Some algorithms for evaluation of Goncharov multilogarithms and related functions

Yahor Dydyshka[a,b]

SANC group

[a] *Dzhelepov Laboratory of Nuclear Problems, JINR, Dubna, Russia*
[b] *Dubna State University, Russia*

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Outline

- Goncharov polylogarithms
- An introductory example: $\mathrm{Li}_2\, z$
- Generalization attempts: $\mathrm{Li}_n\, z$
- Nielsen polylogarithms: $S_{p,n}(z)$
- Generating function: ${}_2F_1(z)$
- Extension to 2 variables: Appell $F_1(x, y)$
- Solvable special case

Dilogarithm $\mathrm{Li}_2\, z$

$$\mathrm{Li}_2\, z = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad |z| < 1, \qquad \mathrm{Li}_2\, 1 = \zeta_2 = \frac{\pi^2}{6}$$

$$\mathrm{Li}_2\, z = -\int_0^z \frac{\ln(1-t)}{t} dt = -\int_0^1 \frac{\ln(1-zt)}{t} dt,$$

Analytic continuations

$$\mathrm{Li}_2\, z = -\,\mathrm{Li}_2\, \frac{1}{z} - \tfrac{1}{2} \log^2(-z) - \zeta_2,$$

$$\mathrm{Li}_2\, z = -\,\mathrm{Li}_2(1-z) - \log(1-z) \log z + \zeta_2$$

Mobius group

$$I(z) \stackrel{\text{def}}{=\!=} \frac{1}{z}, \qquad R(z) \stackrel{\text{def}}{=\!=} 1 - z$$

$$I \circ R = \frac{1}{1-z}, \qquad R \circ I = 1 - \frac{1}{z} = \frac{z-1}{z}$$

$$R \circ I \circ R = I \circ R \circ I = \frac{z}{z-1}$$

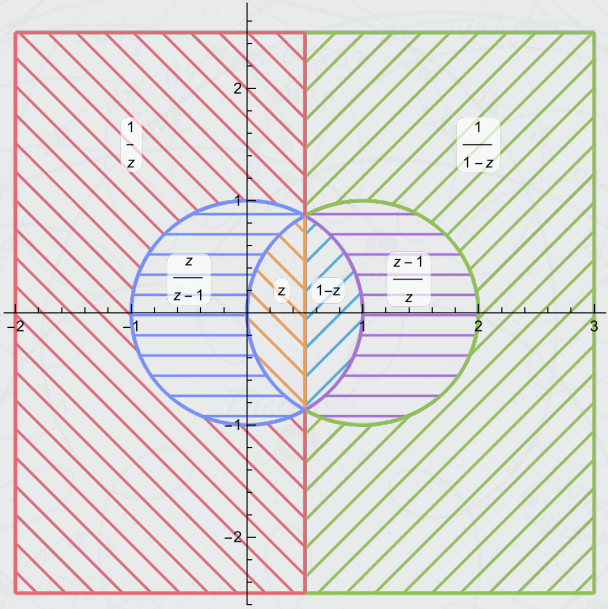
Extra identities

$$\operatorname{Li}_2 z = \operatorname{Li}_2 \frac{1}{1-z} + \tfrac{1}{2} \log^2 \frac{z-1}{z} - \tfrac{1}{2} \log^2(-z) - \zeta_2,$$

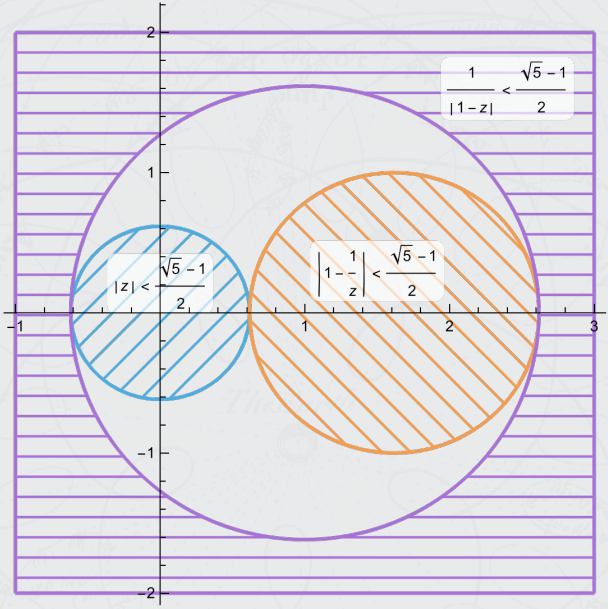
$$\operatorname{Li}_2 z = \operatorname{Li}_2 \frac{z-1}{z} + \tfrac{1}{2} \log^2 z - \log(1-z) \log z + \zeta_2$$

$$\operatorname{Li}_2 z = -\operatorname{Li}_2 \frac{z}{z-1} - \tfrac{1}{2} \log^2(1-z)$$

Poincare Regions



Golden-ration Regions for real axis



Change of variable

$$\mathrm{Li}_2\, z = \int_0^\eta du \frac{u}{e^u - 1}, \qquad \text{where} \qquad \eta \stackrel{\text{def}}{=\!=} \log \frac{1}{1-z}$$

Bernulli expansion

$$\mathrm{Li}_2\, z = \sum_{k=0}^{\infty} B_k \frac{\eta^{k+1}}{(k+1)!}, \qquad \text{where Bernulli numbers:} \qquad \frac{u}{e^u - 1} \stackrel{\text{def}}{=\!=} \sum_{k=0}^{\infty} B_k \frac{u^k}{k!}$$

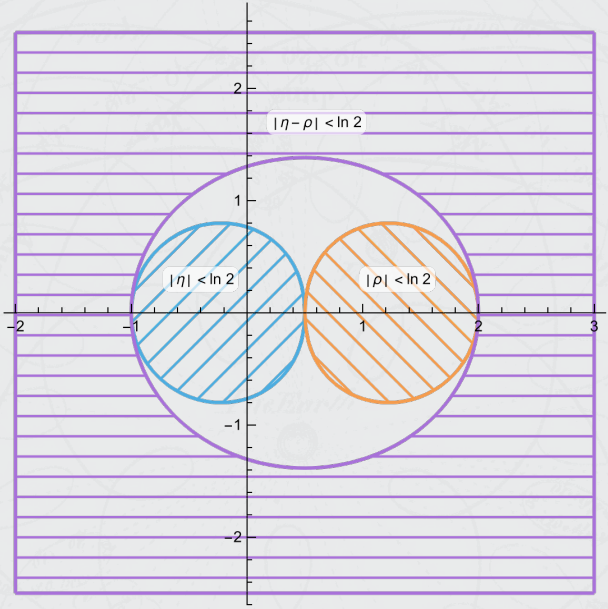
series converges for $|\eta| < \pi$

Exponential argument

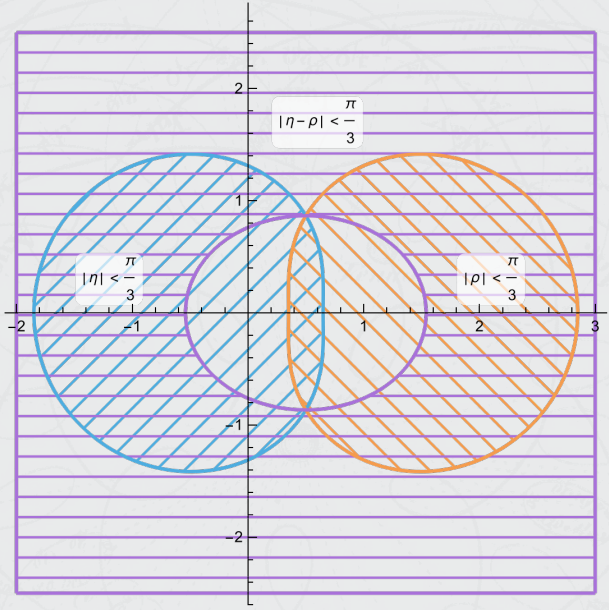
$$\mathrm{Li}_2\, z = -\sum_{k=0}^{\infty} B_k \frac{\rho^{k+1}}{(k+1)!} + \rho \log(1 - e^{-\rho}) + \zeta_2, \qquad \text{where} \qquad \rho \stackrel{\text{def}}{=\!=} \log \frac{1}{z}$$

$\zeta_{-n} = -B_{n+1}/(n+1)$

Regions η,ρ for real axis



Regions η,ρ for whole plane



Polylogarithms

$$\mathrm{Li}_n\, z = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \qquad |z| < 1$$

are not closed with respect to $z \to 1 - z$ transformation

Nielsen generalized polylogarithms

$$S_{n,p}(z) = \frac{-1}{n!p!} \int_0^1 \log^p \frac{1}{1-zt} d \log^n \frac{1}{t}, \qquad \mathrm{Li}_n(z) \equiv S_{n-1,1}(z)$$

closed with respect Mobius group, but grades are mixed

$$S_{2,1}(1-z) = -\mathrm{Li}_3\, z + \log z\, \mathrm{Li}_2\, z + \tfrac{1}{2} \log(1-z) \log^2 z + \zeta_3,$$

$$S_{2,2}(1-z) = -S_{2,2}(z) + S_{1,2}(z) - \mathrm{Li}_3\, z + \log z\, \mathrm{Li}_2\, z + \tfrac{1}{4} \log^2(1-z) \log^2 z + \zeta_3 \log(1-z) + \frac{\zeta_4}{4}$$

Generating function

$$F(\epsilon,\delta,z)=\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}S_{n,p}(z)\epsilon^n\delta^p=\int_0^1(1-zt)^{-\delta}\epsilon t^{-\epsilon-1}dt=-{}_2F_1\left[\begin{smallmatrix}-\epsilon & \delta\\ 1-\epsilon\end{smallmatrix}\middle|z\right]$$

Gauss hypergeometric functions are closed under Monius group

$${}_2F_1\left[\begin{smallmatrix}a,b\\ c\end{smallmatrix}\middle|z\right]=(1-z)^{c-a-b}{}_2F_1\left[\begin{smallmatrix}c-a,c-b\\ c\end{smallmatrix}\middle|z\right],$$

$${}_2F_1\left[\begin{smallmatrix}a,b\\ c\end{smallmatrix}\middle|z\right]=(1-z)^{-a}{}_2F_1\left[\begin{smallmatrix}a,c-b\\ c\end{smallmatrix}\middle|\frac{z}{z-1}\right],$$

$${}_2F_1\left[\begin{smallmatrix}a,b\\ c\end{smallmatrix}\middle|z\right]=\Gamma\left[\begin{smallmatrix}b-a,c\\ b,c-a\end{smallmatrix}\right](-z)^{-a}{}_2F_1\left[\begin{smallmatrix}a,a-c+1\\ a-b+1\end{smallmatrix}\middle|\frac{1}{z}\right]+\Gamma\left[\begin{smallmatrix}a-b,c\\ a,c-b\end{smallmatrix}\right](-z)^{-b}{}_2F_1\left[\begin{smallmatrix}b,b-c+1\\ b-a+1\end{smallmatrix}\middle|\frac{1}{z}\right],$$

$${}_2F_1\left[\begin{smallmatrix}a,b\\ c\end{smallmatrix}\middle|z\right]=\Gamma\left[\begin{smallmatrix}c,a+b-c\\ a,b\end{smallmatrix}\right]{}_2F_1\left[\begin{smallmatrix}1-a,1-b\\ c-a-b+1\end{smallmatrix}\middle|1-z\right]+\Gamma\left[\begin{smallmatrix}c,c-a-b\\ c-a,c-b\end{smallmatrix}\right]{}_2F_1\left[\begin{smallmatrix}a,b\\ a+b-c+1\end{smallmatrix}\middle|1-z\right]$$

Change of variable

now let $u = -\log(1-zt)$ and $\eta = -\log(1-z)$ so $t = \frac{1-e^{-u}}{z}$

$$F(\epsilon,\delta,z) = \int_0^1 (1-zt)^{-\delta}\epsilon t^{-\epsilon-1}dt = \int_0^{-\log(1-z)} e^{u\delta}\epsilon(1-e^{-u})^{-\epsilon-1}\tau^{\epsilon+1}\frac{e^{-u}du}{z}$$
$$= \epsilon z^{\epsilon}\int_0^{\eta} e^{u(\delta+\epsilon)}u^{-1-\epsilon}\Big(\frac{u}{e^u-1}\Big)^{\epsilon+1}du$$

Generalized Bernulli polynomials we defined by series

$$e^{au}\Big(\frac{u}{e^u-1}\Big)^{1+b} = \sum_{k=0}^{\infty} B_k(a,b)\frac{u^k}{k!}$$

then ordinary Bernulli polynomials $B_k(a) = B_k(a,0)$ and *Bernulli numbers* $B_k = B_k(0,0)$. Not that series is converging for $|u| < \pi$.

Let's apply it to generating function $F(\epsilon, \delta, \tau)$

$$F(\epsilon, \delta, z) = \epsilon e^{-\rho\epsilon} \int_0^\eta u^{-1-\epsilon} e^{u\delta} e^{u\epsilon} \left(\frac{u}{e^u - 1}\right)^{\epsilon+1} du$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{\epsilon}{k+p-\epsilon} e^{-\rho\epsilon} \frac{B_k(\epsilon, \epsilon)}{k!} \frac{\delta^p}{p!} \eta^{k+p-\epsilon}$$

Special case $n = 0$

$$S_{1,p}(z) = \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{p!(k+p)} \frac{B_k}{k!}$$

In particular

$$S_{1,1}(z) = \mathrm{Li}_2\, z = \sum_{k=0}^{\infty} \frac{B_k}{(k+1)!} \eta^{k+1}$$

in agreement with previous slides.

For NNLO $e^+e^- \to \gamma\gamma$ we also need $S_{2,2} = G_{0011}$

$$S_{2,p}(z) = \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{(k+p)^2 p!} \frac{B_k}{k!} + \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{p!(k+p)} \frac{[B'_k - \xi B_k]}{k!}$$

$$S_{3,p}(z) = \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{(k+p)^3 p!} \frac{B_k}{k!} + \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{p!(k+p)^2} \frac{[B'_k - \xi B_k]}{k!} + \sum_{k=0}^{\infty} \frac{\eta^{k+p}}{p!(k+p)} \frac{[B''_k - 2\xi B'_k + \xi^2 B_k]}{k!}$$

with $\xi = \rho + \log\eta$ and $B'_k \equiv \partial_\epsilon B_k(\epsilon,\epsilon)|_{\epsilon\to 0}$, $B''_k \equiv \partial^2_\epsilon B_k(\epsilon,\epsilon)|_{\epsilon\to 0}$

$$\frac{\partial}{\partial\epsilon} e^{\epsilon u}\Big(\frac{u}{e^u-1}\Big)^{1+\epsilon}\Big|_{\epsilon\to 0} = \sum_{k=0}^{\infty} B'_k \frac{u^k}{k!} = \frac{u}{2} - \frac{7u^2}{24} + \frac{u^3}{16} - \frac{u^4}{320} - \frac{u^5}{1152} + \dots$$

$$\frac{\partial^2}{\partial\epsilon^2} e^{\epsilon u}\Big(\frac{u}{e^u-1}\Big)^{1+\epsilon}\Big|_{\epsilon\to 0} = \sum_{k=0}^{\infty} B''_k \frac{u^k}{k!} = \frac{u^2}{8} - \frac{u^3}{12} + \frac{25u^4}{1152} - \frac{23u^5}{11520} - \frac{7u^6}{34560} + \dots$$

Polynomial evaluation remark

Horner scheme is not an optimal for polynomial of fixed degree. Better performance and numerical stability obtained by factorization. For example:

$$
\begin{aligned}
&-\frac{\mu^2}{24}+\frac{\mu^4}{2880}-\frac{\mu^6}{181440}+\frac{\mu^8}{9676800}-\frac{\mu^{10}}{479001600}\\
&=-2.08767569878681\times 10^{-9}\mu^2\times\\
&\times(4812.760013587034+\mu^2(-108.0104255181556+\mu^2))\times\\
&\times(4146.975943877296+\mu^2(58.51042551815557+\mu^2)),
\end{aligned}
$$

Goncharov

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{d\xi_1}{\xi_1 - a_1}\int_0^{\xi_1}\frac{d\xi_2}{\xi_2-a_2}\cdots\int_0^{\xi_{n-1}}\frac{d\xi_n}{\xi_n-a_n}$$
$$= \mathbf{L}_{a_1}\dots\mathbf{L}_{a_n}[1],$$

$$\mathbf{L}_a[f] = \int_0^x \frac{d\xi}{\xi-a} f(\xi),$$
$$\mathbf{L}_a^2[f] = \int_0^x \frac{d\xi}{\xi-a}\log\frac{x-a}{\xi-a} f(\xi)$$
$$\mathbf{L}_a^n[f] = \frac{1}{(n-1)!}\int_0^x \frac{d\xi}{\xi-a}\log^{n-1}\frac{x-a}{\xi-a} f(\xi), \qquad \mathbf{L}_a^0[f] = f(x),$$
$$\mathbf{L}_a^n[1] = \frac{1}{n!}\log^n\frac{x-a}{-a}, \qquad \mathbf{L}_0^n[1] \stackrel{\text{def}}{=\!=} \frac{1}{n!}\log^n x$$

Generating operator

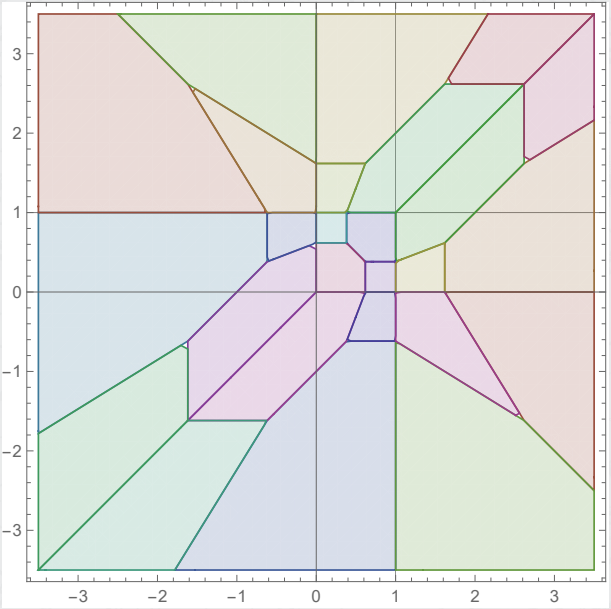
$$\mathbb{L}_a^{\lambda}[f] = \sum_{n=0}^{\infty} \lambda^n \mathbf{L}_a^n[f] = f(x) + \int_0^x \frac{\lambda(x-a)^{\lambda}}{(\xi-a)^{\lambda+1}} f(\xi) d\xi,$$

$$\mathbb{L}_a^{\lambda}[1] = \left(1 - \frac{x}{a}\right)^{\lambda}$$

Nielsen case

$$= (-1)^p S_{n,p}(\tau) = G_{\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{p}}(\tau) = G_{0^n 1^p}(\tau) = \mathbf{L}_1^p \mathbf{L}_0^n[1]$$

Regions for Appell F_1



Thank you