Analytical and Numerical Calculations within the Framework of the Basic Element Method

N.D. Dikusar

JINR, MLIT, dnd@jinr.ru

MMCP‘ 2026

Content

I. Basic Element Method (BEM).

II. Important Properties of the Basis Functions.

III. BEM Polynomials and BEM Coefficients.

IV. Comparison of BEM vs Taylor Errors.

V. BEM-Applications.

1. Shape Approximation and Contour analysis.

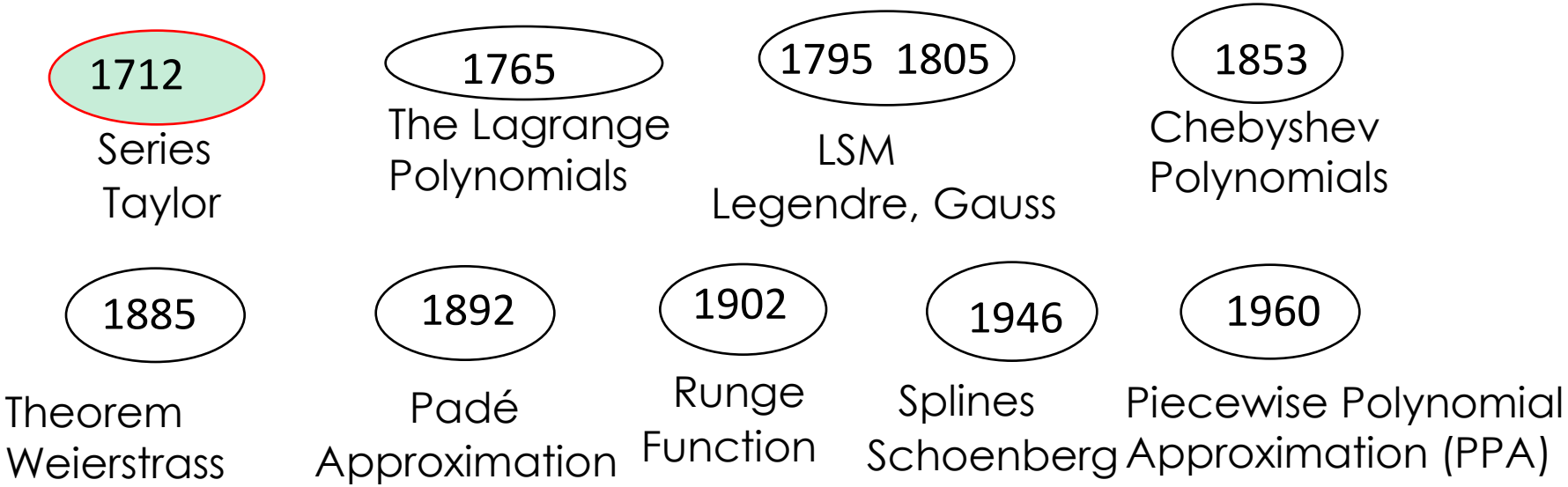
2. Numerical Solution of the Cauchy Problem.

3. IBR-2M Spectrum Normalization.

VI. Conclusion. References.

Historical reference (Backgrounds)

Stages of development of the theory and practice of polynomial approximation



General remarks

Main goal consists in **reducing the complexity** of computational problems while **ensuring the accuracy**, **reliability**, **stability and efficiency** of calculations.

The power of modern computers allows for operations with higher-degree polynomials.

An increasing in the degree polynomials leads to

- ill-conditioned problems, that leads to a loss of precision and often leads to incorrect results;
- increase of computational complexity;
- considerable emission of the predicted values at extrapolation; considerable emission of the predicted values at extrapolation;
- increase the order of derivatives, etc.

To reduce these difficulties, the Basic Element Method (**BEM**) was developed in MLIT [2-6].

BEM-Polynomials

The *nth* degree BEM-polynomial is expressed as

The BEM-polynomial is based on Chebyshev`s idea about the approximation of a smooth function by the polynomial on an interval of the limited length (1853) [1].

$$\mathbf{w} = [w_1, w_2, w_3], \qquad \mathbf{b} = [w_1, w_3, w_2, Qw_1, Qw_3, Qw_2].$$

Basis functions: $\mathbf{b}_j = Q^j \mathbf{w}, \quad \mathbf{b}_j = Q\mathbf{b}_{j-1}, \; j = \overline{0, m}$

The Rule for Calculating Coefficients $\mathbf{r}_j$

BEM-coefficients $\mathbf{r}_j$ are determined from the conditions

$$[f(\tau)-\sum_{i=0}^{j} Q^i \Pi_i]_{\tau=\nu}^{(j)} = 0,\ j=\overline{1,m},\ \tau \equiv \nu = \alpha,\beta,0$$

where the vector components $\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{j-1}$ were obtained

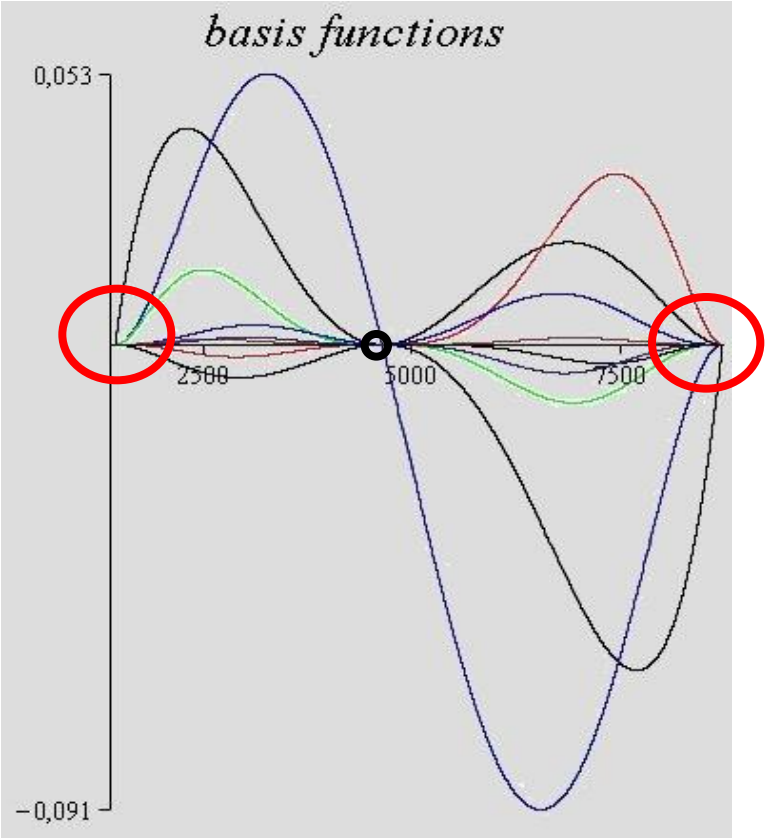
using $f_\nu^{(j-1)}$, non-zero values $Q_\nu^{(j-1)}$, $w_\nu^{(j-1)}$ and $\alpha, \beta, 0$.

$$\Delta_{j\nu} = \Delta_{j\nu}(\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{j-1}, \alpha, \beta, 0),\ j=1,m;\ \nu = \alpha,\ \beta,\ 0$$

$$\mathbf{r}_j = \mathbf{r}_j(f_\nu^{(j)}, \mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{j-1}, \alpha, \beta, x_0)$$

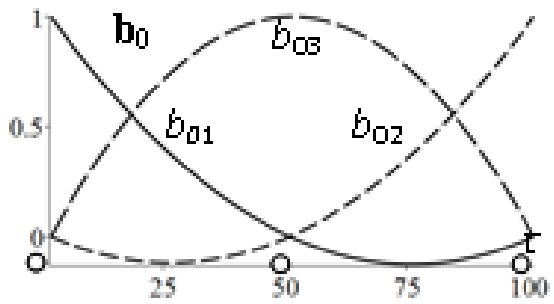
$P_{n\downarrow m}(x,...)$ **is a hierarchical polynomial**

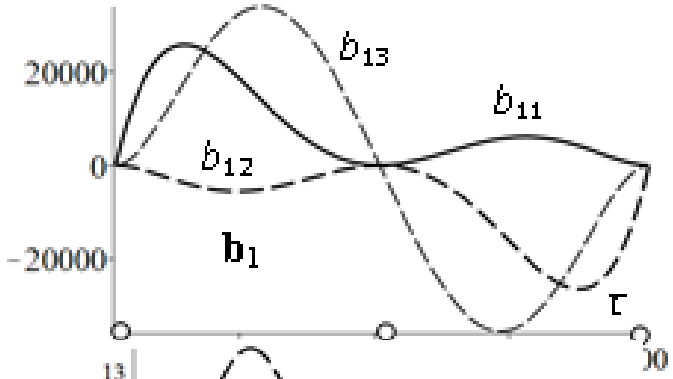
An Important Property of Basis Functions

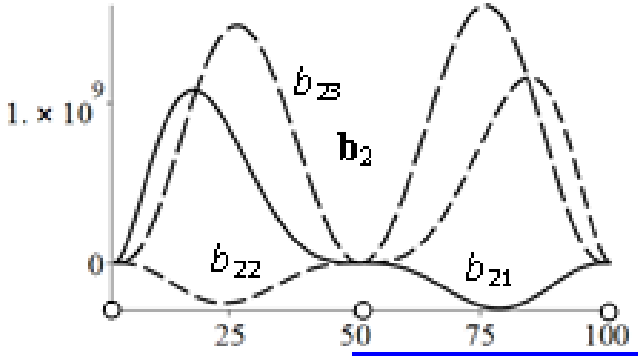


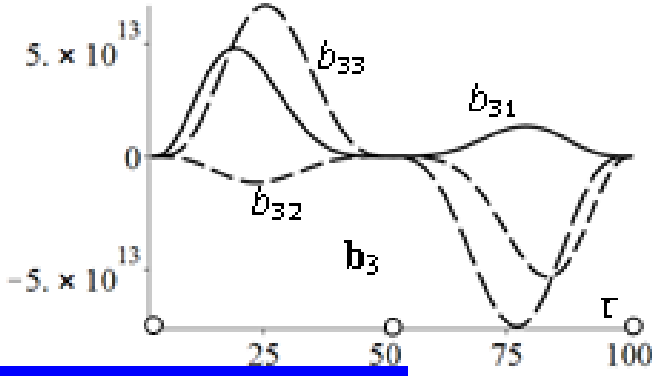
Basis functions are polynomials of degree $3j+2$ in τ with **zeros at the nodes** of the three-point grid.

"But very often it is important for an error to be reduced to zero within the limits of the interval."
P. L. Chebyshev [1]

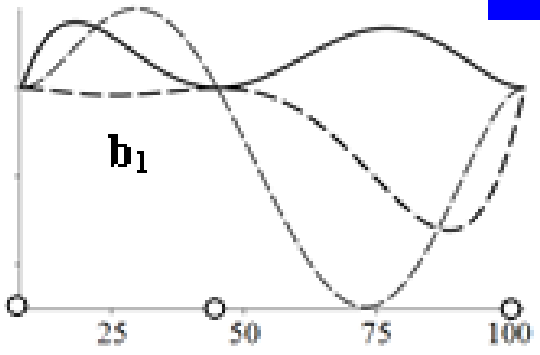


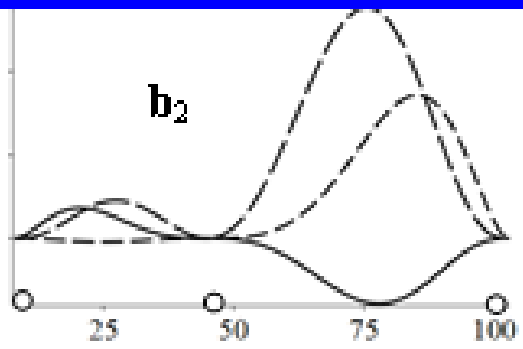


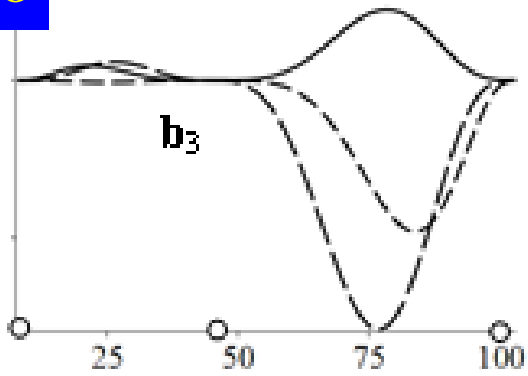




$$\alpha = -49,\ x_0 = 51,\ \beta = 50$$



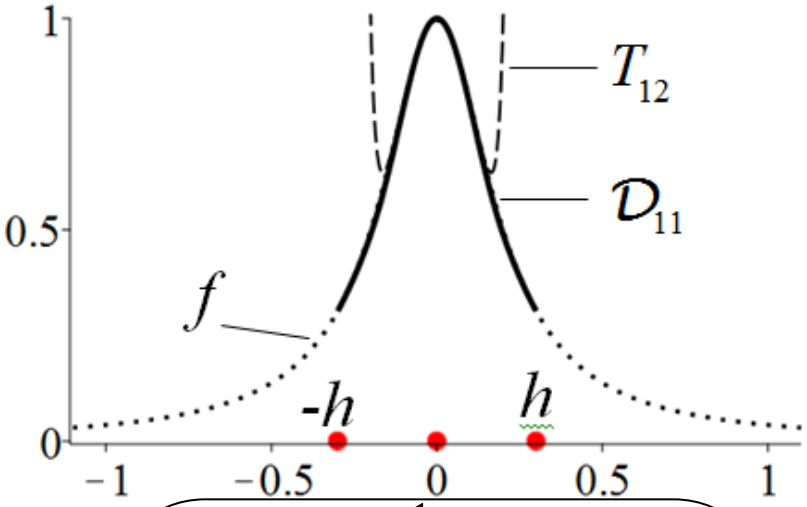


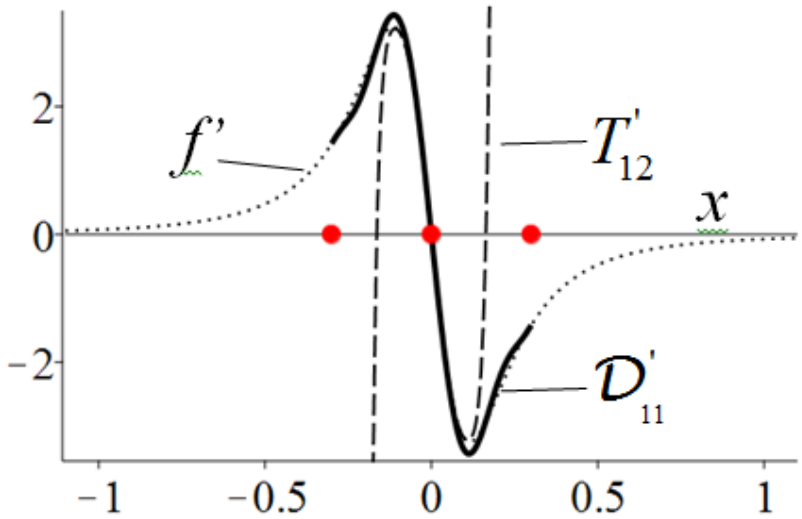


$$\alpha = -37,\ x_0 = 45,\ \beta = 58$$

Example:

BEM-Aapproximation





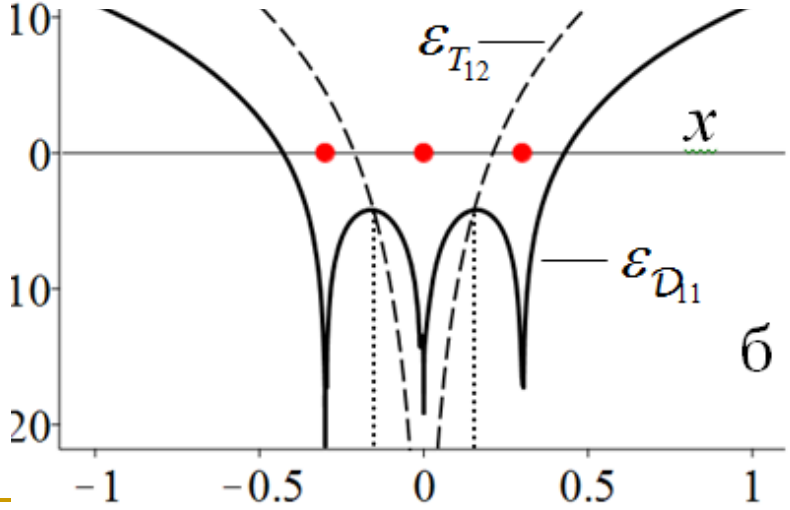
$$f(x) = \frac{1}{25x^2 + 1}, \; x_0 = 0$$

$$-h < x < h,$$

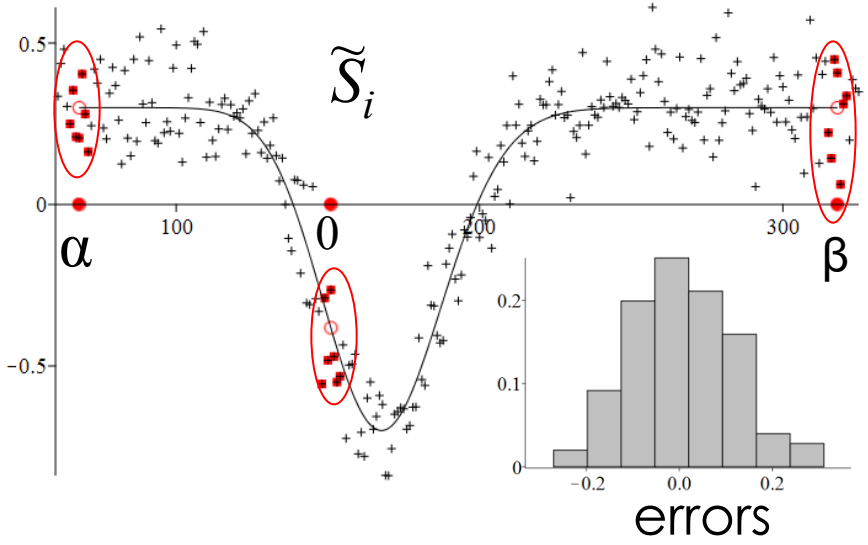
$$\hat{f}(x) \approx T_{11} = \sum_{i=0}^{11} a_i x^i$$

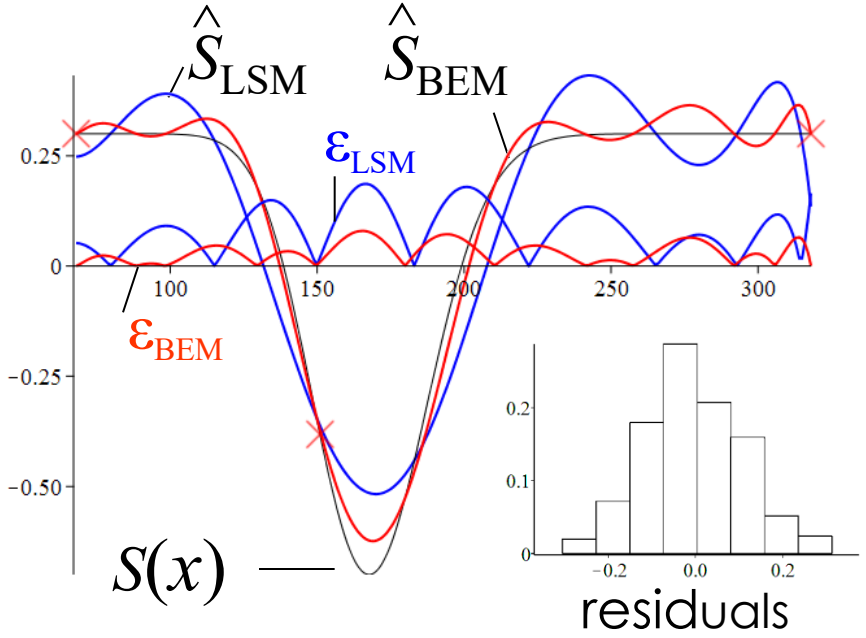
$$\hat{f}(x) \approx D_{11} = \sum_{i=0}^{3} Q^j \Pi_j$$

$$\varepsilon_P = \log|f - P_{11}|$$



BEM: DATA SMOOTHING (ONE SEGMENT)



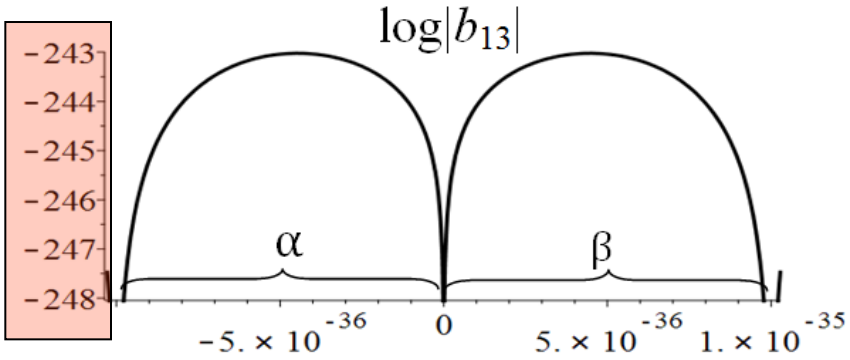


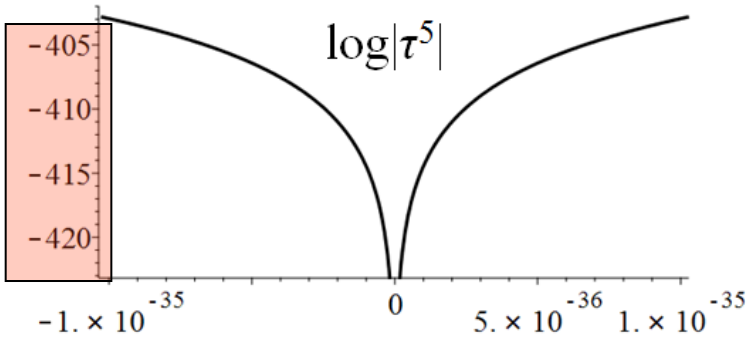
N= 500, NS=1, σ=0.1, α=-0.025, x_0=1.295, β=0.05.

Stability of Calculations

Functions $b_{jk}(\tau;\alpha,\beta)$ zero out errors at the nodes Δ_α^β and ensure stability of calculations **for extremely small values** of the variable $\tau \in [\alpha, \beta]$. For example, in the vicinity of zero, on the interval , where we obtain

$$-248 < \log|b_{13}| < -243 \quad \text{and} \quad -420 < \log|x^5| < -400$$





$$\alpha = -10^{(-35)}, \ \beta = 10^{(-35)}$$

BEM-Applications

The BEM-polynomials were used for effective solution several current tasks. Among them are

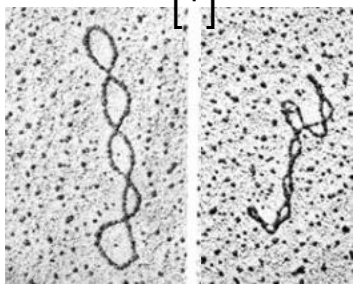
1. The method for the segmentation of 2D-contours with a complex geometrical form [3].

2. A fundamentally new explicit method for numerical solving of the stiff Cauchy problem [4].

3. Neutron noise analysis and normalization of the spectra during investigation of pulse energy fluctuations of the IBR-2M reactor [5,6] etc.

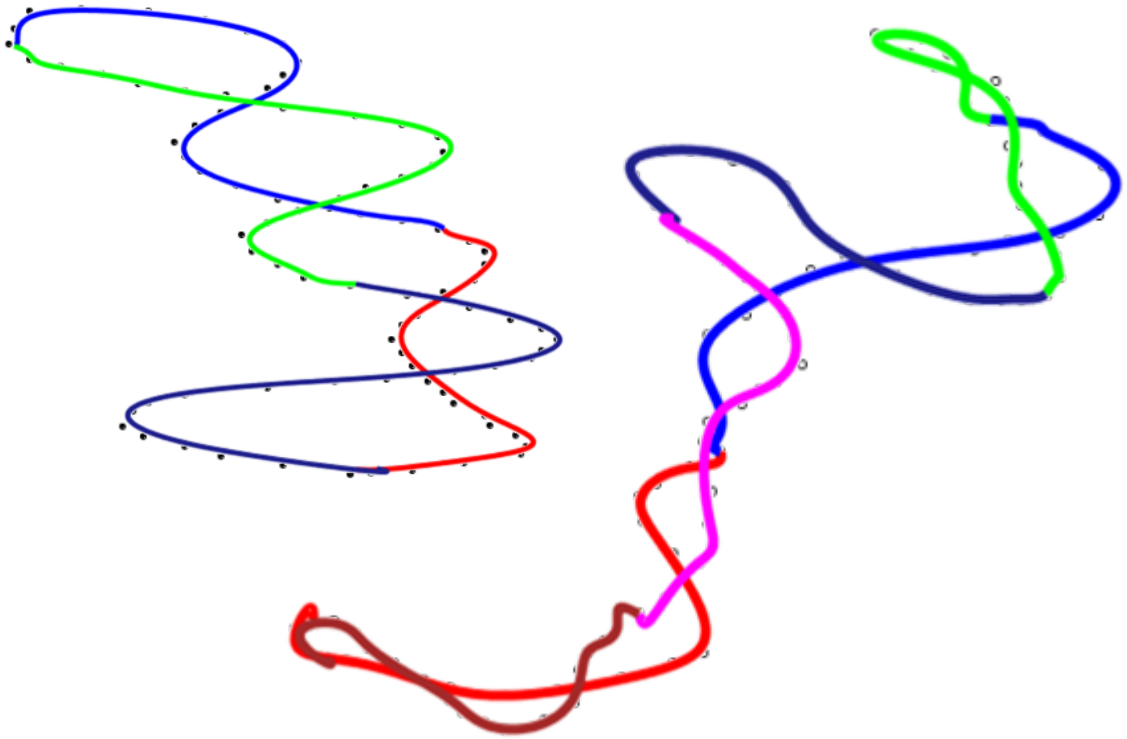
Shape Approximation

Examples:



[*]

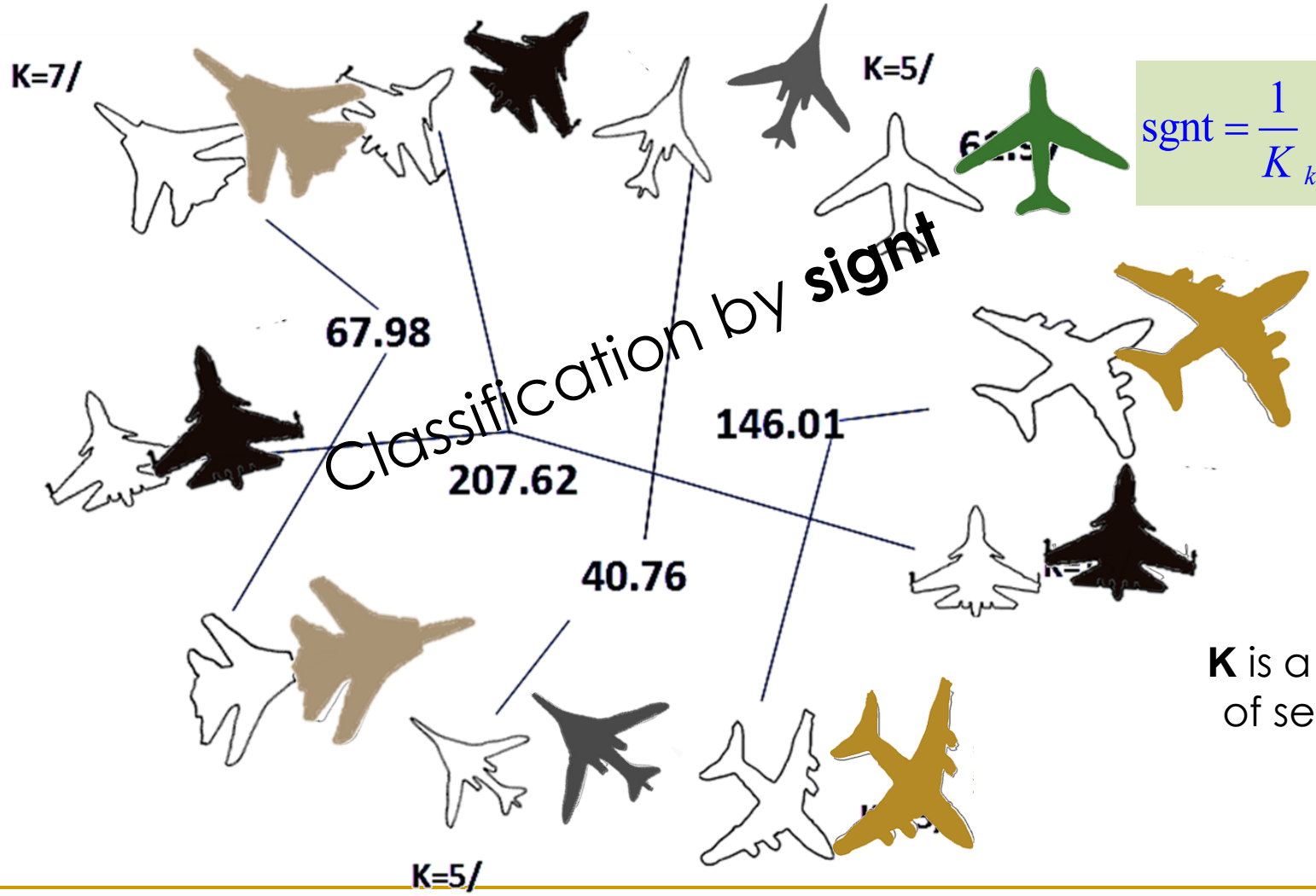
[*] D.Calabi, E. Olver, P.J., Shakiban, C. Tannenbaum, A., and Haker. Differential and numerically invariant signature curves applied to object recognition. 1998.



Signature Curves In Classifying DNA Supercoils

$$\text{sgnt} = \frac{1}{K}\sum_{k=1..K}\frac{\kappa(s_{0_k})}{\kappa_s(s_{0_k})}, \qquad \kappa(s) = \sqrt{\ddot{x}(s)^2 + \ddot{y}(s)^2}$$

BEM Classification



$$\mathrm{sgnt} = \frac{1}{K} \sum_{k=1..K} \frac{\kappa(s_{0_k})}{\kappa_s(s_{0_k})},$$

K is a number of segments

Numerical solution of the Cauchy problem

$$y'(x) = f(x,y), \quad y(x_0) = y_0 \qquad \Delta_a^b : a = x_1 < \cdots < x_N = b$$

A **fundamentally new approach** to the numerical solution of the Cauchy problem.

The approach is based on an **explicit Predictor-Corrector (PC) scheme**.

The fifth order of the method **is confirmed by comparison with the classical methods and the known test for a stiff problem.**

PC-BEM Algorithm

$$p_\nu = \mathbf{k}_\nu(h,1)\mathbf{u} \longrightarrow A_K = P_p = \mathbf{b}(x_K;h,1)\mathbf{y}_{i-2}^{i} \longrightarrow y_K^{'} = f(x_K, A_K)$$

$$c_\nu = \mathbf{k}_\nu(h,K)\mathbf{v} \longrightarrow y_{i+1} = P_c = \mathbf{b}(x_{i+1};h,K)\mathbf{f}_{i-1}^{K} \longrightarrow y_{i+1}^{'} = f(x_{i+1}, y_{i+1}).$$

$$\nu = 1,2,3$$

The coefficients of a polynomial $P_5(\tau;\mathbf{d},\alpha,\beta) \approx y(x)$, $x \in [x_\alpha, x_\beta]$ are found from the conditions

$$\left[\Pi_0 + Q\Pi_1 - y(\tau)\right]_{\tau=\alpha,0,\beta}^{(j)} = 0, \quad j = 0,1$$

$$\mathbf{u}_1 = [y_{i-2}, y_{i-2}, y_{i-1}, y_i] \quad \mathbf{u}_2 = [y_{i-1}, y_{i-2}, y_{i-1}, y_i] \quad \mathbf{u}_3 = [y_i, y_{i-2}, y_{i-1}, y_i]^T$$

$$\mathbf{v}_1 = [y_{i-1}^{'}, y_{i-1}, y_i, A_K]^T \quad \mathbf{v}_2 = [y_i^{'}, y_{i-1}, y_i, A_K]^T \quad \mathbf{v}_3 = [y_K^{'}yy_{i-1}, y_i, A_K]^T$$

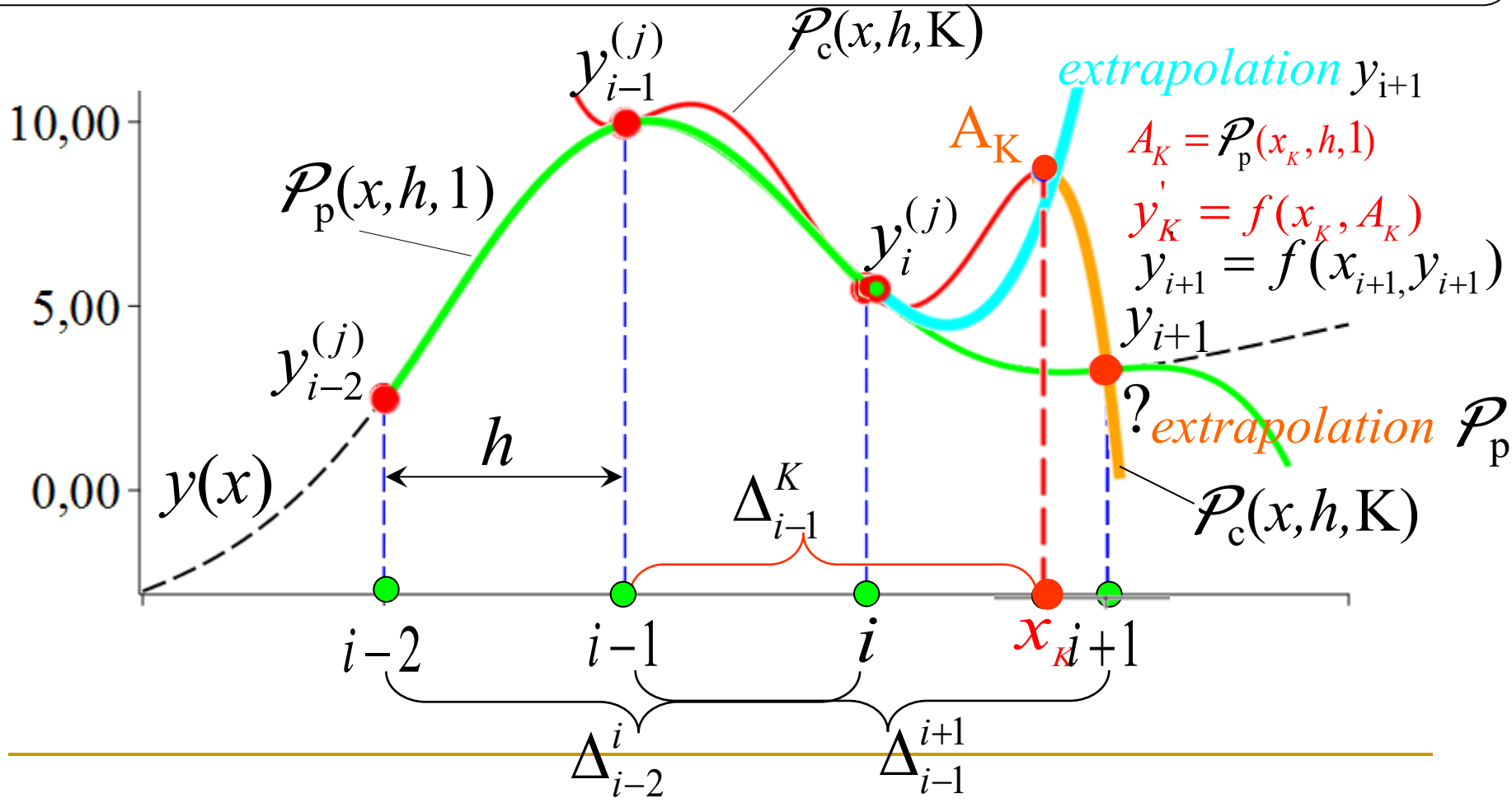
$$\mathbf{y}_{i-2}^{i} = [y_{i-2}, y_{i-1}, y_i, p_1, p_2, p_3]^T$$

$$A_K = \mathcal{P}_{\mathrm{p}}(x_K, h, 1)$$

$$\mathbf{f}_{i-1}^{K} = [y_{i-1}, y_i, A_K, c_1, c_2, c_3]^T$$

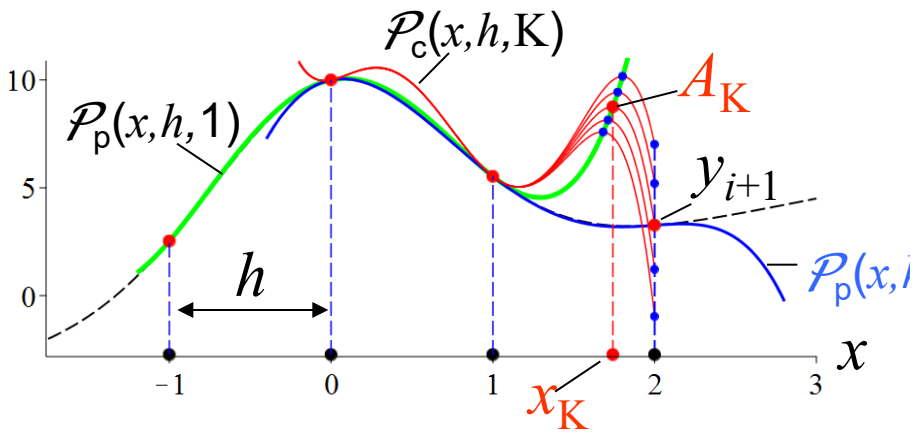
PREDICTION y_{i+1}

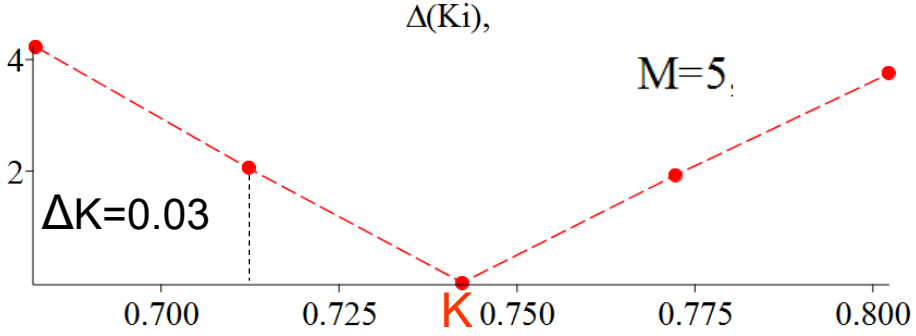
The hypothesis of the existence of a point x_K, which ensures the accuracy of the predicting y_{i+1}:



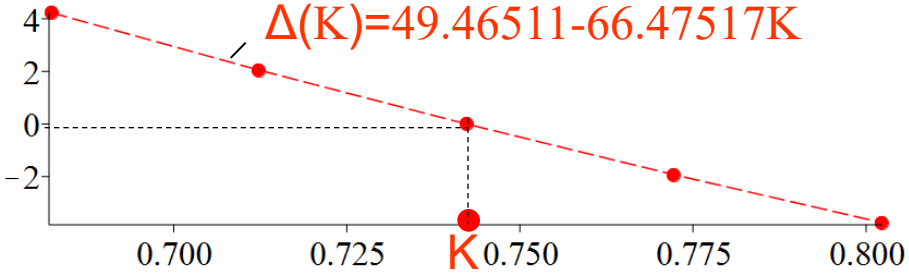
Finding the Number K

For pick up the K the four initial values $y_1,\ldots,y_4$, obtained by the fourth-order RK method are used: $\Delta_i = |y_4 - \mathcal{P}_5(K_i)|$



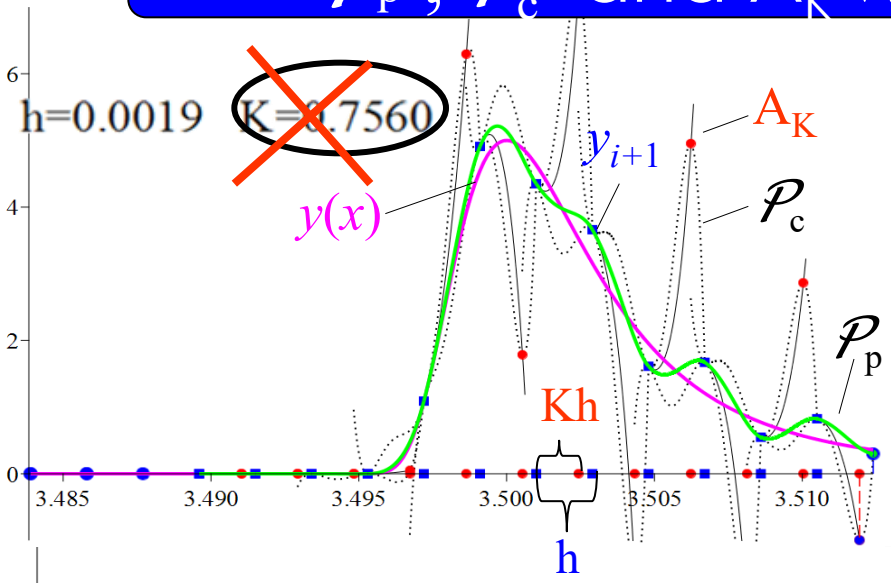


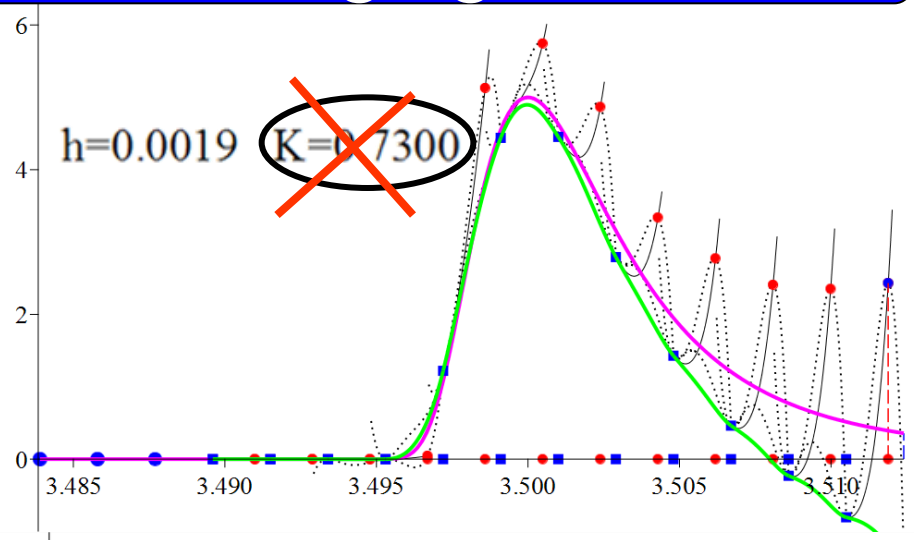
K=0.7423 Δ(K)= −0.000695791679806934

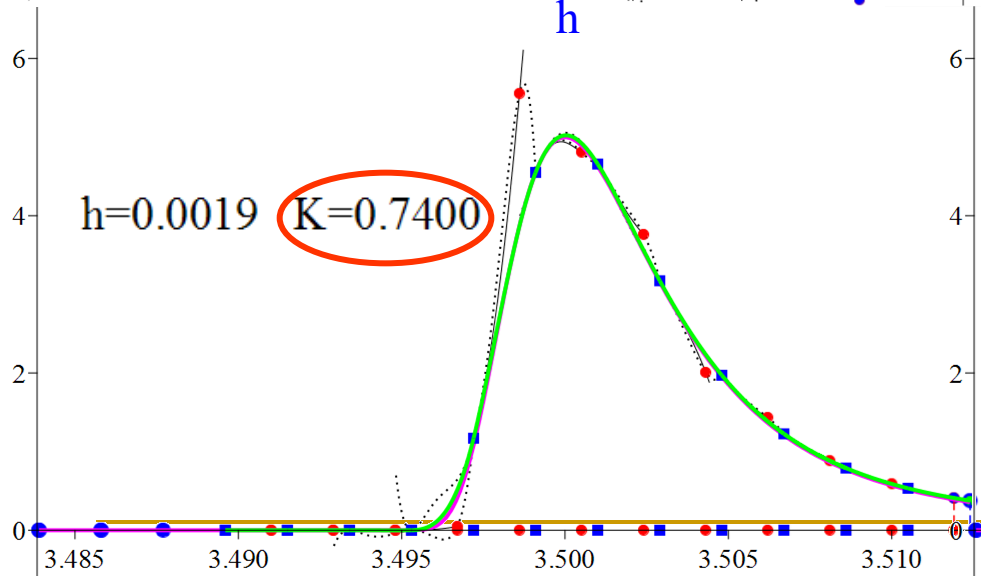


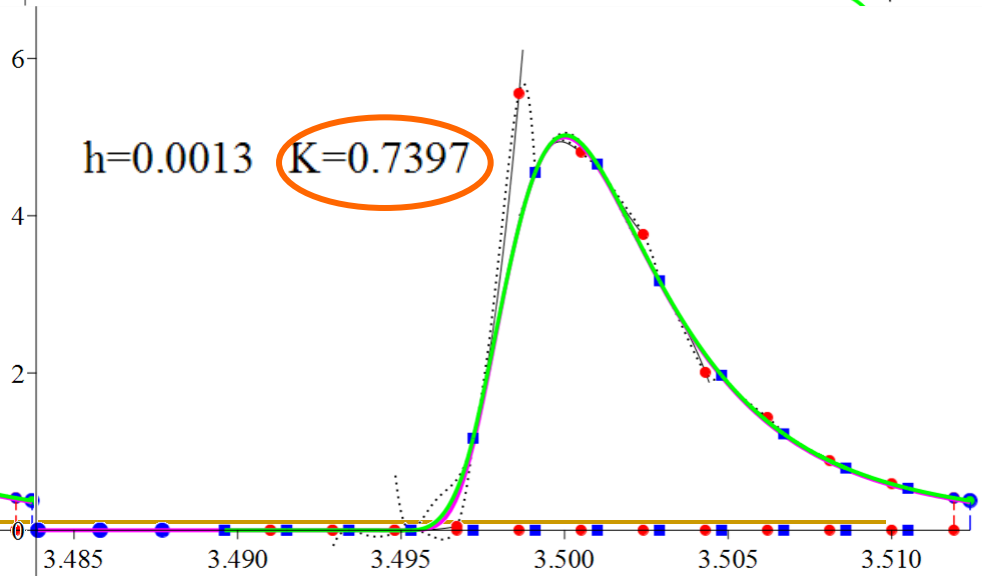
K=0.7441140736 Δ(K)= −1.49540113625335 10^{-9}

$\mathcal{P}_p$, $\mathcal{P}_c$ and A_K when changing h and K





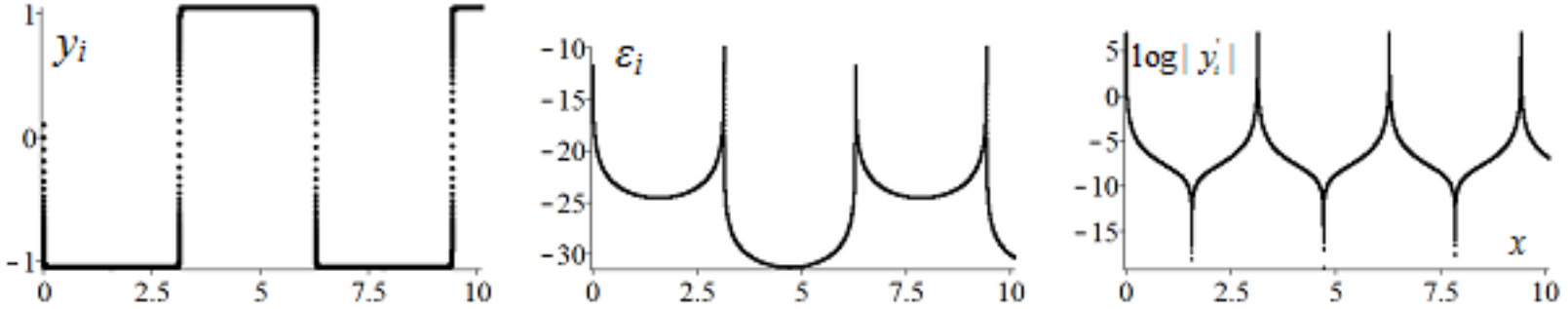




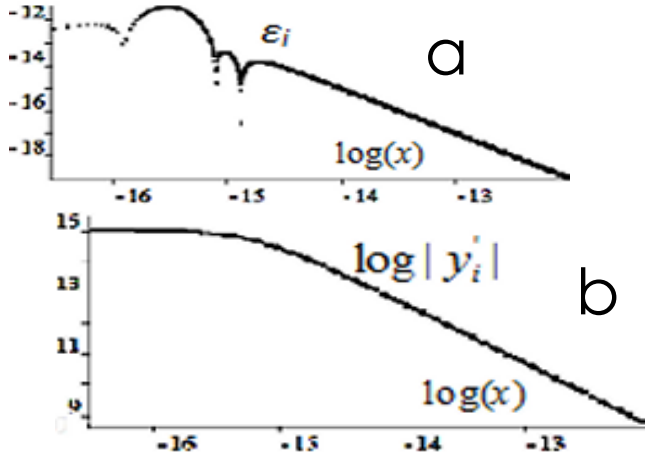
SOLVING A STIFF CAUCHY PROBLEM [*] BY THE PC-BEM

Example:

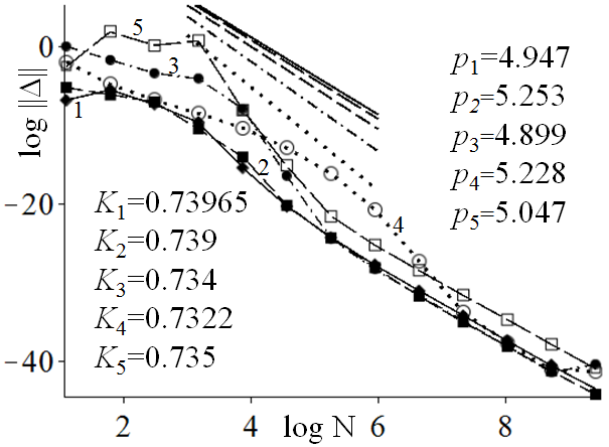
[*] E. K. Zholkovskii, A. A. Belov, and N. N. Kalitkin, “Solution of stiff Cauchy problems with explicit schemes with geometrical-adaptive step selection,” KIAM Preprint No. 227 (Keldysh Inst. Appl. Math. RAS, Moscow, 2018).



The results of testing PC-BEM at $\lambda = 10^3$ $h = 9\times10^{-5}$ $K = 0.740$



$\lambda = 10^{15}$ $h = 5\times10^{-17}$ $K = 0.7385$



Errors calculated by the Richardson

IBR-2M Spectrum Normalization Algorithm

The higher-degree BEM-polynomials are used to determine baselines (BL).

Higher-degree polynomials have advantageous in terms of **accuracy** and approximation **quality**, as they yield **lower residual variance and more closely approximate experimental values**.

The **conditioning** problem in BEM is solved using parameters n, M, x_0, α, β.

The IBR-2M spectrum baseline extraction algorithm

In the IBR-2M spectra $\{\tilde{S}_i\}_{i=1}^{N}$ the BL are nonlinear and the higher-degree BEM polynomial P_{BL} is it model, presented in the form of a "matryoshka ".

$$P_{BL} = \Pi_0 + Q(\Pi_1 + Q(\Pi_2 + \ldots + Q\Pi_m))\ldots), \; m = \lfloor n/3 \rfloor$$

The dynamics of the spectral structure of the **pulsed energy noise** of the IBR-2M reactor was studied by subtracting the resonant baseline from the full array of data on pulse energy fluctuations **recorded at the IBR-2M from 2011 to the end of 2021 [6]**.

Two stages of spectrum baseline extraction algorithm :

I. Pre-processing of the original spectrum (Fig. 1a, b).

At the pre-processing stage, a data array is created in which large peaks are “cut” to specified levels in order to reduce their impact on the accuracy and quality of smoothing (Fig. 1b).

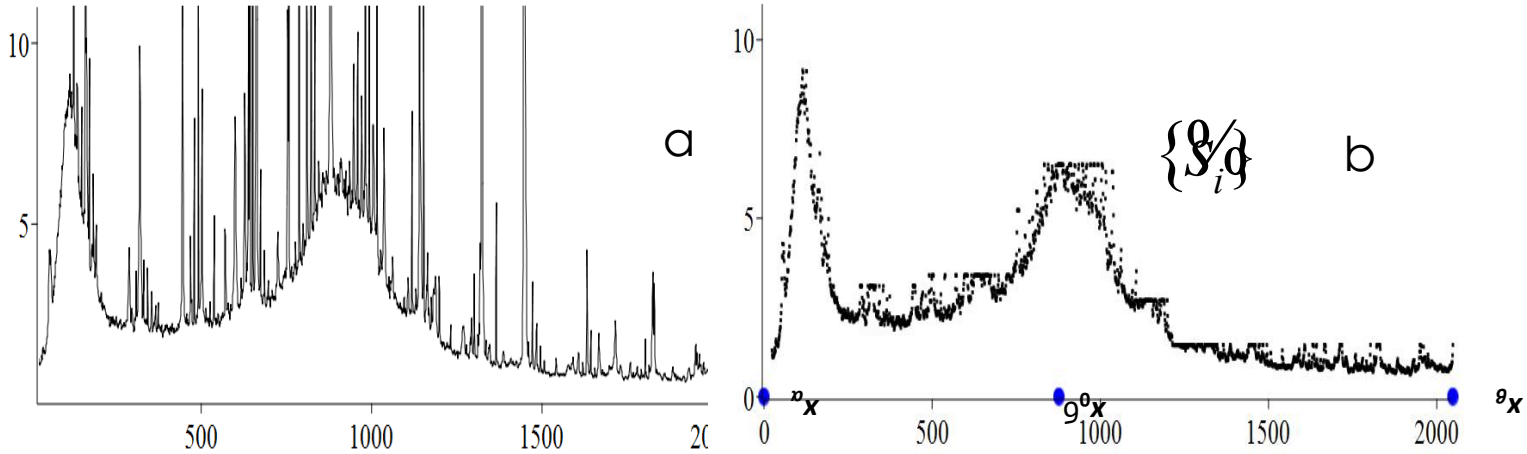


Fig.1.Original spectrum (a); Pre-processing of the original spectrum (b).

II. Calculating the coefficients of the baseline (Fig. 2a,b):

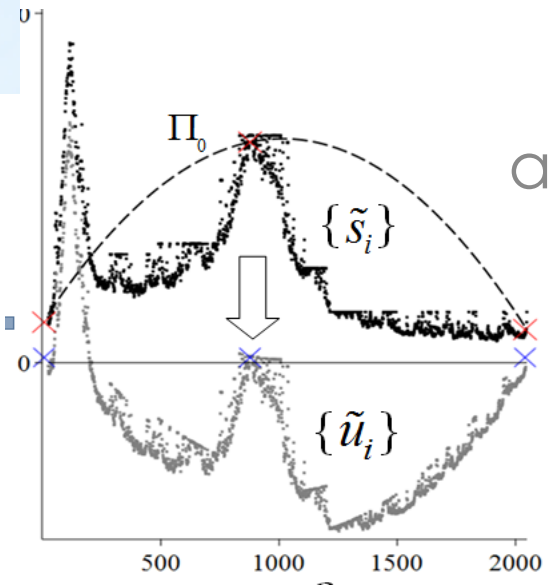
$$N = 2048,\ n = 30,\ M = 5,\ x_0 = 877,\ \alpha = -876,\ \beta = 1171$$

$$\hat{c}_{k0} = (2M+1)^{-1} \sum_{i=-M}^{M} S_{i0},\ k = \overline{1,3}, \qquad \Pi_0 = \sum_{k=1}^{3} w_k \hat{c}_{k0}$$

$$S_{i0} \xrightarrow{\hat{c}_{0k}} u_{i0} \quad \left\{ u_{i0} = S_{i0} - \Pi_{i0} \right\}_{i=1}^{N},\ i = \overline{1,N} \qquad 5 << n < \mathrm{N}$$

$$\partial \sum_{i=1}^{N} \left[u_{i0} - \sum_{j=1}^{m} \sum_{k=1}^{3} b_{ijk} c_{jk} \right]^2 / \partial c_{jk} = 0$$

$$\hat{\mathbf{c}} = [\mathbf{B}^T \mathbf{B}]^{-1} \mathbf{B}^T \tilde{\mathbf{u}}$$



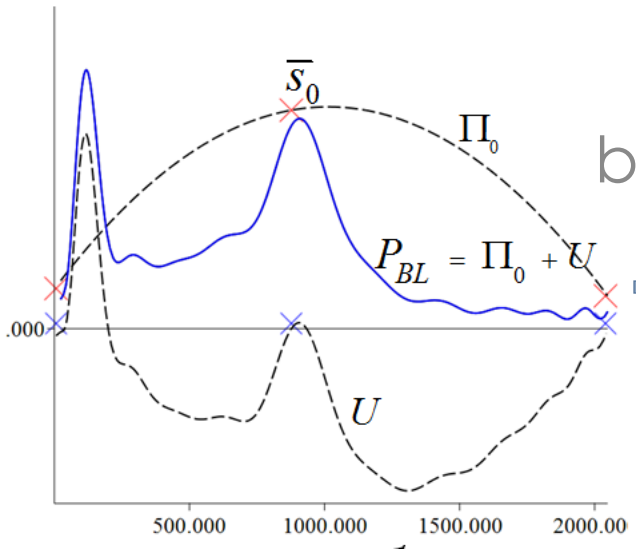
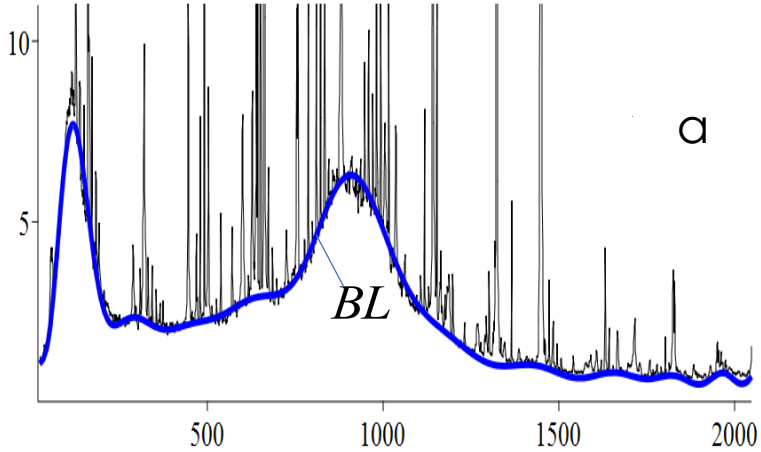


Fig.2. Data transform $\{\tilde{s}_i\} \Rightarrow \{\tilde{u}_i\}$ (a) and BEM-Polynomials U , P_{BL} (b)



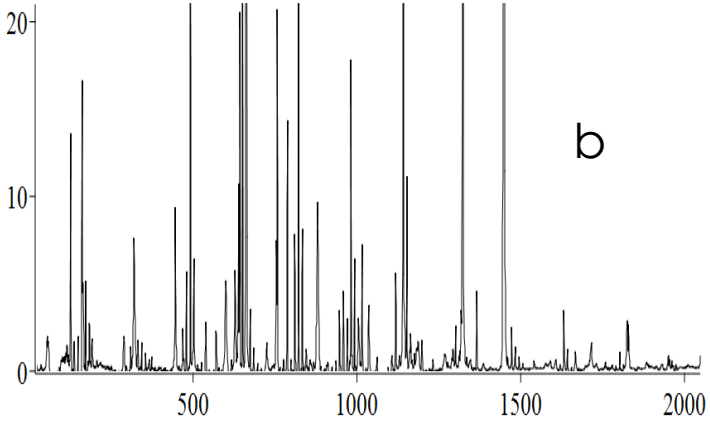


Fig. 3. Baseline (a) and line spectrum (b)

$$n = 30,\ M = 5,\ x_0 = 877,\ \alpha = -876,\ \beta = 1171$$

Conclusion

A fundamentally new type of the polynomials (BEM-polynomials) has been developed for functions approximation and for smoothing of noisy data.

The BEM- polynomial provide on the three-point grid:

- Reduction the order of derivatives more then three times .
- Reduction sensitivity to conditionality of a task.
- Reduction of the computational complexity.
- Stability of calculations vicinity zero.
- Control over the quality of the regression matrix.
- Robustness of calculations to errors in data.
- Expanding the possibility of existing classical methods. …

By the BEM-polynomials has been solve several current tasks: the segmentation of 2D-contours, a numerical solution the stiff Cauchy problem, neutron noise analysis by higher-degree polynomials, etc.

References

[1] P. L. Chebyshev, *Selected Works* (Akademia Nauk SSSR, Moscow, 1955), page 639, [in Russian].

[2] N.D. Dikusar. The Basic Element Method, Math. Models and Comp. Simulat., 2011, Vol. 3, No.4, pp. 492–507. https://doi.org/10.1134/S2070048211040053

[3] Nikolay Dikusar. Shape Approximation Based on Higher-Degree Polynomials, Proceedings of MMCP2017, The European Physical Journal, EPJ Web of Conferences “Mathematical Modeling and Computational Physics”, DOI:10.1051/epjconf/201817301003, www.epj-conferences.org, 173(4):01003 .

[4] N.D. Dikusar, Numerical Solution of the Cauchy Problem Based on the Basic Element Method. ISSN 2070-0482, Mathematical Models and Computer Simulations, 2023, Vol. 15, No. 6, pp. 1024–1036. DOI: 10.20948/mm-2023-05-06.

[5] N.V. Korepanova, N.D. Dikusar, Yu.N. Pepelyshev, M. Dima. Neutron Noise Analysis using the Basic Element Method . Annals of Nuclear Energy, 131, (2019). 475-482. https://doi.org/10.1016/j.anucene.2019.04.008

[6] Pepelyshev Yu.N., Dikusar N.D., Tsogtsaikhan Ts., Sumkhuu D. Investigation of Pulse Energy Fluctuations of the IBR-2M Reactor Using the Basic Element Method. JINR R13-2025-49, acsepted in the journal "Problems of Atomic Science and Technology. Series: Physics of Nuclear Reactors" (in Russian).

Thanks for Attention!

