Numerical solution of elliptic boundary value problems using edge-based approximation

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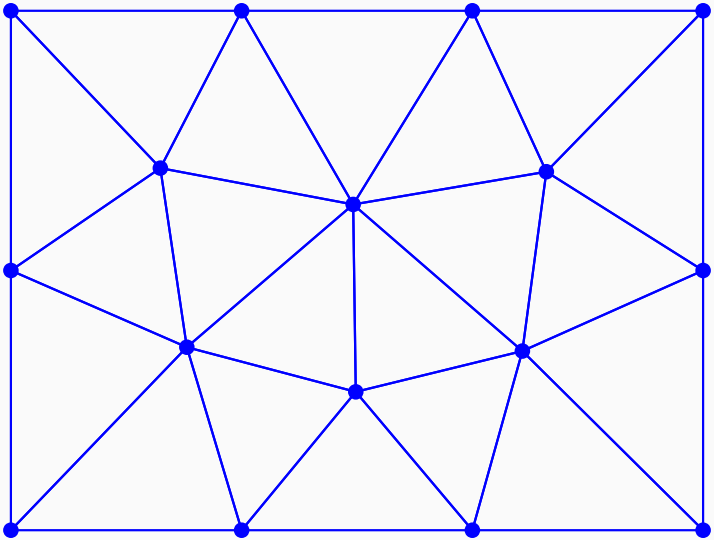
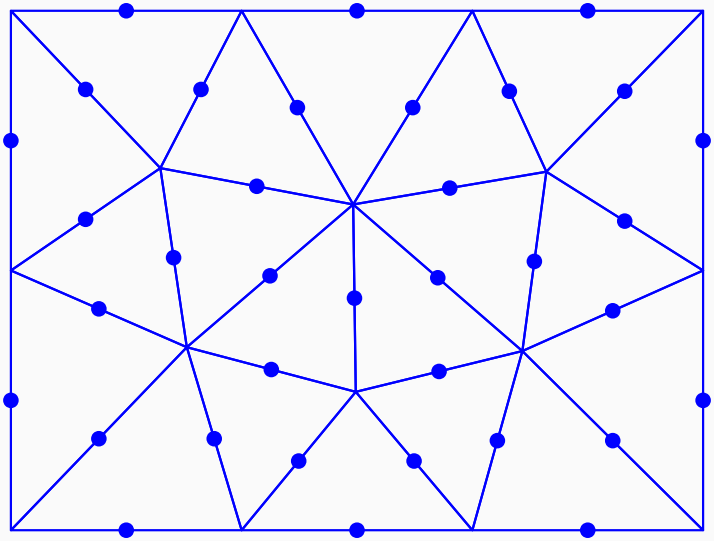
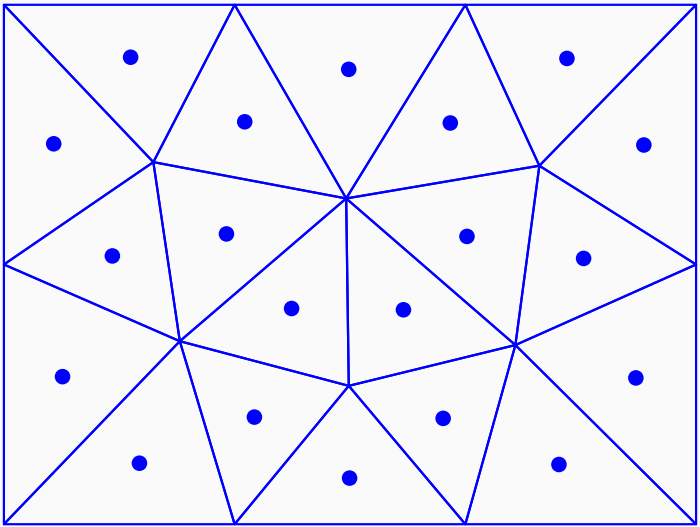
Selection of approximation nodes: mesh nodes



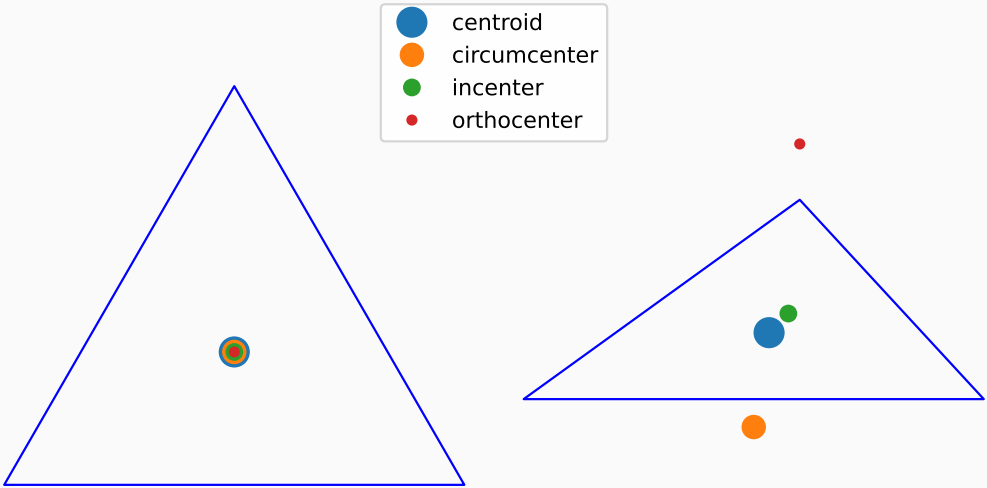
Selection of approximation nodes: edge midpoints



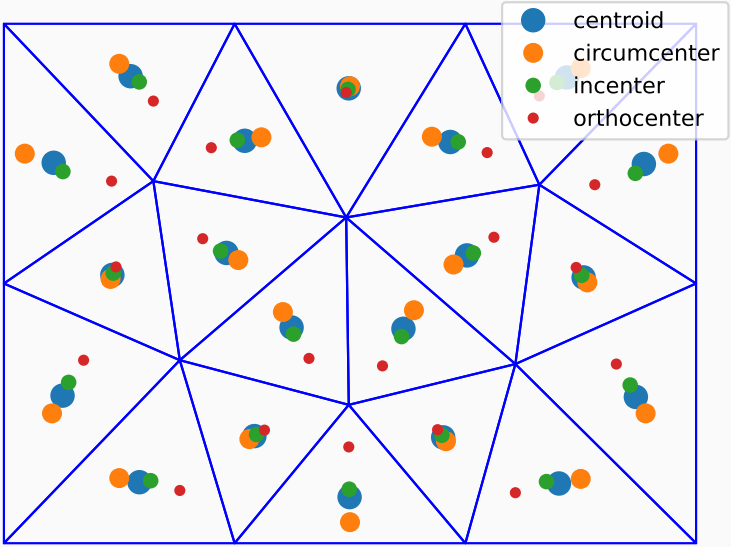
Selection of approximation nodes: cell centers



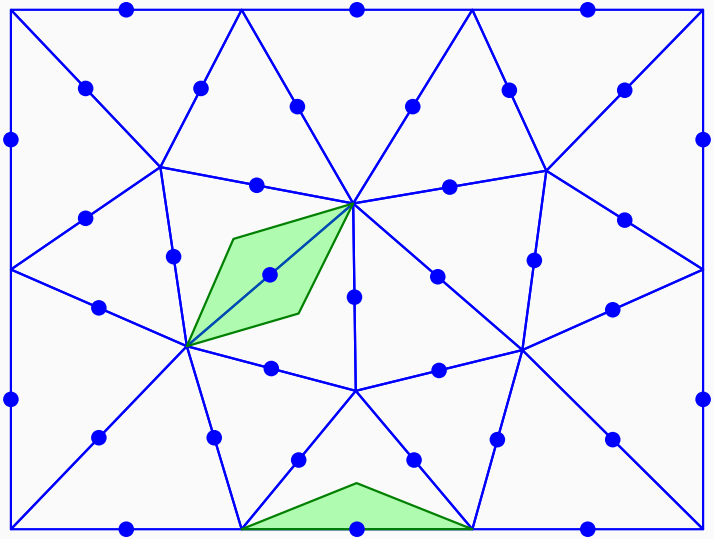
Different choices of cell centers



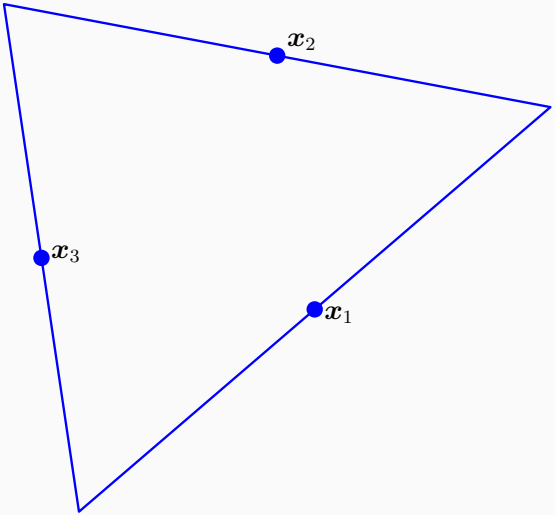
Different choices of cell centers



Finite volumes



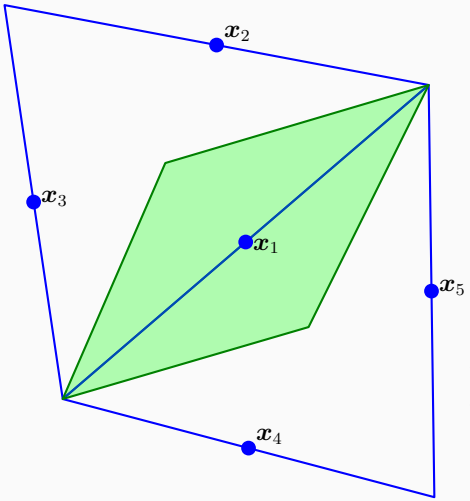
Gradient approximation



$$\begin{aligned}\operatorname{grad} u &= \lim_{V\to 0} \frac{1}{V} \int_S u\boldsymbol{n}\ dx \\ &\approx \frac{1}{V} \int_S u\boldsymbol{n}\ dx \\ &\approx \frac{1}{V} \sum_{i=1}^{3} l_i u_i \boldsymbol{n}_i\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \lim_{V\to 0} \frac{1}{V} \int_S u\ dy \\ \frac{\partial u}{\partial y} &= -\lim_{V\to 0} \frac{1}{V} \int_S u\ dx\end{aligned}$$

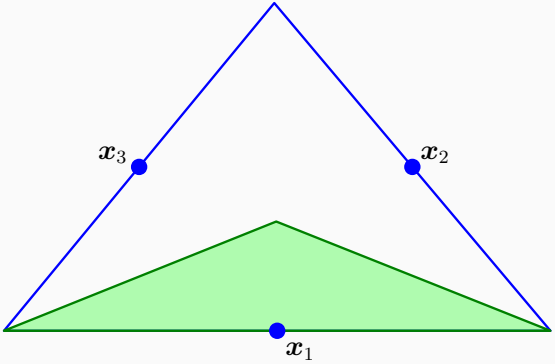
Divergence approximation



$$\begin{aligned}\operatorname{div}\varphi &= \lim_{V\to 0}\frac{1}{V}\int_S \varphi\cdot\boldsymbol{n}\,dx\\ &\approx \frac{1}{V}\int_S \varphi\cdot\boldsymbol{n}\,dx\\ &\approx \frac{1}{V}\sum_{i=1}^{4} l_i\varphi\cdot\boldsymbol{n}_i,\end{aligned}$$

where $\varphi = \varphi_k$ when the i-th edge belongs to the k-th triangle

Robin boundary condition



$$\frac{\partial u}{\partial \nu} + \gamma u = \psi, \quad \boldsymbol{x} \in \partial\Omega$$

$$\int \frac{\partial u}{\partial \nu}\, dx = \int -\gamma u + \psi \, dx \approx (-\gamma u_1 + \psi)\, l_1$$

Test problem

$$
\begin{aligned}
-\operatorname{div}(K \operatorname{grad} u) + ru &= f, && \boldsymbol{x} \in \Omega \\
\frac{\partial u}{\partial \nu} + \gamma u &= \psi, && \boldsymbol{x} \in \partial\Omega
\end{aligned}
$$

Conormal derivative

$$
\frac{\partial u}{\partial \nu} = K \operatorname{grad} u \cdot \boldsymbol{n}
$$

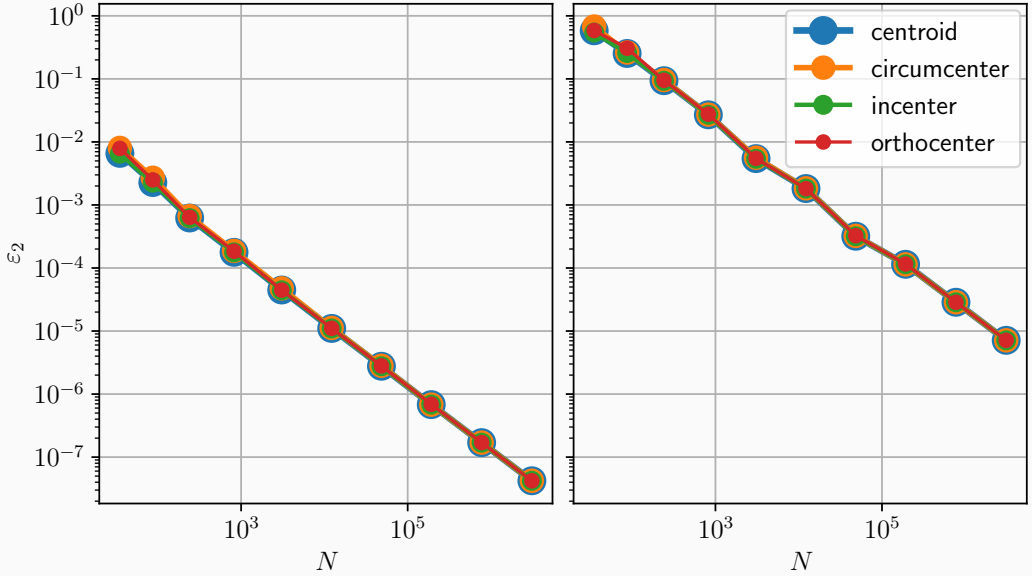
Domain

$$
\Omega = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2),\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 0.75\}
$$

Parameters

$$
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}, \quad r = 1, \quad \gamma = 1, \quad u = e^{\sqrt{1+x_1}x_2}
$$

Computational experiment: different centers



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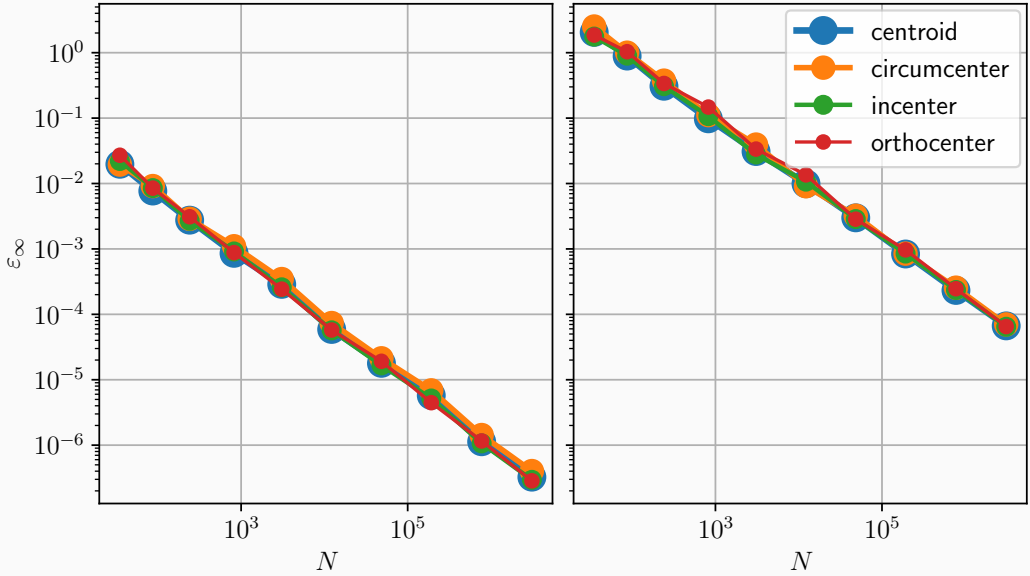
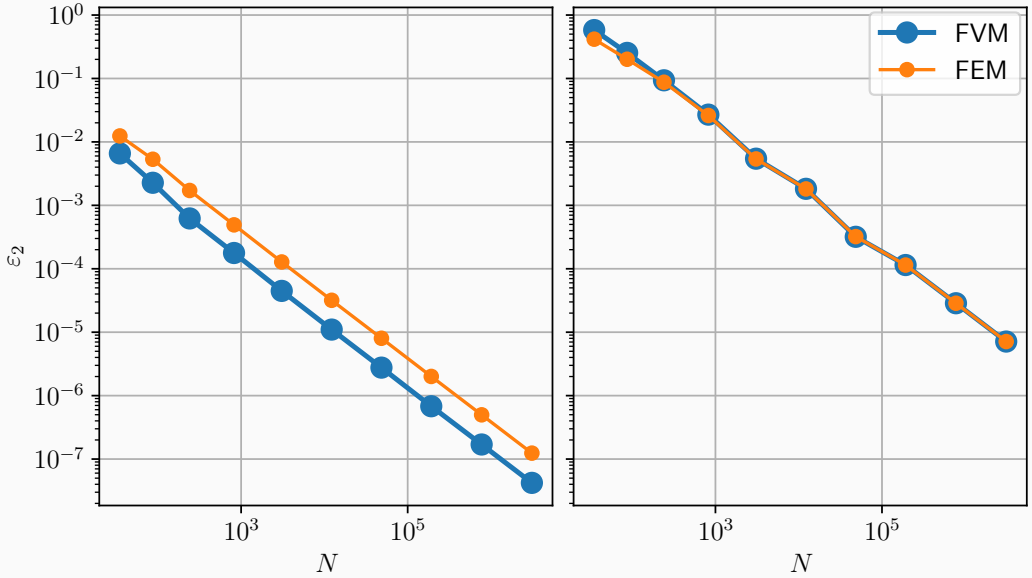


Computational experiment: FVM and FEM



Computational experiment: FVM and FEM

