A NUMERICAL STUDY OF THE FREE-FALL THREE-BODY PROBLEM WITH PYTHAGOREAN MASSES

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What is the three-body problem?

Three bodies move in space under their mutual Newtonian gravitational attraction. Given their initial conditions (initial positions and velocities), determine their subsequent motion.

What are the free-fall initial conditions?

Free-fall initial conditions mean that the bodies are initially at rest (i.e. have zero velocities at $\boldsymbol{t = 0}$). The bodies remain in the plane determined by the initial positions - the motion is planar.

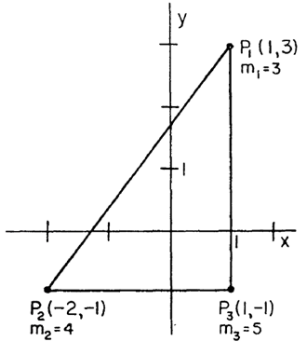
How do periodic free-fall orbits look like?

The time-reversal symmetry of Newton's equations implies that if a free-fall solution is periodic with period $\boldsymbol{T}$, then at $\boldsymbol{t = T/2}$ the bodies are again at rest. The bodies move back and forth between two different "stop" triangles (see the animation later).

Pythagorean three-body problem

This work is motivated by the famous Pythagorean three-body problem (also known as Burrau's problem of three bodies) which became very popular after the works of Victor Szebehely and Frederick Peters in the late 1960s. In fact, the problem became a benchmark problem for numerical computations in celestial mechanics.

In the Burrau's problem the bodies are initially located at the vertices of a right triangle with sides 3, 4, 5, so that the corresponding masses and sides are opposite one another. In 1893 a person named Meissel suggested, that with these initial conditions the solution is periodic, but Szebehely and Peters showed that this is not the case.



The figure from the original paper of Szebehely and Peters

Goals

We consider 3:4:5 (Pythagorean) masses, but arbitrary free-fall initial conditions. The main goals of this work are:

1) To investigate the decay times in the Agekyan-Anosova initial conditions' domain.

2) To conduct an intensive numerical search for periodic orbits in the Agekyan-Anosova domain and to expand the database of known periodic orbits. Only about 100 such orbits are currently known.

3) To study the found periodic orbits for stability.

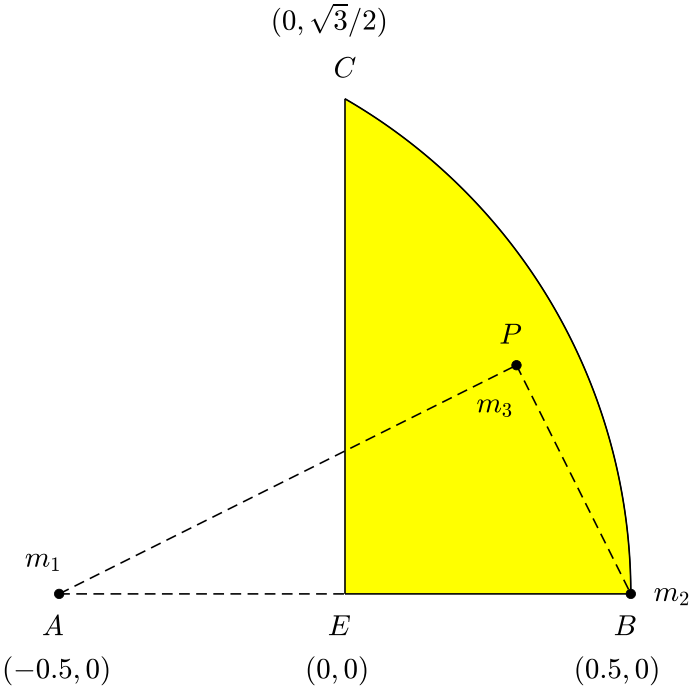
Importance of periodic orbits

Henri Poincare emphasized the importance of periodic orbits with his famous dictum:

" They are the only breach in the fortress that is the three-body problem"



Free-fall initial conditions: Agekyan-Anosova domain



To generate all possible free-fall initial conditions (all possible shapes of the initial triangle), we consider the Agekyan-Anosova domain (in yellow). We have two parameters $\boldsymbol{p_x}, \boldsymbol{p_y}$ - the coordinates of the point $\boldsymbol{P(p_x, p_y)}$. For each point $\boldsymbol{P(p_x, p_y)}$ we need to consider all 6 permutations of the 3, 4, 5 masses in the initial triangle.

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Numerical methods

The free-fall three-body problem is chaotic. Chaos is characterized by a sensitive dependence on initial conditions, expressed in an exponential divergence of neighboring trajectories.

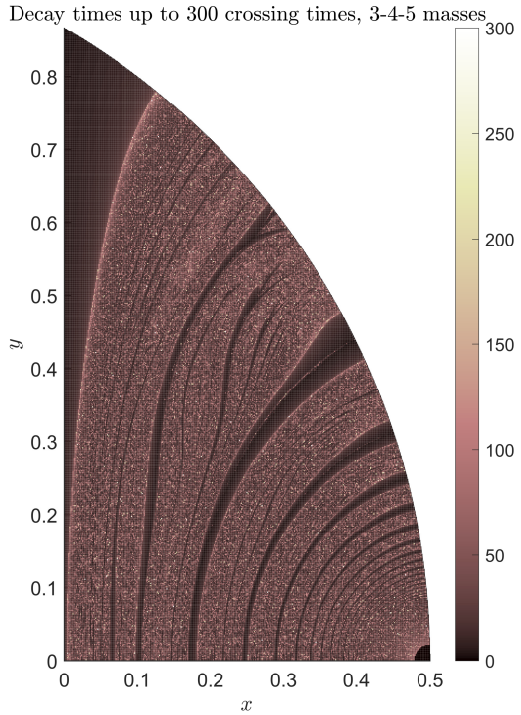
Because of the inevitable rounding and truncation errors, we have an exponential error growth. To overcome this problem, we use as an ODE-solver the high order Taylor series method with arbitrary precision floating-point arithmetic. The search for periodic orbits is based on Newton's method.

Floating-point operations become expensive and also a very large number of initial value problems (Cauchy problems) have to be solved. So, we need a serious computational resource.

How to perform the computations in parallel?

The initial value problems are independent and can be solved in parallel. A static (equal portions) scheduling of the initial conditions can be made by solving independent tasks without parallel programming. A more flexible dynamic scheduling of the initial conditions can also be made with MPI and the Master-Worker pattern.

Some results for the decay times



In dark brown are the regions with very short decay times. The chaotic regions are in inhomogeneous light brown. Many (infinitely many) periodic orbits are expected to exist in these regions. An extensive search for new periodic orbits is currently underway.

ACKNOWLEDGMENTS

We are grateful for the opportunity to use the computational resources of JINR, Dubna, Russia. We thank the people who maintain these resources!

The intensive high-precision computations are performed on the GOVORUN supercomputer and using the resources of the JINR Member States distributed information and computing environment (JINR DICE).

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[3] I. Hristov, R. Hristova, T. Puzynina, Z. Sharipov, Z. Tukhliev, "Numerical search for three-body periodic free-fall orbits with central symmetry." Communications in Nonlinear Science and Numerical Simulation 151, 109066 (2025).

THANK YOU FOR YOUR ATTENTION!

Differential equations

The differential equations for the three-body problem are derived from Newton's second law and Newton's law of gravity:

$$\boldsymbol{m_i \ddot{r}_i = \sum_{j=1, j\neq i}^{3} G m_i m_j \frac{(r_j - r_i)}{\|r_j - r_i\|^3}, i = 1, 2, 3,}$$

where $\boldsymbol{r_i(t)}$ are the vectors of positions in space and $\boldsymbol{G}$ is the gravitational constant. We consider $\boldsymbol{G = 1}$.

The free-fall problem is planar, so $\boldsymbol{r_i = (x_i, y_i)}$.

We solve the system numerically in the following first order form:

$$\boldsymbol{\dot{x}_i = vx_i,\ \dot{y}_i = vy_i}$$

$$\boldsymbol{\dot{vx}_i = \sum_{j=1, j\neq i}^{3} m_j \frac{(x_j - x_i)}{\|r_j - r_i\|^3},\ \dot{vy}_i = \sum_{j=1, j\neq i}^{3} m_j \frac{(y_j - y_i)}{\|r_j - r_i\|^3}}$$

We have a vector of 12 unknown functions:

$$\boldsymbol{X(t) = (x_1, y_1, x_2, y_2, x_3, y_3, vx_1, vy_1, vx_2, vy_2, vx_3, vy_3)^\top}$$

The model treats the bodies as point masses.

Periodicity equation. Return proximity function

To search for periodic orbits we use Newton's method with initial approximations obtained by the grid-search method.

Let $\boldsymbol{X(p_x, p_y, t)}$ be the solution to the initial problem corresponding to a point $\boldsymbol{P(p_x, p_y)}$ in the Agekyan-Anosova domain.

When we search for periodic orbits with periods $\boldsymbol{T > 0}$, we actually search for triplets $\boldsymbol{(p_x, p_y, T)}$ for which the periodicity equation:

$$\boldsymbol{X(p_x, p_y, T) = X(p_x, p_y, 0)}$$

is satisfied. The function

$$\boldsymbol{R(t) = \|X(p_x, p_y, t) - X(p_x, p_y, 0)\|_2}$$

is called the return proximity function. The triplet $\boldsymbol{(p_x, p_y, T)}$ corresponds to a periodic solution only if $\boldsymbol{R(T) = 0.}$

To avoid misunderstandings, let us note that we will further use the same notations for $\boldsymbol{p_x, p_y, T}$ and for their approximations.

Stages of the numerical procedure for searching for periodic orbits

The search for periodic orbits generally consists of three main stages:

(I) Scanning the initial conditions' grid for possible periodic orbit candidates - initial conditions corresponding to some point $\boldsymbol{P(p_x, p_y)}$ in a grid introduced in the Agekyan-Anosova domain for which at some time $\boldsymbol{T > 0}$ (an approximation of the period) the return proximity

$$\boldsymbol{R(T) = \|X(p_x, p_y, T) - X(p_x, p_y, 0)\|_2}$$

is small. These candidates are the initial approximations for Newton's method at stage II.

(II) Capturing the periodic orbits using Newton's method. A convergence of Newton's method means a captured periodic orbit.

(III) Computing the orbits with many correct digits using Newton's method with a higher precision.

Newton's method for the periodicity equation

The approximations $(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{T})$ are improved with corrections $\boldsymbol{\Delta p_x}, \boldsymbol{\Delta p_y}, \boldsymbol{\Delta T}$ by expanding the periodicity equation:

$$\boldsymbol{X}(p_x+\Delta p_x, p_y+\Delta p_y, T+\Delta T) = \boldsymbol{X}(p_x+\Delta p_x, p_y+\Delta p_y, 0)$$

in a multivariable linear approximation:

$$\begin{pmatrix} x_1(T) \\ y_1(T) \\ x_2(T) \\ y_2(T) \\ x_3(T) \\ y_3(T) \\ vx_1(T) \\ vy_1(T) \\ vx_2(T) \\ vy_2(T) \\ vx_3(T) \\ vy_3(T) \end{pmatrix} + \begin{pmatrix} \frac{\partial x_1}{\partial p_x}(T) & \frac{\partial x_1}{\partial p_y}(T) & \dot{x}_1(T) \\ \frac{\partial y_1}{\partial p_x}(T) & \frac{\partial y_1}{\partial p_y}(T) & \dot{y}_1(T) \\ \frac{\partial x_2}{\partial p_x}(T) & \frac{\partial x_2}{\partial p_y}(T) & \dot{x}_2(T) \\ \frac{\partial y_2}{\partial p_x}(T) & \frac{\partial y_2}{\partial p_y}(T) & \dot{y}_2(T) \\ \frac{\partial x_3}{\partial p_x}(T) & \frac{\partial x_3}{\partial p_y}(T) & \dot{x}_3(T) \\ \frac{\partial y_3}{\partial p_x}(T) & \frac{\partial y_3}{\partial p_y}(T) & \dot{y}_3(T) \\ \frac{\partial vx_1}{\partial p_x}(T) & \frac{\partial vx_1}{\partial p_y}(T) & \dot{vx}_1(T) \\ \frac{\partial vy_1}{\partial p_x}(T) & \frac{\partial vy_1}{\partial p_y}(T) & \dot{vy}_1(T) \\ \frac{\partial vx_2}{\partial p_x}(T) & \frac{\partial vx_2}{\partial p_y}(T) & \dot{vx}_2(T) \\ \frac{\partial vy_2}{\partial p_x}(T) & \frac{\partial vy_2}{\partial p_y}(T) & \dot{vy}_2(T) \\ \frac{\partial vx_3}{\partial p_x}(T) & \frac{\partial vx_3}{\partial p_y}(T) & \dot{vx}_3(T) \\ \frac{\partial vy_3}{\partial p_x}(T) & \frac{\partial vy_3}{\partial p_y}(T) & \dot{vy}_3(T) \end{pmatrix} \begin{pmatrix} \Delta p_x \\ \Delta p_y \\ \Delta T \end{pmatrix} = \begin{pmatrix} x_1(0) \\ y_1(0) \\ x_2(0) \\ y_2(0) \\ x_3(0)+\Delta p_x \\ y_3(0)+\Delta p_y \\ vx_1(0) \\ vy_1(0) \\ vx_2(0) \\ vy_2(0) \\ vx_3(0) \\ vy_3(0) \end{pmatrix}$$

Computing and solving the linear systems in Newton's method

To compute the elements of the $\mathbf{12 \times 3}$ matrices of the linear systems, we have to add to the 12 equations in the original system, the 24 differential equations for the partial derivatives with respect to the parameters $\boldsymbol{p_x, p_y}$:

$$\frac{\partial x_i}{\partial p_x}(t), \frac{\partial y_i}{\partial p_x}(t), \frac{\partial x_i}{\partial p_x}(t), \frac{\partial vy_i}{\partial p_x}(t), i = 1, 2, 3.$$

$$\frac{\partial x_i}{\partial p_y}(t), \frac{\partial y_i}{\partial p_y}(t), \frac{\partial vx_i}{\partial p_y}(t), \frac{\partial vy_i}{\partial p_y}(t), i = 1, 2, 3.$$

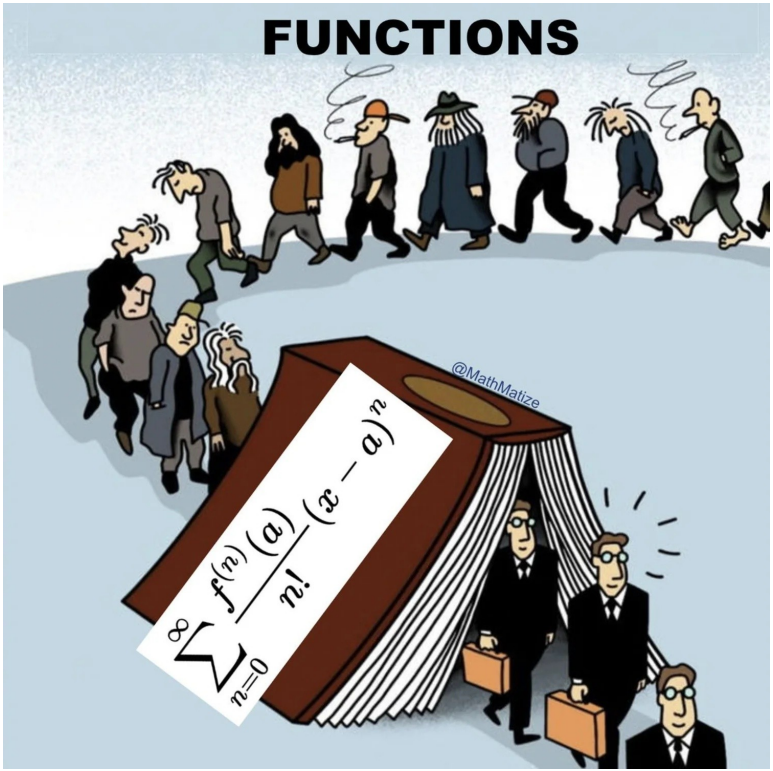
These equations can be obtained by differentiation of the original system with respect to the parameters $\boldsymbol{p_x, p_y}$, but we do not need them in explicit form for the ODE solver we will used.

A crucial decision for the success of finding periodic orbits is the choice of the numerical algorithm for solving this system of 36 ODEs. We use high order high precision Taylor Series Method (TSM). The linear systems are solved in least square sense by QR-decomposition!

Taylor Series Method (TSM)

Joseph-Louis Lagrange termed Taylor's theorem

"The main foundation of differential calculus."



Taylor Series Method (TSM)

For the initial value problem $\dot{\boldsymbol{u}}(t) = \boldsymbol{f}(\boldsymbol{u}, t),\ \boldsymbol{u_0} = \boldsymbol{u}(0)$, the N-th order Taylor series method for finding an approximate solution $\boldsymbol{U}(t) \approx \boldsymbol{u}(t)$ is given by:

$$\boldsymbol{U}(t+\tau) = \sum_{i=0}^{N} \boldsymbol{U}^{[i]} \tau^i,\ \ \boldsymbol{U}^{[i]} = \frac{1}{i!} \frac{d\boldsymbol{U}^{(i)}(t)}{dt^i},$$

where the coefficients $\boldsymbol{U}^{[i]}$ are called normalized derivatives.

The use of an adaptive step-size strategy is crucial for the three-body problem. The time stepsize τ is determined this way:

$$\tau = \frac{e^{-0.7/(N-1)}}{e^2} \min \left\{ \left(\frac{1}{\|\mathbf{U}^{[\mathbf{N-1}]}\|_\infty} \right)^{\frac{1}{N-1}}, \left(\frac{1}{\|\mathbf{U}^{[\mathbf{N}]}\|_\infty} \right)^{\frac{1}{N}} \right\}$$

Jorba, Angel, and Maorong Zou *Experimental Mathematics 14.1 (2005): 99-117*

Normalized derivatives are computed with the rules of automatic differentiation!

The choice of Taylor's order and the precision

The order N of the method controls the local error and is chosen according to the absolute local error tolerance ε by the formula

$$N = \lceil -0.5 \ln(\varepsilon) + 1 \rceil,$$

where $\lceil \cdot \rceil$ is the ceiling function. If we take ε to be the precision corresponding to $64k$ bits of mantissa, we obtain:

$$M = \lceil -0.5 \ln(2^{-64k}) + 1 \rceil = \lceil \approx 22.1807k + 1 \rceil$$

In practice we use 64 bit of precision for every 22 Taylor's order.

THE GMP LIBRARY

The GNU Multiple Precision Arithmetic Library (GMP Library) is used for the arbitrary high precision floating-point arithmetic.

