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Trajectory analysis for the Moyal equation

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Introduction

The Moyal equation

$$\frac{\partial}{\partial t}W+\frac{\vec{p}}{m}\cdot\nabla_r W-\nabla_r U\cdot\nabla_p W=\sum_{l=1}^{+\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l+1)!}U\left(\overleftarrow{\nabla}_r\cdot\vec{\nabla}_p\right)^{2l+1}W.$$

The Wigner function

$$W(\vec{r},\vec{p},t)=\frac{1}{(2\pi\hbar)^3}\int_{(\infty)}\left\langle\vec{r}-\frac{\vec{s}}{2}\right|\hat{\rho}\left|\vec{r}+\frac{\vec{s}}{2}\right\rangle e^{-i\frac{\vec{p}\cdot\vec{s}}{\hbar}}d^3s.$$

The second Vlasov equation

$$\frac{\partial}{\partial t}f^{1,2}+\vec{v}\cdot\nabla_r f^{1,2}+\mathrm{div}_v\left[f^{1,2}\left\langle\dot{\vec{v}}\right\rangle_{1,2}\right]=0.$$

The Vlasov-Moyal approximation for $\langle\dot{v}_\mu\rangle_{1,2}$

$$\left\langle\dot{v}_\mu\right\rangle_{1,2}=\sum_{l=0}^{+\infty}\frac{(-1)^{l+1}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!}\frac{\partial^{2l+1}U}{\partial x_\mu^{2l+1}}\frac{1}{f^{1,2}}\frac{\partial^{2l}f^{1,2}}{\partial v_\mu^{2l}},$$

where $f^{1,2}(\vec{r},\vec{v},t)=m^3W(\vec{r},\vec{p},t)$, $U_N(x,y,z,t)=\sum_{k_1,k_2,k_3=0}^{N}\left(a_{k_1}x^{k_1}+b_{k_2}y^{k_2}+c_{k_3}z^{k_3}\right)$.

Introduction

$$\frac{\partial}{\partial t} f^{1,2} + \vec{v} \cdot \nabla_r f^{1,2} + \mathrm{div}_v \left[f^{1,2} \left\langle \dot{\vec{v}} \right\rangle_{1,2} \right] = 0.$$

$\times \frac{1}{2} v^2, \int\limits_{(\infty)} d^3v$ $\qquad$ $\int\limits_{(\infty)} d^3v$ $\qquad$ $\times v_\mu, \int\limits_{(\infty)} d^3v$

The first Vlasov equation

$$\frac{\partial}{\partial t} f^1 + \mathrm{div}_r \left[f^1 \left\langle \vec{v} \right\rangle_1 \right] = 0.$$

The Equation of motion in a hydrodynamic approximation

$$\frac{d}{dt} \left\langle v_\mu \right\rangle_1 = \left(\frac{\partial}{\partial t} + \left\langle v_\lambda \right\rangle_1 \frac{\partial}{\partial x_\lambda} \right) \left\langle v_\mu \right\rangle_1 = -\frac{1}{f^1} \frac{\partial P_{\mu\lambda}}{\partial x_\lambda} + \left\langle \dot{v}_\mu \right\rangle_1 .$$

Law of conversation of Energy

$$\frac{\partial}{\partial t} \left[\frac{f^1}{2} \left| \left\langle \vec{v} \right\rangle_1 \right|^2 + \frac{1}{2} \mathrm{Tr}\, \mathrm{P}_{\lambda\lambda} \right] + \frac{\partial}{\partial x_\mu} \left[\frac{f^1}{2} \left| \left\langle \vec{v} \right\rangle_1 \right|^2 \left\langle v_\mu \right\rangle_1 + \frac{1}{2} \left\langle v_\mu \right\rangle_1 \mathrm{Tr}\, \mathrm{P}_{\lambda\lambda} + \left\langle v_\lambda \right\rangle_1 \mathrm{P}_{\lambda\mu} + \frac{1}{2} \mathrm{Tr}\, \mathrm{P}_{\lambda\lambda\mu} \right] = \int\limits_{(\infty)} f^{1,2} \left\langle \dot{v}_\lambda \right\rangle_{1,2} v_\lambda \, d^3v,$$

where $\mathrm{P}_{\lambda \ldots \mu} \overset{\det}{=} \int\limits_{(\infty)} f^{1,2} \left(v_\lambda - \left\langle v_\lambda \right\rangle_1 \right) \ldots \left(v_\mu - \left\langle v_\mu \right\rangle_1 \right) d^3v,$

$$f^1 \left\langle v_\mu \right\rangle_1 = \int\limits_{(\infty)} f^{1,2} v_\mu d^3v, \quad f^1 \left\langle \dot{v}_\mu \right\rangle_1 = \int\limits_{(\infty)} f^{1,2} \left\langle \dot{v}_\mu \right\rangle_{1,2} d^3v, \quad f^1 = \int\limits_{(\infty)} f^{1,2} d^3v.$$

Vector field restoration

The current density vector field

$$\vec{J}^{2}_{1,2} = f^{1,2}\left\langle \dot{\vec{v}} \right\rangle_{1,2} \Rightarrow \operatorname{div}_v \vec{J}^{2}_{1,2} = \operatorname{div}_v \left[f^{1,2}\left\langle \dot{\vec{v}} \right\rangle_{1,2} \right] = \sum_{l=0}^{+\infty} \frac{(-1)^{l+1}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!} U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^{2l+1} f^{1,2}.$$

$$\vec{J}^{2}_{1,2} = \vec{J}^{2(p)}_{1,2} + \vec{J}^{2(s)}_{1,2} = -\nabla_v \Pi^{1,2} + \operatorname{curl}_v \vec{\Upsilon}_{1,2}, \Rightarrow \operatorname{div}_v \vec{J}^{2}_{1,2} = -\Delta_v \Pi^{1,2}, \Rightarrow \Pi^{1,2}(\vec{r},\vec{v},t) = \frac{1}{4\pi} \int_{(\infty)} \frac{1}{|\vec{v}-\vec{v}'|} \operatorname{div}_v \vec{J}^{2}_{1,2} d^3v'.$$

Theorem 1. Let condition for the $\operatorname{div}_v \vec{J}^{2}_{1,2}$ be satisfied and the series admits term-by-term integration, then the expression for the scalar potential $\Pi^{1,2}(\vec{r},\vec{v},t)$ has the form:

$$\Pi^{1,2}(\vec{r},\vec{v},t) = \sum_{l=0}^{+\infty} \frac{(-1)^{l+1}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!} U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^{2l+1} \vartheta^{1,2}, \text{ where } \vartheta^{1,2}(\vec{r},\vec{v},t) \overset{\text{det}}{=} \frac{1}{4\pi} \int_{(\infty)} \frac{f^{1,2}(\vec{r},\vec{v}',t)}{|\vec{v}-\vec{v}'|} d^3v', \text{ and } \Delta_v \vartheta^{1,2} = -f^{1,2}.$$

$$J^{2(p)}_{\mu} = -\frac{\partial}{\partial v_\mu} \Pi^{1,2} = -\sum_{l=0}^{+\infty} \frac{(-1)^{l+1}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!} U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^{2l+1} \frac{\partial \vartheta^{1,2}}{\partial v_\mu}.$$

Vector field restoration

$$\Pi^{1,2} \mapsto \Pi^{1,2} + \mathrm{P}^{1,2}, \quad \Delta_v \mathrm{P}^{1,2} = 0,$$

Adding field $\mathrm{P}^{1,2}, \Delta_v \mathrm{P}^{1,2} = 0$ to expansion will not change the expressions for $\mathrm{div}_v \vec{J}^2_{1,2}$ and $\mathrm{curl}_v \vec{J}^2_{1,2}$. The statement of the problem of restoring field $\vec{J}^2_{1,2}$ with respect to div and curl has three types :

(1) an inner problem (in a bounded domain Ω_i with smooth boundary Γ_i and a condition on it);

(2) an external problem (in unbounded domain with a notch $\Omega_e = \mathbb{R}^3 / \Omega_{cut}$ and a boundary condition at Γ_e);

(3) a problem for the entire space $\mathbb{R}^3$ (it is assumed that the Vlasov condition is satisfied at infinity)

Multiple-valuedness and «quantum» indeterminacy

$$\mathrm{div}_v \vec{J}^2_{1,2} = \mathrm{div}_v \left[f^{1,2} \left\langle \dot{\vec{v}} \right\rangle_{1,2} \right] = \sum_{l=0}^{+\infty} \frac{(-1)^{l+1} (\hbar/2)^{2l}}{m^{2l+1} (2l+1)!} U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^{2l+1} f^{1,2}.$$

$$\hbar^0 : U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right) f^{1,2} = \frac{\partial U}{\partial x} \frac{\partial f^{1,2}}{\partial v_x} + \frac{\partial U}{\partial y} \frac{\partial f^{1,2}}{\partial v_y} + \frac{\partial U}{\partial z} \frac{\partial f^{1,2}}{\partial v_z},$$

$$\frac{\partial}{\partial v_\mu} \left[f^{1,2} \left\langle \dot{v}_\mu \right\rangle_{1,2} \right] \mapsto \frac{\partial U}{\partial x_\mu} \frac{\partial f^{1,2}}{\partial v_\mu} = \frac{\partial}{\partial v_\mu} \left(f^{1,2} \frac{\partial U}{\partial x_\mu} \right) \quad \Rightarrow \quad \left\langle \dot{v}_\mu \right\rangle_{1,2} \mapsto \frac{\partial U}{\partial x_\mu}.$$

Multiple-valuedness and «quantum» indeterminacy

$$\mathrm{div}_v \vec{J}^2_{1,2} = \mathrm{div}_v \left[f^{1,2} \left\langle \dot{\vec{v}} \right\rangle_{1,2} \right] = \sum_{l=0}^{+\infty} \frac{(-1)^{l+1} (\hbar/2)^{2l}}{m^{2l+1} (2l+1)!} U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^{2l+1} f^{1,2}.$$

$$\hbar^2 : U \left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v \right)^3 f^{1,2} = \frac{\partial^3 f^{1,2}}{\partial v_x^3} \frac{\partial^3 U}{\partial x^3} + \frac{\partial^3 f^{1,2}}{\partial v_y^3} \frac{\partial^3 U}{\partial y^3} + \frac{\partial^3 f^{1,2}}{\partial v_z^3} \frac{\partial^3 U}{\partial z^3} +$$
$$+ 3 \frac{\partial^3 f^{1,2}}{\partial v_x \partial v_y^2} \frac{\partial^3 U}{\partial x \partial y^2} + 3 \frac{\partial^3 f^{1,2}}{\partial v_x \partial v_z^2} \frac{\partial^3 U}{\partial x \partial z^2} + 2 \frac{\partial^3 f^{1,2}}{\partial v_x \partial v_y \partial v_z} \frac{\partial^3 U}{\partial x \partial y \partial z} +$$
$$+ 3 \frac{\partial^3 f^{1,2}}{\partial v_x^2 \partial v_y} \frac{\partial^3 U}{\partial x^2 \partial y} + 3 \frac{\partial^3 f^{1,2}}{\partial v_y \partial v_z^2} \frac{\partial^3 U}{\partial y \partial z^2} + 2 \frac{\partial^3 f^{1,2}}{\partial v_x \partial v_y \partial v_z} \frac{\partial^3 U}{\partial x \partial y \partial z} +$$
$$+ 3 \frac{\partial^3 f^{1,2}}{\partial v_x^2 \partial v_z} \frac{\partial^3 U}{\partial x^2 \partial z} + 3 \frac{\partial^3 f^{1,2}}{\partial v_y^2 \partial v_z} \frac{\partial^3 U}{\partial y^2 \partial z} + 2 \frac{\partial^3 f^{1,2}}{\partial v_x \partial v_y \partial v_z} \frac{\partial^3 U}{\partial x \partial y \partial z}.$$

$$f^{1,2} \left\langle \dot{v}_\mu \right\rangle_{1,2} \mapsto \frac{\partial^2 f^{1,2}}{\partial v_x^2} \frac{\partial^3 U}{\partial x^3} + \frac{\partial^2 f^{1,2}}{\partial v_y^2} \frac{\partial^3 U}{\partial y^3} + \frac{\partial^2 f^{1,2}}{\partial v_z^2} \frac{\partial^3 U}{\partial z^3},$$

$$\frac{\partial^3 f^{1,2}}{\partial v_\lambda^2 \partial v_\mu} ??, \quad \frac{\partial^3 f^{1,2}}{\partial v_\lambda \partial v_\mu^2} ??, \quad \frac{\partial^3 f^{1,2}}{\partial v_\mu \partial v_\lambda \partial v_\upsilon} ??$$

Multiple-valuedness and «quantum» indeterminacy

	The 1st variation	The 2nd variation	The 3rd variation
$U\left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v\right)^3 f^{1,2} \mapsto$	$\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x^2} & \frac{\partial^3 U}{\partial x^3} \\ \frac{\partial^2 f^{1,2}}{\partial v_y^2} & \frac{\partial^3 U}{\partial y^3} \end{pmatrix} + 3\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_y^2} & \frac{\partial^3 U}{\partial x \partial y^2} \\ \frac{\partial^2 f^{1,2}}{\partial v_x^2} & \frac{\partial^3 U}{\partial x^2 \partial y} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x^2} & \frac{\partial^3 U}{\partial x^3} \\ \frac{\partial^2 f^{1,2}}{\partial v_y^2} & \frac{\partial^3 U}{\partial y^3} \end{pmatrix} + 3\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x \partial v_y} & \frac{\partial^3 U}{\partial x^2 \partial y} \\ \frac{\partial^2 f^{1,2}}{\partial v_x \partial v_y} & \frac{\partial^3 U}{\partial x \partial y^2} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial^3 U}{\partial x^3} & \frac{\partial^2 f^{1,2}}{\partial v_x^2} \\ \frac{\partial^3 U}{\partial y^3} & \frac{\partial^2 f^{1,2}}{\partial v_y^2} \end{pmatrix} + \begin{pmatrix} \frac{\partial^3 U}{\partial x \partial y^2} & \frac{\partial^2 f^{1,2}}{\partial v_y^2} \\ \frac{\partial^3 U}{\partial x^2 \partial y} & \frac{\partial^2 f^{1,2}}{\partial v_x^2} \end{pmatrix} + 2\frac{\partial^2 f^{1,2}}{\partial v_x \partial v_y}\begin{pmatrix} \frac{\partial^3 U}{\partial x^2 \partial y} \\ \frac{\partial^3 U}{\partial x \partial y^2} \end{pmatrix}$

$$m\left\langle \dot{v}_\mu \right\rangle_{1,2} = F_\mu^{1,2} = -\frac{\partial U}{\partial x_\mu} + O_\mu\left(\hbar^2\right), \qquad O_\mu\left(\hbar^2\right) = \sum_{l=1}^{+\infty} \frac{\left(-1\right)^l}{4^l\left(2l+1\right)!}\frac{\partial^{2l+1}U}{\partial x_\nu \ldots}\kappa + \ldots.$$

$$\underbrace{\frac{\partial^{2l}W}{\partial p_\lambda \ldots}}_{2l} = \frac{\left(-1\right)^l}{\hbar^{2l}}\frac{1}{\left(2\pi\hbar\right)^3}\int_{(\infty)}\left\langle \ldots. \right\rangle e^{-i\frac{\vec{p}\cdot\vec{s}}{\hbar}}\underbrace{s_\lambda \ldots}_{2l} d^3 s, \qquad O_\mu\left(\hbar^2\right) = \sum_{l=1}^{+\infty}\frac{\left(-1\right)^l}{4^l\left(2l+1\right)!}\frac{\hbar^{2l}}{W}\frac{\partial^{2l}W}{\partial p_\lambda \ldots}\frac{\partial^{2l+1}U}{\partial x_\nu \ldots},$$

Multiple-valuedness and «quantum» indeterminacy

New approximation for the current density vector field

$$\vec{J}^{2}_{1,2} = f^{1,2}\left\langle \dot{v}_{\mu}\right\rangle_{1,2} \overset{\det}{=} \sum_{l=0}^{+\infty} \frac{(-1)^{l+1}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!}\frac{\partial U}{\partial x_{\mu}}\left(\overleftarrow{\nabla}_{r}\cdot\vec{\nabla}_{v}\right)^{2l} f^{1,2}, \quad \operatorname{curl}_{v}\vec{J}^{2}_{1,2} = \sum_{l=0}^{+\infty} \frac{(-1)^{l}(\hbar/2)^{2l}}{m^{2l+1}(2l+1)!} U\left(\overleftarrow{\nabla}_{r}\cdot\vec{\nabla}_{v}\right)^{2l}\left(\overleftarrow{\nabla}_{r}\times\vec{\nabla}_{v}\right) f^{1,2}.$$

Theorem 2. If series for $\vec{J}^{2}_{1,2}$ admits term-by-term integration and differentiation, then the substitution of new approximation for the current density vector field into the second Vlasov equation gives the Moyal equation for the Wigner function $f^{1,2}(\vec{r},\vec{v},t) = m^{3}W(\vec{r},\vec{p},t)$, wherein

$$\left\langle \dot{v}_{\mu}\right\rangle_{1} = \frac{1}{f^{1}}\int\limits_{(\infty)} f^{1,2}\left\langle \dot{v}_{\mu}\right\rangle_{1,2} d^{3}v = -\frac{1}{m}\frac{\partial U}{\partial x_{\mu}},$$

$$\left\langle \dot{v}_{\mu}\right\rangle_{1,2} = -\frac{1}{m}\frac{\partial U}{\partial x_{\mu}} + \frac{1}{f^{1,2}}\frac{\hbar^{2}}{4!m^{3}}\frac{\partial^{3}U}{\partial x_{\mu}\partial x_{\lambda}\partial x_{\nu}}\frac{\partial^{2} f^{1,2}}{\partial v_{\lambda}\partial v_{\nu}} - \ldots..$$

Numerical results

Model example

$$f^1(\vec{r},t)=\frac{1}{(2\pi)^{3/2}\,\sigma_r^3}\exp\left[-\frac{r^2\sin^2\theta+z^2}{2\sigma_r^2}\right],$$

$$\langle\vec{v}\rangle_1(\vec{r})=\frac{\hbar}{2m}\nabla_r\Phi=\frac{\hbar}{m}\nabla_r\varphi=\frac{\hbar}{m}\nabla_r\phi=\frac{\hbar}{m\rho}\vec{e}_\phi=\frac{\hbar}{mr\sin\theta}\vec{e}_\phi,$$

$$i\hbar\frac{\partial}{\partial t}\Psi=-\frac{\hbar^2}{2m}\Delta_r\Psi+U\Psi,\quad f^1=|\Psi|^2,\quad \arg\Psi=\varphi,$$

$$-\hbar\frac{\partial\varphi}{\partial t}=\frac{m}{2}\left|\langle\vec{v}\rangle_1\right|^2+\mathrm{V},\quad \mathrm{V}=U+\mathrm{Q},\quad \mathrm{Q}=-\frac{\hbar^2}{2m}\frac{\Delta_r|\Psi|}{|\Psi|}.$$

Theorem 3. Wave function

$$\Psi(\rho,\phi,z,t)=\frac{1}{(2\pi)^{3/4}\,\sigma_r^{3/2}}\exp\left(-\frac{\rho^2+z^2}{4\sigma_r^2}+i\phi-i\frac{E}{\hbar}t\right),$$

is the solution of the Schrödinger equation with the potential

$$U(\rho,\phi,z)=\frac{\hbar^2}{8m\sigma_r^4}\left(\rho^2+z^2-4\frac{\sigma_r^4}{\rho^2}\right),$$

in this case, the quantum potential has the form

$$\mathrm{Q}=\frac{\hbar^2}{4m\sigma_r^2}\left(3-\frac{\rho^2+z^2}{2\sigma_r^2}\right).$$

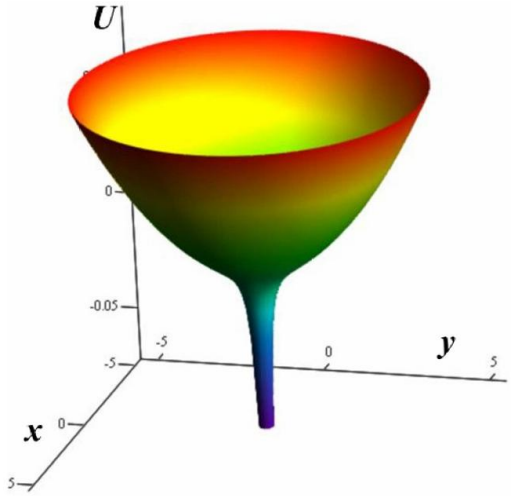


Figure 1. Graph of potential $U(\rho,\phi,0)$.

Numerical results

Theorem 4. The Wigner function W and its derivatives $\dfrac{\partial^2 W}{\partial p_\lambda \partial p_\mu}$ corresponding to the wave function Ψ can be represented in the following form :

$$G(\rho, z, \vec{p}) = \frac{\rho^2}{2\pi^4 \hbar^3 \sigma_r^2} e^{-\frac{\rho^2+z^2}{2\sigma_r^2} - \frac{2\sigma_r^2}{\hbar^2} p_z^2} \int_0^{2\pi} \frac{\Lambda\left(\rho, \bar{\rho}_s, \phi, \bar{\phi}_s, \vec{p}\right)}{\sqrt{1 + k_1^2 \sin^2 \bar{\phi}_s}} d\bar{\phi}_s \int_0^{+\infty} e^{-\rho^2 \frac{\bar{\rho}_s^2}{2\sigma_r^2}} \bar{\rho}_s d\bar{\rho}_s,$$

where $\Lambda\left(\rho, \bar{\rho}_s, \phi, \bar{\phi}_s, \vec{p}\right) = \xi\left(\rho, \bar{\rho}_s, \phi + \bar{\phi}_s\right)\left(1 + ik_1 \sin \bar{\phi}_s\right) e^{-i\left(p_\rho \cos \bar{\phi}_s + p_\phi \sin \bar{\phi}_s\right)\varsigma(\rho, \bar{\rho}_s)}$,

$$\xi\left(\rho, \bar{\rho}_s, \phi\right) = \begin{cases} 1, \ \textit{for} \ G = W, \\ -\zeta^2 \cos^2 \phi, \ \textit{for} \ G = \dfrac{\partial^2 W}{\partial p_x^2}, \\ -\zeta^2 \sin^2 \phi, \ \textit{for} \ G = \dfrac{\partial^2 W}{\partial p_y^2}, \\ -\dfrac{1}{2}\zeta^2 \sin 2\phi, \ \textit{for} \ G = \dfrac{\partial^2 W}{\partial p_x \partial p_y}, \end{cases}$$

$$\zeta\left(\rho, \bar{\rho}_s\right) = 2\frac{\rho \bar{\rho}_s}{\hbar},$$

$$k_1\left(\bar{\rho}_s\right) = \frac{2\bar{\rho}_s}{1 - \bar{\rho}_s^2}, \ k_2\left(\bar{\rho}_s\right) = \frac{2\bar{\rho}_s}{1 + \bar{\rho}_s^2}, \ k_1^2 = \frac{k_2^2}{1 - k_2^2}, \ k_2^2 = \frac{k_1^2}{1 + k_1^2}.$$

Numerical results

$$W(\vec{r},p_z)=\frac{2\rho^2}{\pi^4\hbar^3\sigma_r^2}e^{-\frac{r^2}{2\sigma_r^2}-\frac{2\sigma_r^2}{\hbar^2}p_z^2}\int_0^{+\infty}K(ik_1)e^{-\frac{\rho^2\bar{\rho}_s^2}{2\sigma_r^2}}\bar{\rho}_s d\bar{\rho}_s,\quad where\ \ K(k)=\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

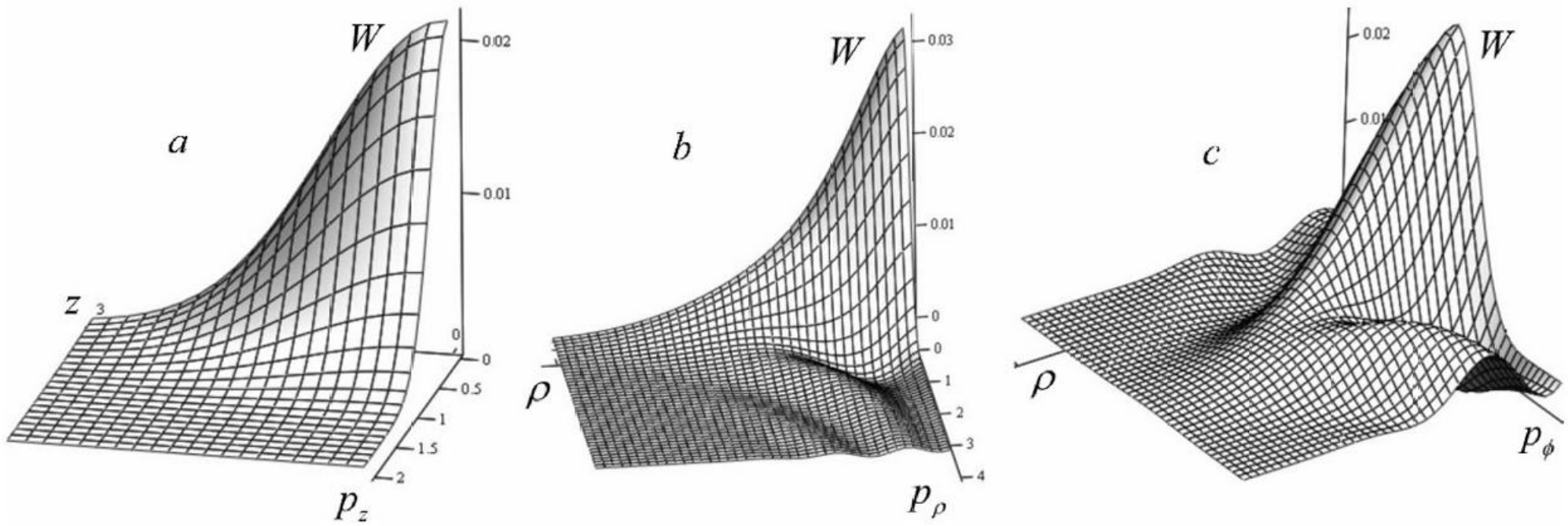


Figure 2. Phase-plane portraits of Wigner function $W(\vec{r},\vec{p})$.

Numerical results

Theorem 5: Let the quantum system be described by the Wigner function, then the equation of motion is valid, in which the pressure tensor has the form:

$$P_{\mu\lambda} = \frac{1}{m^2}\int\limits_{(\infty)} W\left(p_\mu - \left\langle p_\mu \right\rangle_1\right)\left(p_\lambda - \left\langle p_\lambda \right\rangle_1\right)d^3p = f^1\left(\vec{r}\right)\frac{\hbar^2}{4m^2\sigma_r^2}\delta_{\mu\lambda},$$

$$\left\langle \dot{v}_\mu \right\rangle_1 = \frac{1}{f^1}\int\limits_{(\infty)} W\left\langle \dot{v}_\mu \right\rangle_{1,2} d^3p = -\frac{1}{m}\frac{\partial U}{\partial x_\mu}, \quad \frac{d}{dt}\left\langle \vec{v} \right\rangle_1 = -\frac{\hbar^2}{m^2\rho^3}\vec{e}_\rho,$$

in this case, the quantum pressure force is related to the quantum potential as follows:

$$-\frac{\partial \mathrm{Q}}{\partial x_\mu} = \frac{m}{f^1}\frac{\partial P_{\mu\lambda}}{\partial x_\lambda} = \frac{\hbar^2 x_\mu}{4m\sigma_r^4}.$$

Numerical results

$$\nabla_r U = \frac{\hbar^2}{4m\sigma_r^4}\left[\rho\left(1+4\frac{\sigma_r^4}{\rho^4}\right)\vec{e}_\rho + z\vec{e}_z\right]. \qquad \frac{\partial^3 U}{\partial x_\nu \ldots} \sim \frac{\hbar^2}{m}\frac{1}{\rho^5}, \qquad \frac{\partial^3 U}{\partial x^3} = \frac{12\hbar^2}{m\rho^5}\cos 2\phi\cos\phi, \qquad \frac{\partial^3 U}{\partial y^3} = -\frac{12\hbar^2}{m\rho^5}\sin\phi\cos 2\phi,$$

$$\frac{\partial^3 U}{\partial x\partial y^2} = -\frac{4\hbar^2}{m\rho^5}\cos\phi\left(3\cos 2\phi - 2\right), \qquad \frac{\partial^3 U}{\partial y\partial x^2} = \frac{4\hbar^2}{m\rho^5}\sin\phi\left(3\cos 2\phi + 2\right),$$

$$\frac{\partial^3 U}{\partial x\partial y\partial z} = \frac{\partial^3 U}{\partial x\partial z^2} = \frac{\partial^3 U}{\partial y\partial z^2} = \frac{\partial^3 U}{\partial z^3} = 0.$$

	The 1st variation	The 2nd variation	The 3rd variation
$U\left(\overleftarrow{\nabla}_r \cdot \vec{\nabla}_v\right)^3 f^{1,2} \mapsto$	$\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x^2}\frac{\partial^3 U}{\partial x^3} \\ \frac{\partial^2 f^{1,2}}{\partial v_y^2}\frac{\partial^3 U}{\partial y^3} \end{pmatrix} + 3\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_y^2}\frac{\partial^3 U}{\partial x\partial y^2} \\ \frac{\partial^2 f^{1,2}}{\partial v_x^2}\frac{\partial^3 U}{\partial x^2\partial y} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x^2}\frac{\partial^3 U}{\partial x^3} \\ \frac{\partial^2 f^{1,2}}{\partial v_y^2}\frac{\partial^3 U}{\partial y^3} \end{pmatrix} + 3\begin{pmatrix} \frac{\partial^2 f^{1,2}}{\partial v_x\partial v_y}\frac{\partial^3 U}{\partial x^2\partial y} \\ \frac{\partial^2 f^{1,2}}{\partial v_x\partial v_y}\frac{\partial^3 U}{\partial x\partial y^2} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial^3 U}{\partial x^3}\frac{\partial^2 f^{1,2}}{\partial v_x^2} \\ \frac{\partial^3 U}{\partial y^3}\frac{\partial^2 f^{1,2}}{\partial v_y^2} \end{pmatrix} + \begin{pmatrix} \frac{\partial^3 U}{\partial x\partial y^2}\frac{\partial^2 f^{1,2}}{\partial v_y^2} \\ \frac{\partial^3 U}{\partial x^2\partial y}\frac{\partial^2 f^{1,2}}{\partial v_x^2} \end{pmatrix} + 2\frac{\partial^2 f^{1,2}}{\partial v_x\partial v_y}\begin{pmatrix} \frac{\partial^3 U}{\partial x^2\partial y} \\ \frac{\partial^3 U}{\partial x\partial y^2} \end{pmatrix}$

Numerical results

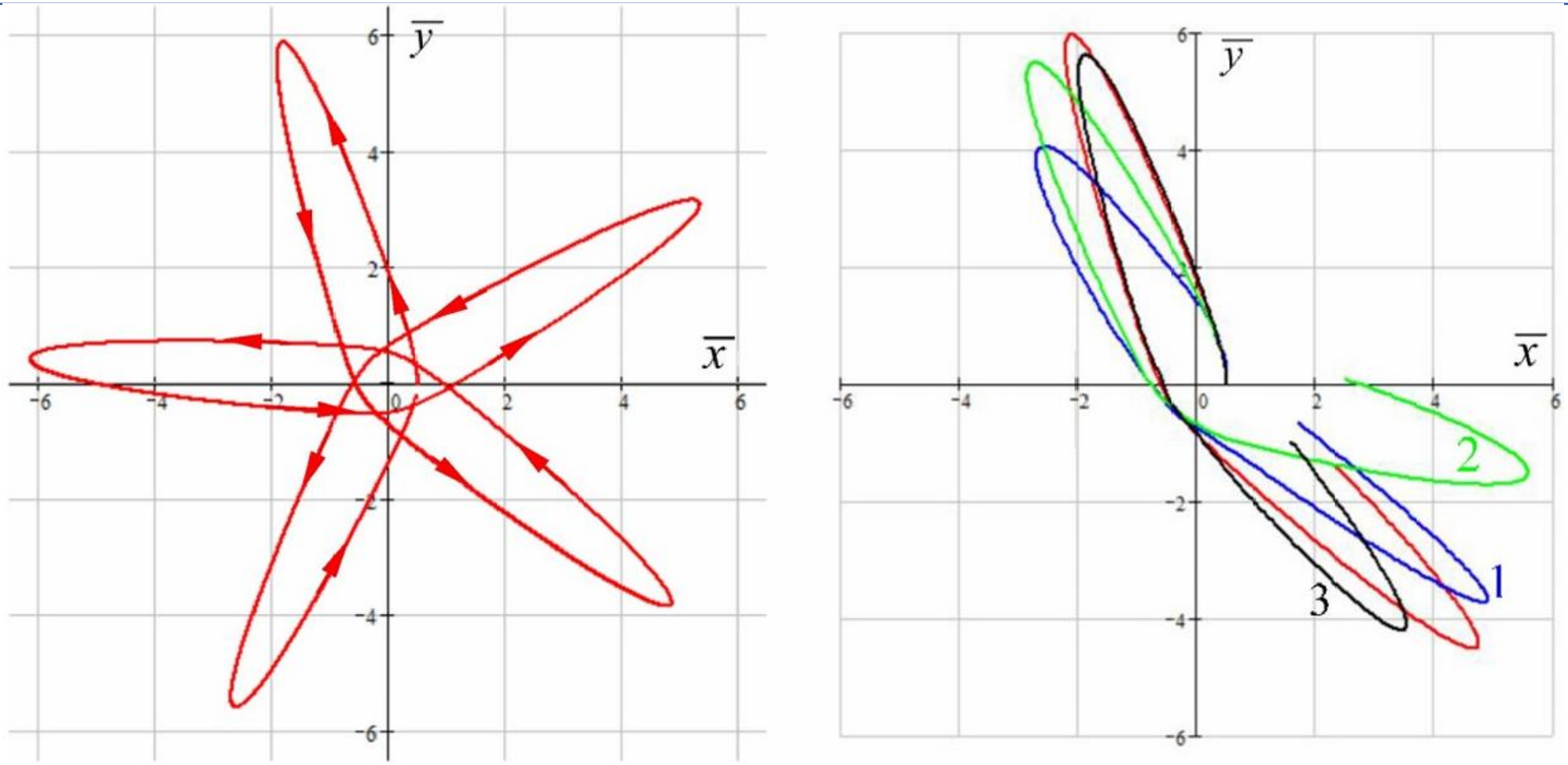


Figure 3. Particle trajectories on the microscale.

Conclusion

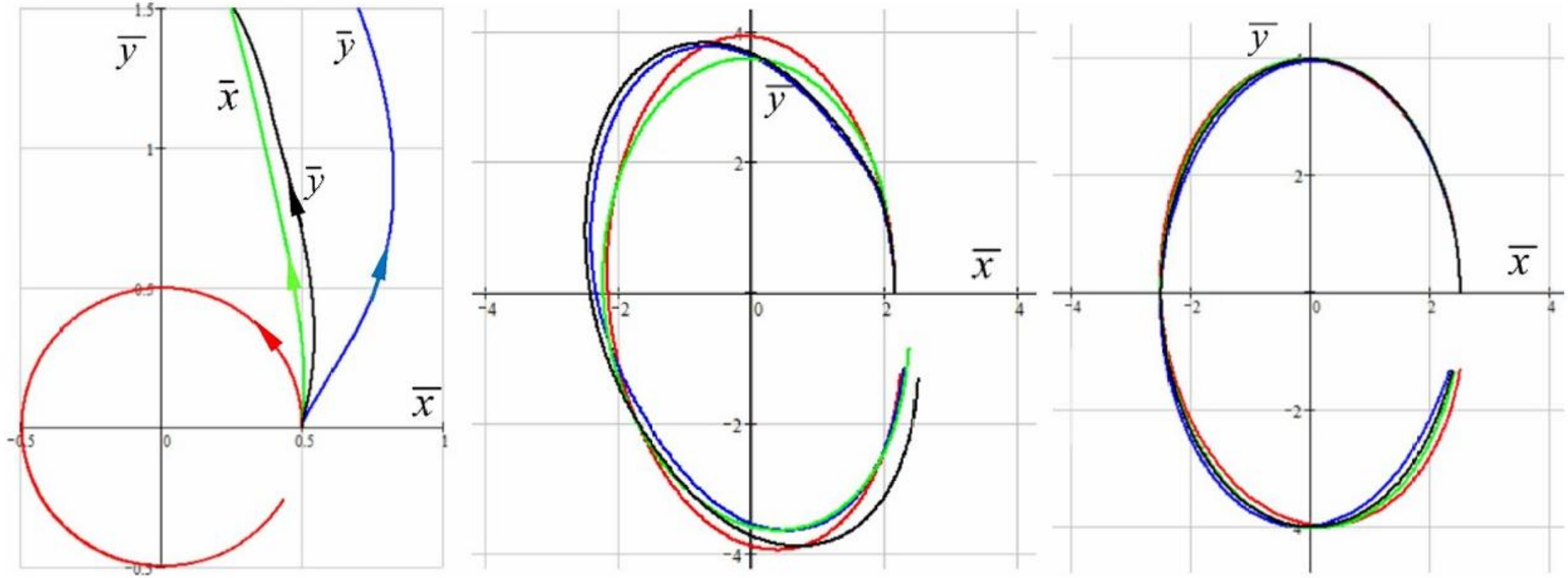


Figure 4. Transition from the microscale to macroscale.