi\rangle_A = \sum_{\substack{m_1 m_2 \\ n_1 n_2}} A_{n_1 n_2 N}^{m_1 m_2 M} |z_{n_1}^{m_1}, z_{n_2}^{m_2}\rangle.$$

We are interested in collinear joint two-photon detection, in Zernike-modes

After such measurement it turns out that the state will be projected at

$$|\phi\rangle = 4 \sum_{\substack{nn'n'' \\ mm'm''}} \Gamma_{n_1 n_2 n}^{m_1 m_2 m} \Gamma_{n' n'' n}^{m' m'' m*} |\tilde{z}_{n'}^{m'}, \tilde{z}_{n''}^{m''}\rangle,$$

Propagation in turbulent atmosphere

In extended Huygens-Fresnel formalism the effect of atmospheric perturbation is accounted for in a form ψ-function describing acuired random phase shift

$$U(\boldsymbol{\rho}, z) = \frac{k e^{ikz}}{2\pi i z} \int_D d^2 s \; U_0(\mathbf{s}, 0) e^{\frac{ik|\mathbf{s}-\boldsymbol{\rho}|^2}{2z} + \psi(\boldsymbol{\rho}, \mathbf{s})}.$$

If additionally the Fraunhofer approximation is applied then the fields are determined by

$$U(\boldsymbol{\rho}, z) = \frac{k e^{ikz} e^{\frac{ik}{2z}\rho^2}}{2\pi i L} \int_D d^2 s \, U_0(\mathbf{s}, 0) e^{-\frac{ik}{z}\boldsymbol{\rho}\cdot\mathbf{s} + \psi(\boldsymbol{\rho}, \mathbf{s})}.$$

The field operators in paraxial approximation in terms of Zernike operators become

$$E^{(+)}(\mathbf{r}) = \frac{k e^{ikz}}{2iz\pi^{\frac{3}{2}}} \sum_{mn} \tilde{z}_n^m \int d^2 s Z_n^m(\mathbf{s}) e^{i\frac{k}{z}\,\mathbf{s}\cdot\boldsymbol{\rho} + \psi(\boldsymbol{\rho}, \mathbf{s})}.$$

The two photon detection amplitude

$$\mathcal{A}_N^M \propto \langle 0 | \hat{E}^{(+)}(\boldsymbol{\rho}_2) \hat{E}^{(+)}(\boldsymbol{\rho}_1) | \psi \rangle = \int d^2 s \, Z_N^M(\mathbf{s}) \, e^{i\frac{k}{z}\mathbf{s}\cdot(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2) + \psi(\mathbf{s}, \boldsymbol{\rho}_1) + \psi(\mathbf{s}, \boldsymbol{\rho}_2)},$$

Probabilty of detection after turbulent medium

The Probability of detection becomes

$$P_N^M\left(z_{N_1}^{M_1}, z_{N_2}^{M_2}\right) = \frac{k^4}{z^4}\iint d^2\rho_1 d^2\rho_2 \widetilde{Z}_{N_1}^{M_1 *}\left(\frac{kR}{z}\boldsymbol{\rho}_1\right)\widetilde{Z}_{N_2}^{M_2 *}\left(\frac{kR}{z}\boldsymbol{\rho}_1\right)\widetilde{Z}_{N_1}^{M_1}\left(\frac{kR}{z}\boldsymbol{\rho}_2\right)\widetilde{Z}_{N_2}^{M_2}\left(\frac{kR}{z}\boldsymbol{\rho}_2\right)$$
$$\times \iint d^2 s_1 d^2 s_2 \, Z_N^M\left(\frac{\mathbf{s}_1}{R}\right) Z_N^{M *}\left(\frac{\mathbf{s}_2}{R}\right) e^{2\pi i \frac{2k}{z}(\mathbf{s}_1\cdot\boldsymbol{\rho}_1 - \mathbf{s}_2\cdot\boldsymbol{\rho}_2)} \left\langle e^{2[\psi(\mathbf{s}_1,\boldsymbol{\rho}_1)+\psi^*(\mathbf{s}_2,\boldsymbol{\rho}_2)]}\right\rangle$$

Concise algebraic form

$$P_N^M\left(z_{N_1}^{M_1}, z_{N_2}^{M_2}\right) = \sum_{n_1 n_2} F_{n_1} F_{n_2}^* \sum_{n_5} G_{n_5}^0 A_{n_1 n_2 n_5}^{m_1, -m_1\, 0}, \tag{1}$$

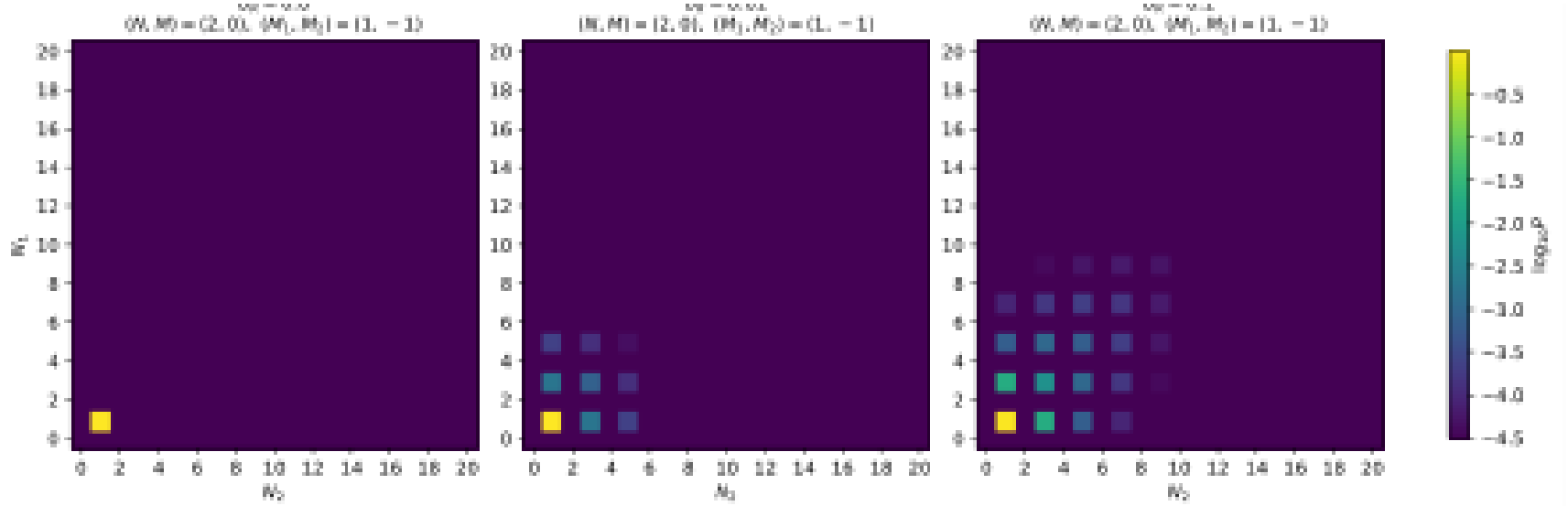
where

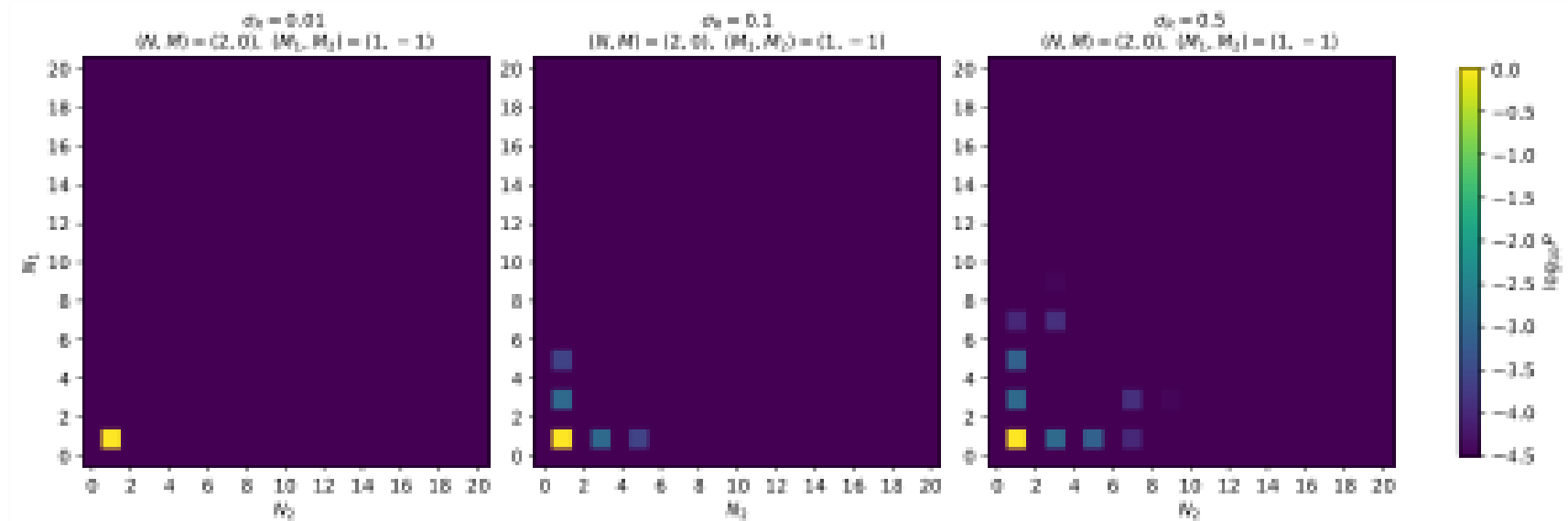
$$F_n = \sum_{n'} \Gamma_{N_1 N_2 n'}^{M_1 M_2, M_1+M_2} \Gamma_{n' N n}^{-M_1-M_2, M, M-M_1-M_2 *}, \tag{2}$$

and

$$\begin{aligned} G_{n_5}^{m_5}(\gamma) &= \int d^2 u\, e^{-2\frac{\gamma k}{z}u^2} \widetilde{Z}_{n_5}^{m_5}\left(\frac{4kR}{z}u\right) \\ &= 2\pi\delta_{m_5 0}\, i^{n_5}\sqrt{n_5+1}\,\frac{\pi z}{4\gamma k}\left(\sqrt{\frac{8\pi^2 kR^2}{\gamma z}}\right)^{n_5} \\ &\times \frac{\Gamma\left(\frac{n_5+2}{2}\right)}{\Gamma(n_5+2)}\, {}_1F_1\left(\frac{n_5+2}{2}; n_5+2; -\frac{8\pi^2 kR^2}{\gamma z}\right), \end{aligned} \tag{3}$$

Results





Thank You!