Transition to a many-body localization state in the disordered spin-1/2 Ising chain in the tensor network approach

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Outline:

1. Many-Body Localization (MBL)
2. Basics of the time evolution of a quantum system in the tensor network
3. Disordered TFIM model : a one-dimensional Ising model with a long-range exchange interaction between spins in a disordered transverse magnetic field
4. Results

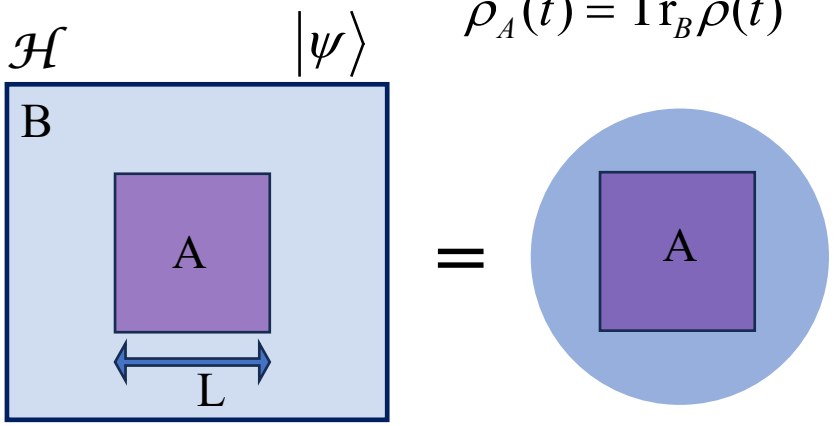
1. Many-Body Localization (MBL)

Complete isolated system A+B: $\mathcal{H}$

$$|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle = \sum_\alpha e^{-i\varepsilon_\alpha t}|\varepsilon_\alpha\rangle\langle\varepsilon_\alpha|\psi(0)\rangle$$

for $t=0$ $\quad \rho(0) = \rho_A \otimes \rho_B$

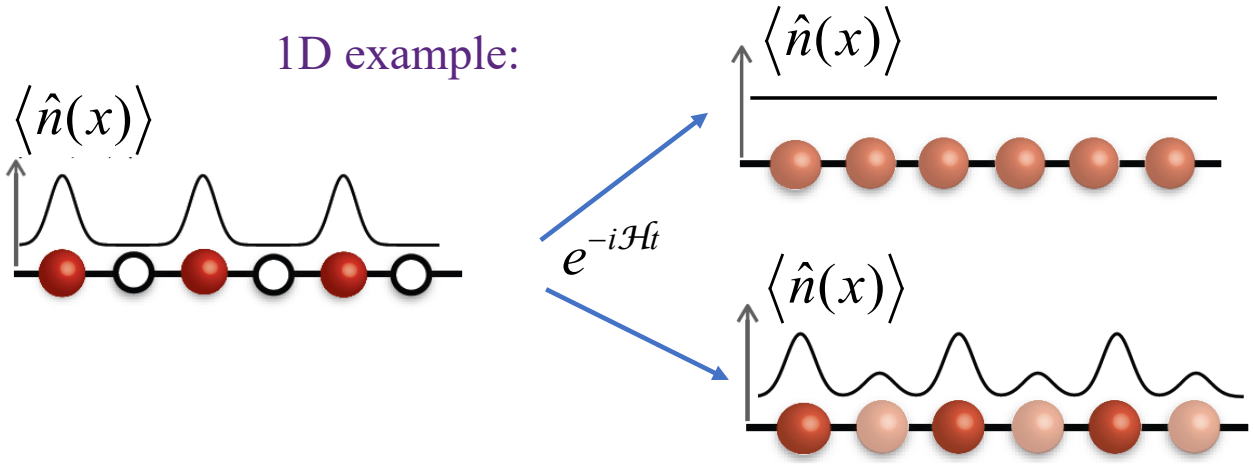
$$\rho_A(t) = \mathrm{Tr}_B \rho(t)$$



Problem:

$$\rho_A(t\to\infty) \stackrel{?}{\approx} \rho_{A,\mathrm{th}} = \frac{1}{Z}e^{-\beta H}$$

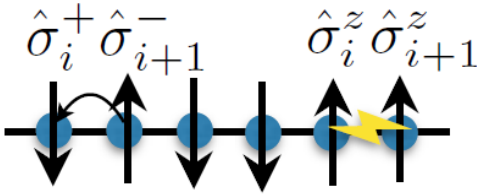
1D example:



thermalized system (memory is lost)

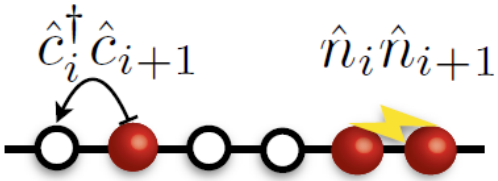
escaping thermalization (memory is preserved)

Heisenberg spin chain

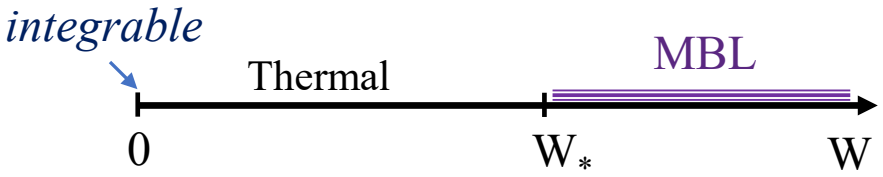


$$\hat{H}_{XXZ} = \frac{J_\perp}{2}\sum_i^L \left(\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y\right) + \sum_i^L \left(\frac{J_z}{2}\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + h_i^z \hat{\sigma}_i^z\right), \qquad h_i^z \in [-W;W]$$

1D spinless fermions



$$\hat{H} = t\sum_i \left(\hat{c}_i^+ \hat{c}_{i+1} + h.c.\right) + V\sum_i \hat{n}_i \hat{n}_{i+1} + \sum_i \varepsilon_i \hat{n}_i, \qquad \varepsilon_i \in [-W;W]$$



Dmitry A. Abanin, et al. Rev. Mod. Phys. 91, 021001, 2019

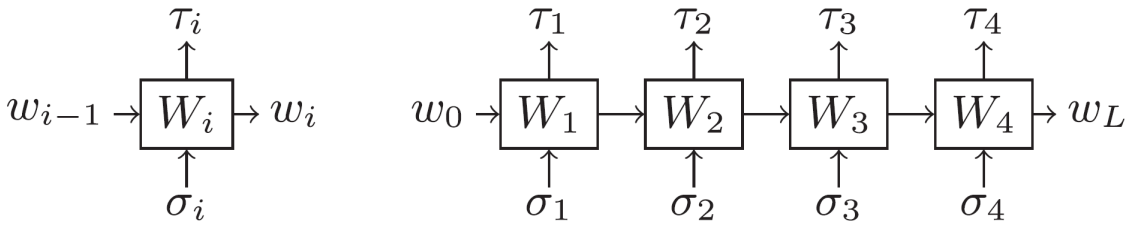
2. Time-evolving a matrix product state with long-ranged interactions

a) Hamiltonian H with dimension L in MPO

For local Hilbert spaces $\mathcal{H}_i, i \in [1, L]$ (d_i-dimension each $\mathcal{H}_i$) operator H in the tensor product $\mathcal{H} = \otimes_i \mathcal{H}_i$:

$$\hat{H} = \sum_{\sigma_1 \tau_1} \cdots \sum_{\sigma_L \tau_L} c^{\sigma_1 \ldots \sigma_L}_{\tau_1 \ldots \tau_L} \left| \tau_1 \ldots \tau_L \right\rangle \left\langle \sigma_1 \ldots \sigma_L \right|$$

$$\sum_{w} \left(W_1^{\sigma_1 \tau_1} \right)_{w_0, w_1} \left(W_2^{\sigma_2 \tau_2} \right)_{w_1, w_2} \cdots \left(W_L^{\sigma_L \tau_L} \right)_{w_{L-1}, w_L} = c^{\sigma_1 \ldots \sigma_L}_{\tau_1 \ldots \tau_L}$$

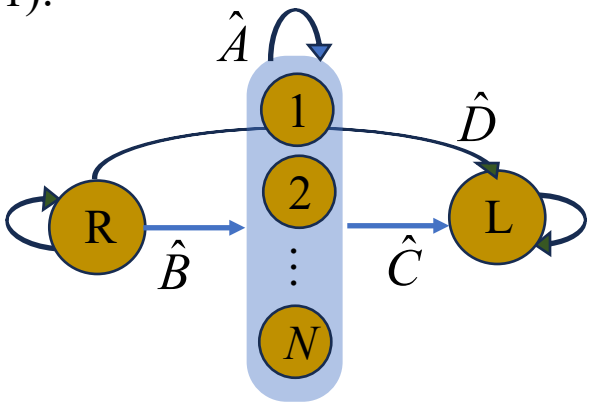


$$\hat{H} = \hat{H}_{L_i} \otimes \hat{1}_{R_i} + \hat{1}_{L_i} \otimes \hat{H}_{R_i} + \sum_{\alpha_i = 1}^{N_i} \hat{h}_{L_i, \alpha_i} \otimes \hat{h}_{R_i, \alpha_i}$$

Paeckel, S. et al. *Ann. Phys.* **411**, 167998 (2019).

a recursion between the decomposition bond $(i-1,i)$ and $(i,i+1)$:

$$\begin{pmatrix} \hat{H}_{R_{i-1}} \\ \hat{h}_{R_{i-1}, \alpha_{i-1}} \\ \hat{\mathbf{1}}_{R_{i-1}} \end{pmatrix} = \underbrace{\begin{matrix} 1 \\ N_{i-1} \\ 1 \end{matrix} \begin{pmatrix} \overset{1}{\hat{\mathbf{1}}} & \overset{N_i}{\hat{\mathbf{C}}} & \overset{1}{\hat{\mathbf{D}}} \\ 0 & \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ 0 & 0 & \hat{\mathbf{1}} \end{pmatrix}}_{\hat{W}_i} \otimes \begin{pmatrix} \hat{H}_{R_i} \\ \hat{h}_{R_i, \alpha_i} \\ \hat{\mathbf{1}}_{R_i} \end{pmatrix}$$



b) Evolution operator in MPO

From simplest Euler approximation of the operator exponential $e^{-i\delta_t \hat{H}} = 1 - i\delta_t \sum_j \hat{H}_j + O(\delta_t^2)$

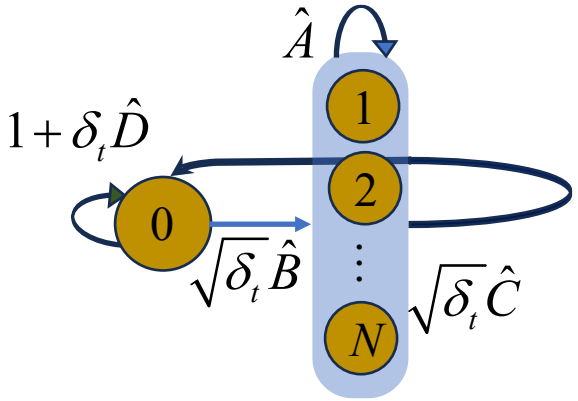
we move to

$$1 - i\delta_t \sum_j \hat{H}_j \Longrightarrow \prod_j \left(1 - i\delta_t H_j\right) \equiv U^{\mathrm{I}}(\delta_t) \Longrightarrow U^{\mathrm{II}}(\delta_t)$$

U^{II} improves U^{I} by keeping more quantum correlations.

Representation of the evolution operator in MPO presentation:

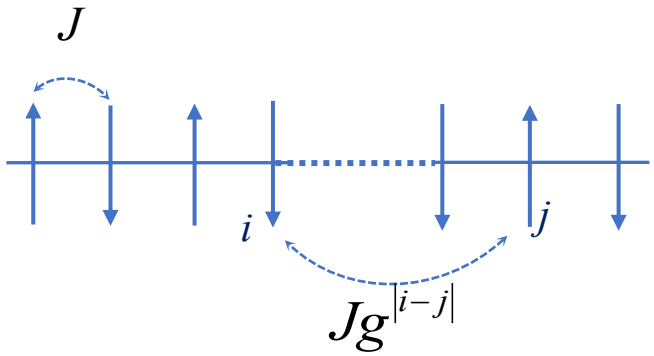
$$W^I_{(j)}(\delta_t) = \begin{array}{c c} & \begin{array}{cc} 1 & N_{j-1} \end{array} \\ \begin{array}{c} 1 \\ N_{j-1} \end{array} & \begin{pmatrix} \hat{\mathbf{1}}_{(j)} + \delta_t \hat{\mathbf{D}}_{(j)} & \sqrt{\delta_t}\,\hat{\mathbf{C}}_{(j)} \\ \sqrt{\delta_t}\,\hat{\mathbf{B}}_{(j)} & \hat{\mathbf{A}}_{(j)} \end{pmatrix} \end{array}$$



3. 1D long-range Ising model in a disordered transverse magnetic field

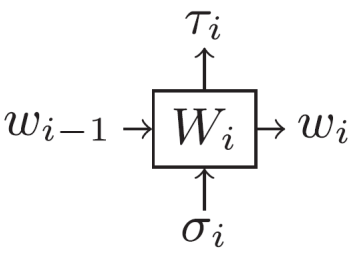
$$H=\sum_{i<j}J_{ij}\sigma_i^x\sigma_i^x+\frac{1}{2}\sum_i\left(B+D_i\right)\sigma_i^z,$$

where $J_{ij}=\dfrac{J}{|i-j|^{\alpha}}, \qquad D_i\in[-W,W]$



$$H=\sum_{i<j}\left(\sum_m k_m e^{-\frac{|i-j|}{\chi_m}}\frac{\tilde{J}_m}{2}\right)S_i^xS_j^x+\sum_i h_z(i)S_i^z,$$

Hamiltonian in MPO:



$$W_i=\begin{pmatrix}\hat{1} & S^x & S^x & -h_iS^z\\ 0 & g_1\cdot\hat{1} & 0 & \frac{1}{2}k_1\tilde{J}S^x\\ 0 & 0 & g_2\cdot\hat{1} & \frac{1}{2}k_2\tilde{J}S^x\\ 0 & 0 & 0 & \hat{1}\end{pmatrix}, \qquad g_l=e^{-\frac{1}{\chi_l}}$$

Problem: $W_*=?$

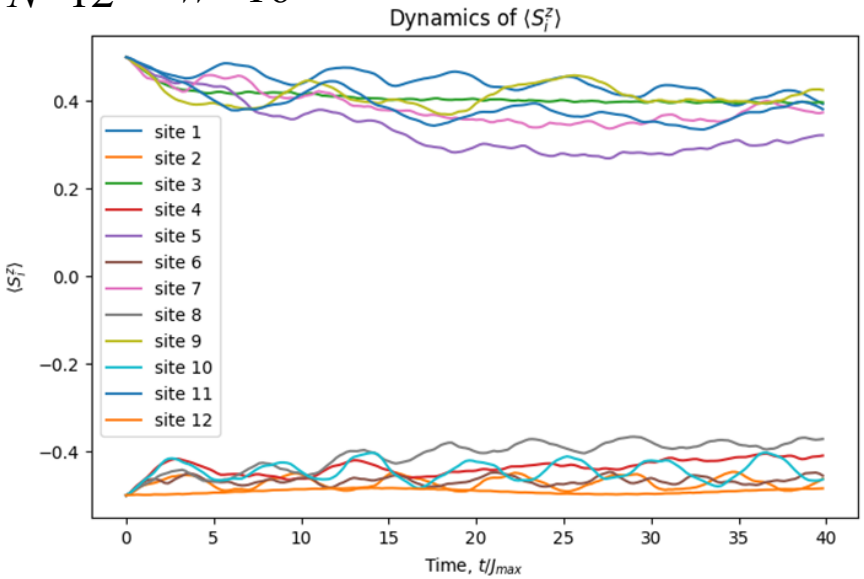


4. Results

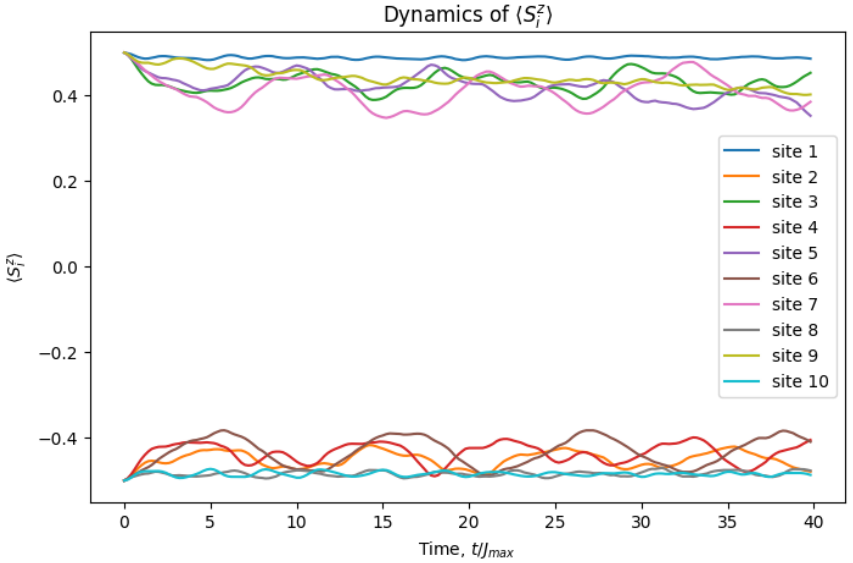
$\alpha = 1.13, \quad B = 4$

a) evolution of local observables

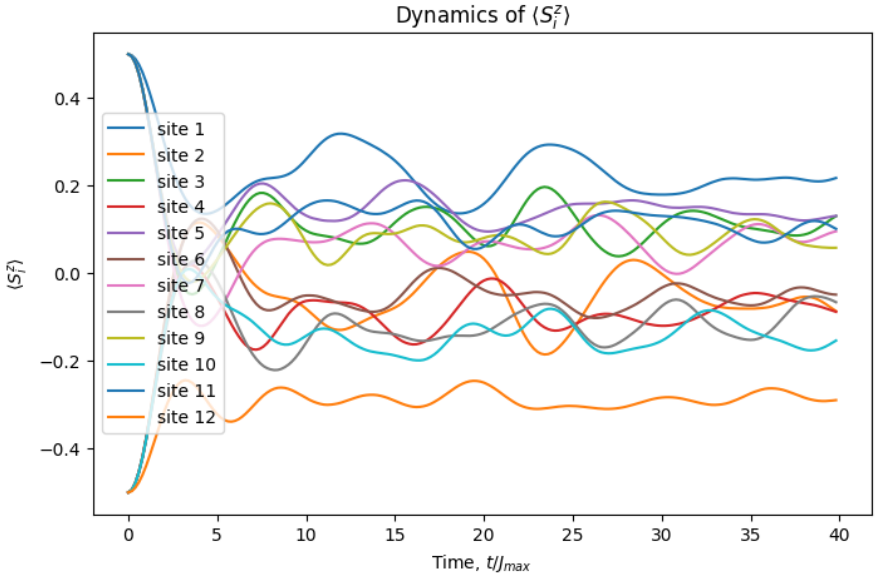
N=12 W=16



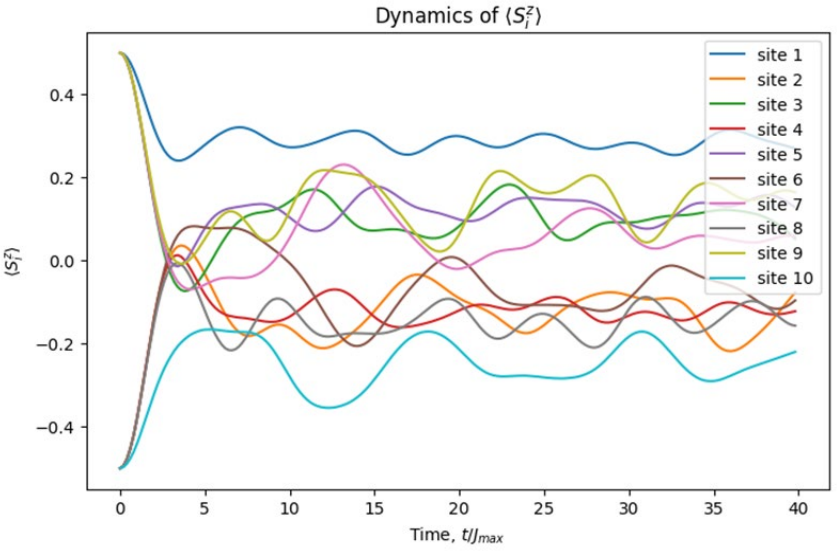
N=10 W=16



W=1



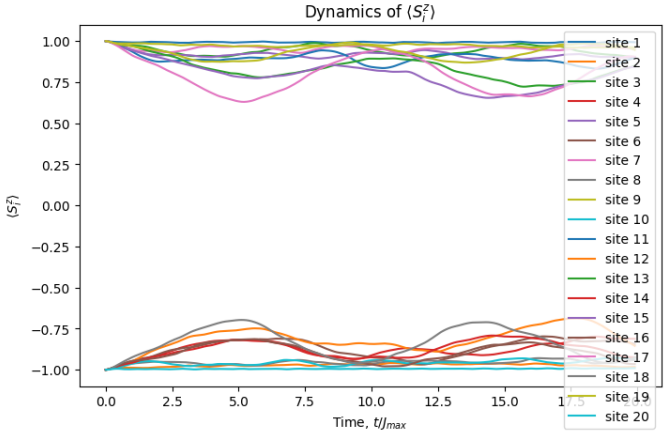
W=1



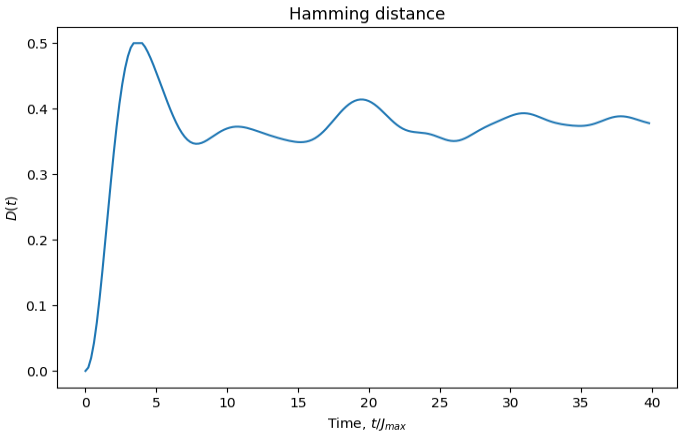
Smith, J. *et al. Nature Phys* **12**, 907–911 (2016).

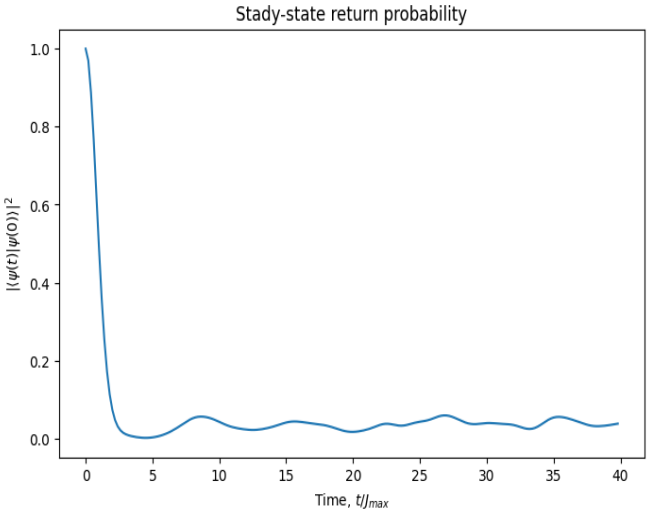
$\alpha = 1.13$

N=20, B=4, W=16

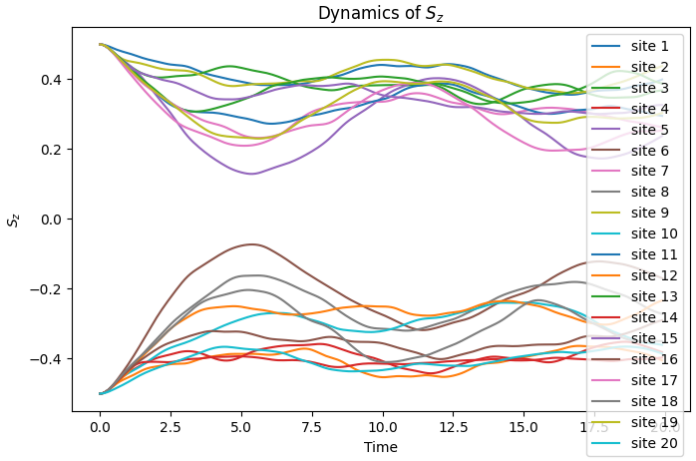


$$D(t) = \frac{1}{2} - \frac{1}{N}\sum_k (-1)^k \left\langle \psi(t) \middle| S_k^z \middle| \psi(t) \right\rangle$$



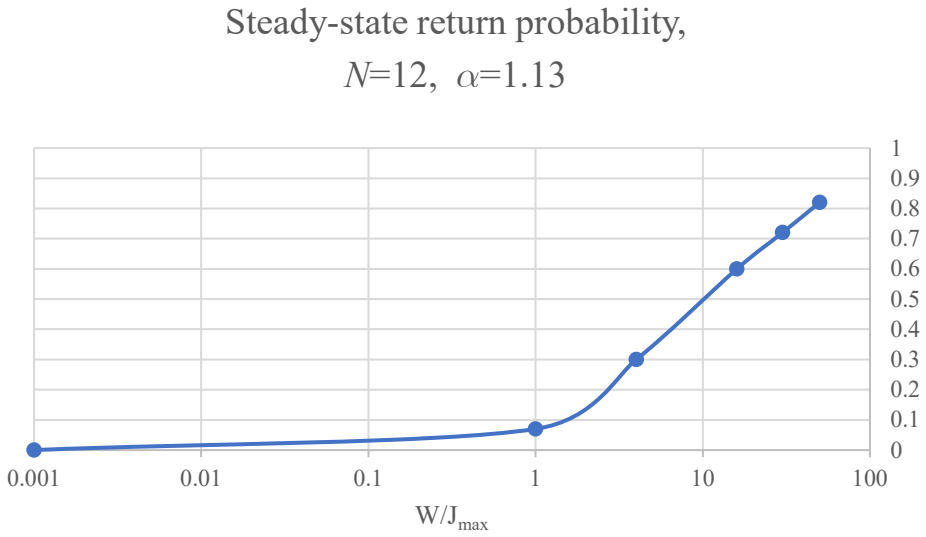


N=20, B=8, W=16



b) crossover behavior as evidence of MBL in finite system

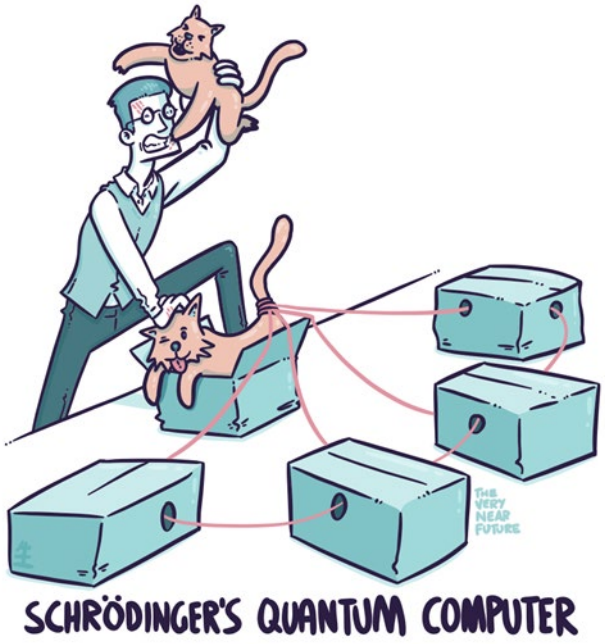
$$\left|\langle\psi(t)|\psi(0)\rangle\right|^2$$



Conclusions and Perspectives

- TN methods allows us to extend computations of MBL to the system size $N > 100$
- Along this way a crossover behavior of small systems transforms into that of MBL phase transition
- Comparison with the finite size measurements of MBL in trapped ion quantum simulator is plausible

Thank you for your attention!



Standard MBL properties

Many-body localization is a property of a part of the excited spectrum of a many-body interacting and disordered system characterized by the following correlated characteristics:

- the existence of a complete set of local integrals of motion, separate for each disorder realization, implying quantum integrability;
- statistical properties of spectra corresponding to the Poisson ensemble;
- area-law entangled eigenstates;
- along-lasting memory of the initial nonstationary state in the time dynamics;
- a logarithmic growth of the entanglement entropy in a quench from an initially low-entanglement state.

Approximation order:

For $(p-1)$ approximation order of method: $U^I(t_1)U^I(t_2)\cdots U^I(t_N) = U^I(t_1) + O(Lt^p)$

where $\{t_a\}$ define from: $\sum_a t_a = t, \quad \sum_{a<b} t_a t_b = \frac{1}{2}t^2, \quad \sum_a t_a^2 = 0$