APPLICATION OF THE CABARET SCHEME FOR MODELING TWOPHASE FLOWS IN PIPELINES

Mathematical Modeling and Computational Physics” (MMCP’2026)

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Introduction

Mathematical modeling of multiphase flows is a crucial task in computational fluid dynamics (CFD), with a wide range of applications: from pipeline transport to heat transfer and chemical engineering processes.

A fundamental problem is the presence of non-conservative terms in multi-velocity models ($\Phi \frac{\partial a}{\partial x}$)

The CABARET Scheme

Advantages of the algorithm:

- High computational efficiency;
- Non-dissipative property;
- Compact spatial stencil;
- Second-order accuracy in space and time;
- Stable for Courant numbers $0 < CFL < 1$;

Two-phases model

$$\frac{\partial a_g}{\partial t} + V\frac{\partial a_g}{\partial x} = f_{ag}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_g \rho_g}{\partial t} + \frac{\partial a_g \rho_g u_g}{\partial x} = f_{\rho g}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_l \rho_l}{\partial t} + \frac{\partial a_l \rho_l u_l}{\partial x} = f_{\rho l}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_g \rho_g u_g}{\partial t} + \frac{\partial \left(a_g \rho_g u_g^2 + a_g P_g\right)}{\partial x} - \bar{P}\frac{\partial a_g}{\partial x} = f_{\rho u g}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_l \rho_l u_l}{\partial t} + \frac{\partial \left(a_l \rho_l u_l^2 + a_g l P_l\right)}{\partial x} - \bar{P}\frac{\partial a_l}{\partial x} = f_{\rho u l}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_g \rho E_g}{\partial t} + \frac{\partial \left(a_g \rho_g E_g u_g + a_g P_g u_g\right)}{\partial x} - \bar{P}V\frac{\partial a_g}{\partial x} = f_{Eg}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a_l \rho_l E_l}{\partial t} + \frac{\partial (a_l \rho_l E_l u_l + a_l P_l u_l)}{\partial x} - \bar{P}V\frac{\partial a_l}{\partial x} = f_{El}(a, \rho, \varepsilon, u)$$

The interface is described by

$V = \frac{a_g \rho_g u_g + a_l \rho_l u_l}{a_g \rho_g + a_l \rho_l}$ $\qquad$ $\bar{P} = a_g P_g + a_l P_l$

Closure relations

$a_l + a_g = 1$

Stiffened gas equation of state (SG EOS)

$P(\rho, \varepsilon) = \rho(\varepsilon(\gamma - 1) + C_o^2) - \gamma P_o$

Two-phases model

The transport of each phase is considered independently

$$\frac{\partial a}{\partial t} + V\frac{\partial a}{\partial x} = f_a(a, \rho, \varepsilon, u)$$

$$\frac{\partial a\rho}{\partial t} + \frac{\partial a\rho u}{\partial x} = f_\rho(a, \rho, \varepsilon, u)$$

$$\frac{\partial a\rho u}{\partial t} + \frac{\partial (a\rho u^2 + aP)}{\partial x} - \bar{P}\frac{\partial a}{\partial x} = f_{\rho ug}(a, \rho, \varepsilon, u)$$

$$\frac{\partial a\rho E}{\partial t} + \frac{\partial (a\rho E u + aPu)}{\partial x} - \bar{P}V\frac{\partial a}{\partial x} = f_E(a, \rho, \varepsilon, u)$$

a – volume fraction
ρ – density
u – velocity
P – pressure
C – speed of sound
$E = \varepsilon + u^2/2$ – specific total energy
ε – specific internal energy
$\bar{P}$ – interfacial pressure
V – interfacial velocity

Simple form

$$\frac{\partial a}{\partial t} + V\frac{\partial a}{\partial x} = f_a$$

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \frac{\rho(u-V)}{a}\frac{\partial a}{\partial x} = \frac{f_\rho - \rho f_a}{a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{(P-\bar{P})}{a\rho}\frac{\partial a}{\partial x} = \frac{f_{\rho u} - u f_\rho}{a\rho}$$

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + \rho c^2\frac{\partial u}{\partial x} + \frac{\rho c^2(u-V)}{a}\frac{\partial a}{\partial x} + \frac{\partial P}{\partial \varepsilon}\frac{(u-V)(\bar{P}-P)}{a\rho}\frac{\partial a}{\partial x} =$$

$$= \frac{\partial P}{\partial \rho}\frac{f_\rho - \rho f_a}{a} + \frac{1}{a\rho}\frac{\partial P}{\partial \varepsilon}\left(f_E - f_\rho\left(\varepsilon + \frac{u^2}{2}\right) - u\left(f_{\rho u} - u f_\rho\right)\right)$$

Eigenpairs for the system of equations

To transition to the characteristic form, it is necessary to find the eigenvalues λ and the left eigenvectors $\overrightarrow{l_1}$ of the matrix form.

$$\frac{\partial\vec{\varphi}}{\partial t} + \boldsymbol{A}\frac{\partial\vec{\varphi}}{\partial x} = \vec{f} \qquad \vec{\varphi}^T = (a, \rho, u, P) \quad \vec{f}^T = \left(f_a, f_\rho, f_{\rho u}, f_E\right)$$

The system of equations is hyperbolic

$$\mathbf{A} = \begin{pmatrix} V & 0 & 0 & 0 \\ \frac{\rho(u-V)}{a} & u & \rho & 0 \\ \frac{(P-\bar{P})}{a\rho} & 0 & u & \frac{1}{\rho} \\ \frac{\rho^2 c(u-V)}{a} - \frac{(P-\bar{P})(u-V)}{a\rho}\frac{\partial P}{\partial \varepsilon} & 0 & \rho c^2 & u \end{pmatrix}$$

$$\lambda_1 = V \qquad \overrightarrow{l_1} = (1,0,0,0)$$

$$\lambda_2 = u + c \qquad \overrightarrow{l_2} = \left(\frac{(P-\bar{P})(u-V)}{a\rho^2 c(V-u-c)}\frac{\partial P}{\partial \varepsilon} - \frac{(P-\bar{P})}{a\rho(V-u-c)} - \frac{c(u-V)}{a(V-u-c)}, 0, 1, \frac{1}{\rho c}\right)$$

$$\lambda_3 = u - c \qquad \overrightarrow{l_3} = \left(\frac{c(u-V)}{a(V-u+c)} - \frac{(P-\bar{P})}{a\rho(V-u+c)} - \frac{(P-\bar{P})(u-V)}{a\rho^2 c(V-u+c)}\frac{\partial P}{\partial \varepsilon}, 0, 1, -\frac{1}{\rho c}\right)$$

$$\lambda_4 = u \qquad \overrightarrow{l_4} = \left(-\frac{(P-\bar{P})}{a\rho}\frac{\partial P}{\partial \varepsilon}, -c^2, 0, 1\right)$$

Characteristic form

The characteristic form can be written as:

$$\left\{\boldsymbol{\Omega}\frac{\partial\vec{\varphi}}{\partial t}\right\}+\boldsymbol{\Lambda}\left\{\boldsymbol{\Omega}\frac{\partial\vec{\varphi}}{\partial x}\right\}=\boldsymbol{\Omega}\vec{f} \qquad \vec{\varphi}^T=(a,\rho,u,P) \qquad \vec{f}^T=\left(f_a,f_\rho,f_u,f_P\right)$$

where $\vec{f}$ is the right-hand side vector from the primitive form

$$\boldsymbol{\Omega}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{(P-\bar{P})(u-V)}{a\rho^2c(V-u-c)}\frac{\partial P}{\partial\varepsilon}-\frac{(P-\bar{P})}{a\rho(V-u-c)}-\frac{c(u-V)}{a(V-u-c)} & 0 & 1 & \frac{1}{\rho c} \\ \frac{c(u-V)}{a(V-u+c)}-\frac{(P-\bar{P})}{a\rho(V-u+c)}-\frac{(P-\bar{P})(u-V)}{a\rho^2c(V-u+c)}\frac{\partial P}{\partial\varepsilon} & 0 & 1 & -\frac{1}{\rho c} \\ -\frac{(P-\bar{P})}{a\rho}\frac{\partial P}{\partial\varepsilon} & -c^2 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{pmatrix}=\begin{pmatrix}V&0&0&0\\0&u+c&0&0\\0&0&u-c&0\\0&0&0&u\end{pmatrix}$$

Stage 1 (Predictor)

Conservative variables a, ρ, u, E of the gas phase at the intermediate time level $n + 1/2$

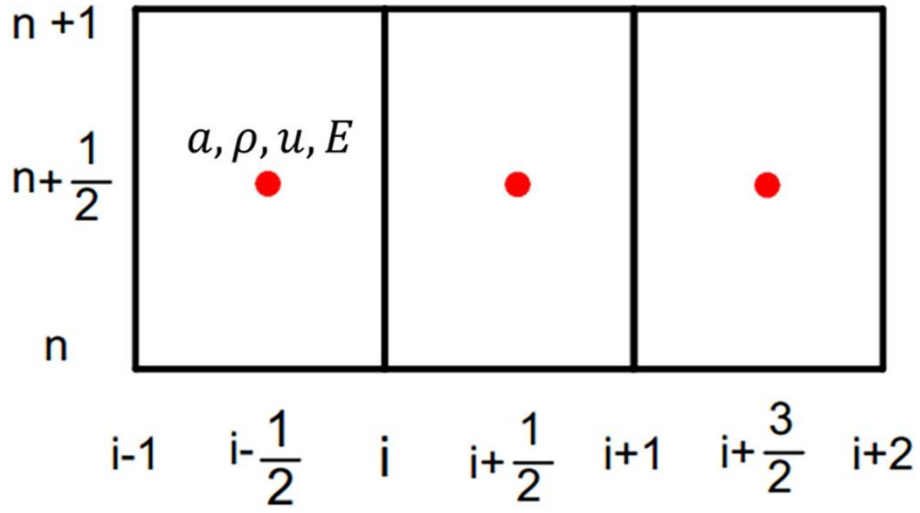
$$a_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} = a_{g,i+\frac{1}{2}}^{n} + \frac{\tau}{2}\left(f_a - V\left(\frac{a_{g,i+1}^{n} - a_{g,i}^{n}}{\Delta x}\right)\right)$$

$$\rho_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{a_{g,i+\frac{1}{2}}^{n+\frac{1}{2}}}\left((a\rho)_{g,i+\frac{1}{2}}^{n} + \frac{\tau}{2}\left(f_\rho - \frac{(a\rho u)_{g,i+1}^{n} - (a\rho u)_{g,i}^{n}}{\Delta x}\right)\right)$$

$$u_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{(a\rho)_{g,i+\frac{1}{2}}^{n+\frac{1}{2}}}\left((a\rho u)_{g,i+\frac{1}{2}}^{n} + \frac{\tau}{2}\left(f_{\rho u} + \bar{P}\left(\frac{a_{g,i+1}^{n} - a_{g,i}^{n}}{\Delta x}\right) - \frac{(a\rho u^2)_{g,i+1}^{n} - (a\rho u^2)_{g,i}^{n}}{\Delta x} - \frac{(aP)_{g,i+1}^{n} - (aP)_{g,i}^{n}}{\Delta x}\right)\right)$$

$$E_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{(a\rho)_{g,i+\frac{1}{2}}^{n+\frac{1}{2}}}\left((a\rho E)_{g,i+\frac{1}{2}}^{n} + \frac{\tau}{2}\left(f_E + \bar{P}V\left(\frac{a_{g,i+1}^{n} - a_{g,i}^{n}}{\Delta x}\right) - \frac{(a\rho u E)_{g,i+1}^{n} - (a\rho u E)_{g,i}^{n}}{\Delta x} - \frac{(aPu)_{g,i+1}^{n} - (aPu)_{g,i}^{n}}{\Delta x}\right)\right)$$

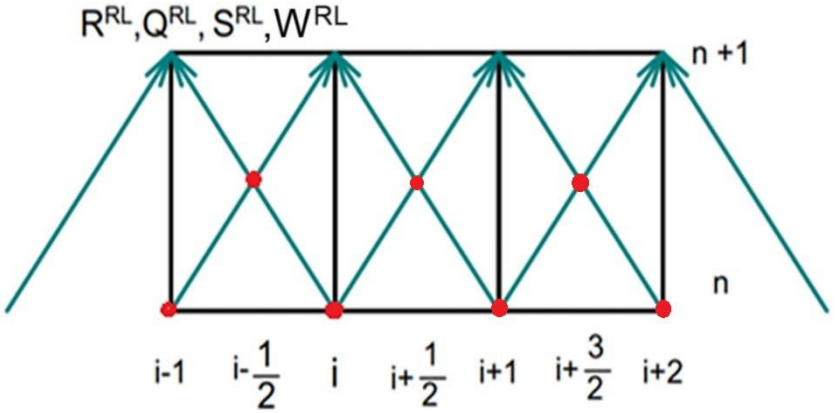
$$\tau_g = \frac{CFL_g \cdot \Delta x}{|u_{g,i}| + c_{g,i}} \qquad \tau_l = \frac{CFL_l \cdot \Delta x}{|u_{l,i}| + c_{l,i}}$$



Spatial step Δx is equal to cell width

Time step $\tau = min(\tau_g, \tau_l)$

Local Riemann invariants



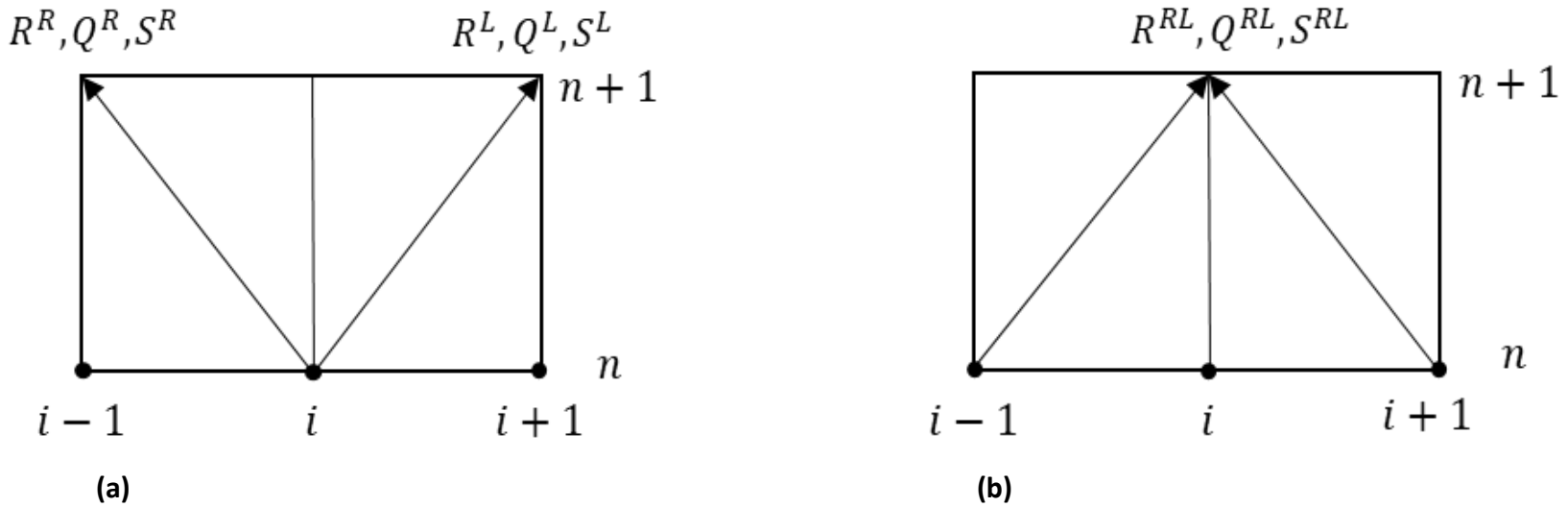
$$W = a$$

$$R = \left[-\frac{(P-\bar{P})(u-V)}{a\rho^2 c(u+c-V)}\frac{\partial P}{\partial \varepsilon} + \frac{\rho c(u-V)+(P-\bar{P})}{a\rho(u+c-V)} \right]_{i+\frac{1}{2}}^{n+\frac{1}{2}} a + u + \frac{P}{[\rho c]_{i+\frac{1}{2}}^{n+\frac{1}{2}}}$$

$$Q = \left[\frac{(P-\bar{P})(u-V)}{a\rho^2 c(u-c-V)}\frac{\partial P}{\partial \varepsilon} - \frac{\rho c(u-V)-(P-\bar{P})}{a\rho(u-c-V)} \right]_{i+\frac{1}{2}}^{n+\frac{1}{2}} a + u - \frac{P}{[\rho c]_{i+\frac{1}{2}}^{n+\frac{1}{2}}}$$

$$S = \left[-\frac{(P-\bar{P})}{a\rho}\frac{\partial P}{\partial \varepsilon} \right]_{i+\frac{1}{2}}^{n+\frac{1}{2}} a + P - \rho[c^2]_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

The "sonic points" problem



Schematic representation of "sonic points": (a) – transonic rarefaction, (b) – flow collision.

Stage 2 (Flux generation)

Estimated velocity values $\tilde{u}_i^{n+1}$ and the new speed of sound $\tilde{c}_i^{n+1}$ for finding new values of flux variables.

- The estimated velocity values are calculated as:

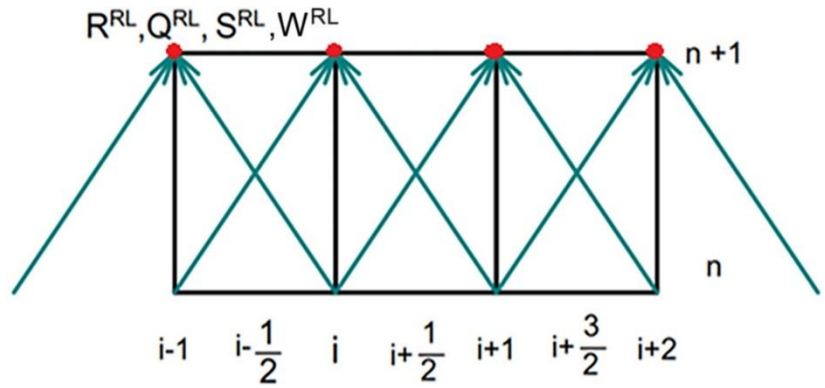
$$\tilde{u}_i^{n+1} = \sigma_i^{n+1/2} \cdot \tilde{c}_i^{n+1}, \quad \text{где} \quad \sigma_i^{n+1/2} = \frac{1}{2}\left(\frac{u_{i-1/2}^{n+1/2}}{c_{i-1/2}^{n+1/2}} + \frac{u_{i+1/2}^{n+1/2}}{c_{i+1/2}^{n+1/2}}\right)$$

- The new speed of sound is calculated by the formula:

$$\tilde{c}_i^{n+1} = c_{i+1/2}^{n+1/2} + c_{i-1/2}^{n+1/2} - c_i^n$$

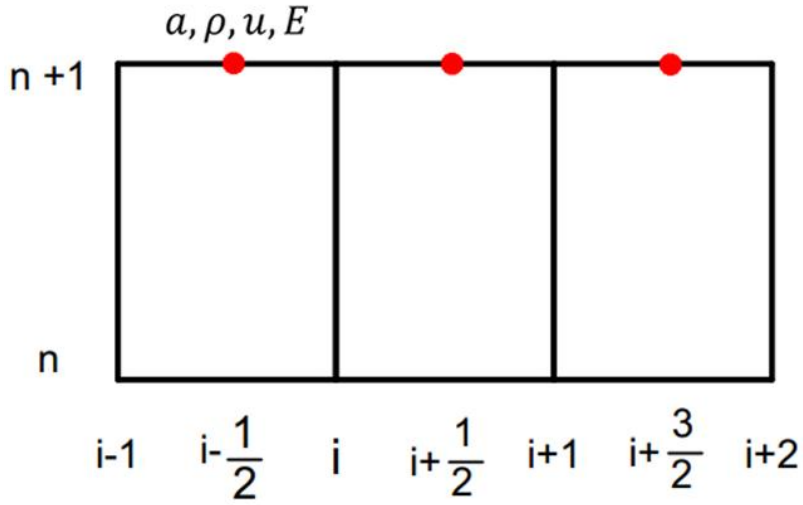
- Interfacial velocity:

$$\tilde{V}_i^{n+1} = V_{i+1/2}^{n+1/2} + V_{i-1/2}^{n+1/2} - V_i^n$$



$$(\tilde{\lambda}_1)_i^{n+1} = \tilde{V}_i^{n+1} \quad (\tilde{\lambda}_2)_i^{n+1} = \tilde{u}_i^{n+1} + \tilde{c}_i^{n+1} \quad (\tilde{\lambda}_3)_i^{n+1} = \tilde{u}_i^{n+1} - \tilde{c}_i^{n+1} \quad (\tilde{\lambda}_3)_i^{n+1} = \tilde{u}_i^{n+1}$$

Stage 3 (Corrector)



Conservative variables a, ρ, u, E at nodes $(i + 1/2, n + 1)$:

$$a_{g,i+\frac{1}{2}}^{n+1} = a_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\tau}{2}\left(f_a - V\left(\frac{a_{g,i+1}^{n+1} - a_{g,i}^{n+1}}{\Delta x}\right)\right)$$

$$\rho_{g,i+\frac{1}{2}}^{n+1} = \frac{1}{a_{g,i+\frac{1}{2}}^{n+1}}\left((a\rho)_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\tau}{2}\left(f_\rho - \frac{(a\rho u)_{g,i+1}^{n+1} - (a\rho u)_{g,i}^{n+1}}{\Delta x}\right)\right)$$

$$u_{g,i+\frac{1}{2}}^{n+1} = \frac{1}{(a\rho)_{g,i+\frac{1}{2}}^{n+1}}\left((a\rho u)_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\tau}{2}\left(f_{\rho u} + \bar{P}\left(\frac{a_{g,i+1}^{n+1} - a_{g,i}^{n+1}}{\Delta x}\right) - \frac{(a\rho u^2)_{g,i+1}^{n+1} - (a\rho u^2)_{g,i}^{n+1}}{\Delta x} - \frac{(aP)_{g,i+1}^{n+1} - (aP)_{g,i}^{n+1}}{\Delta x}\right)\right)$$

$$E_{g,i+\frac{1}{2}}^{n+1} = \frac{1}{(a\rho)_{g,i+\frac{1}{2}}^{n+1}}\left((a\rho E)_{g,i+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\tau}{2}\left(f_E + \bar{P}V\left(\frac{a_{g,i+1}^{n+1} - a_{g,i}^{n+1}}{\Delta x}\right) - \frac{(a\rho uE)_{g,i+1}^{n+1} - (a\rho uE)_{g,i}^{n+1}}{\Delta x} - \frac{(aPu)_{g,i+1}^{n+1} - (aPu)_{g,i}^{n+1}}{\Delta x}\right)\right)$$

Next, the calculation is performed for the following time step.

Initial conditions

- Domain $x \in \Omega, \quad \Omega = \{ x | \, 0 \leq x \leq L \}$

$$0 < t \leq \mathrm{T}$$

- Initial conditions $a_{g,l}, \rho_{g,l}, \mathrm{u}_{g,l}, \mathrm{P}_{g,l}(x,0) = \begin{cases} a_{Left}, \rho_{Left}, \mathrm{u}_{Left}, \mathrm{P}_{Left}, & 0 < x < L/2 \\ a_{Right}, \rho_{Right}, \mathrm{u}_{Right}, \mathrm{P}_{Right}, & L/2 < x \geq L \end{cases}$

Test	ρ_L	u_L	P_L	ρ_R	u_R	P_R	t	N
1	1	−2	1	1	2	1	0.12	100
2	1	2	1	1	-2	1	0.5	100

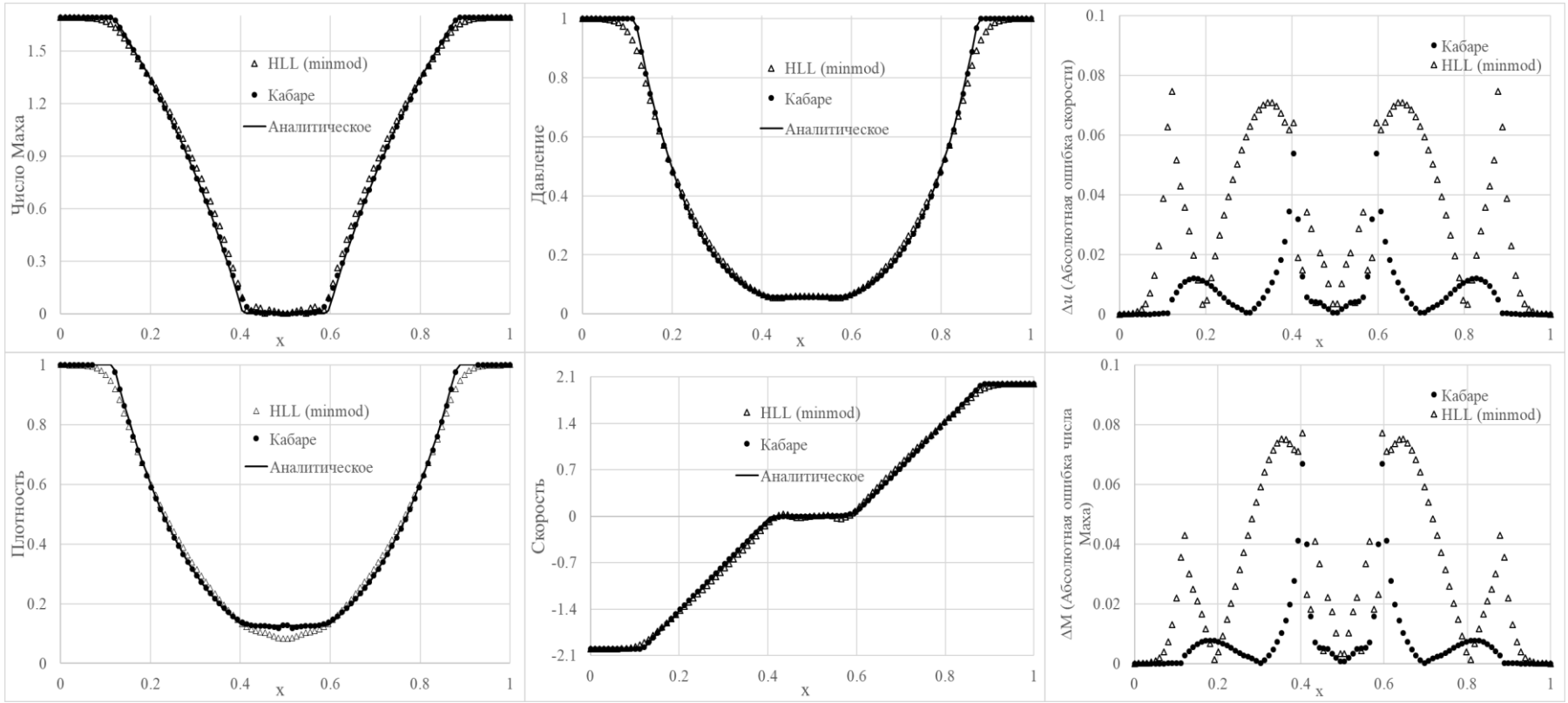
Comparison of the numerical solution U_i^n for N cells and the analytical solution $\widetilde{U}(x, t_n)$ at the final time t_n is performed using the L_1 norm:

$$\varepsilon_N(U) = \|\Delta U\|_1 = \frac{1}{N} \sum_{i=1,N} \left| U_i^n - \widetilde{U}(x_i, t_n) \right|$$

Transonic rarefaction problem

Error table for Test No. 1

Method	ρ	u	P	e	C	$Mach\ numer$
CABARET	0.002772	0.007456	0.002709	0.012774	0.004269	0.007392
minmod	0.010519	0.033027	0.014737	0.08808	0.027888	0.029641

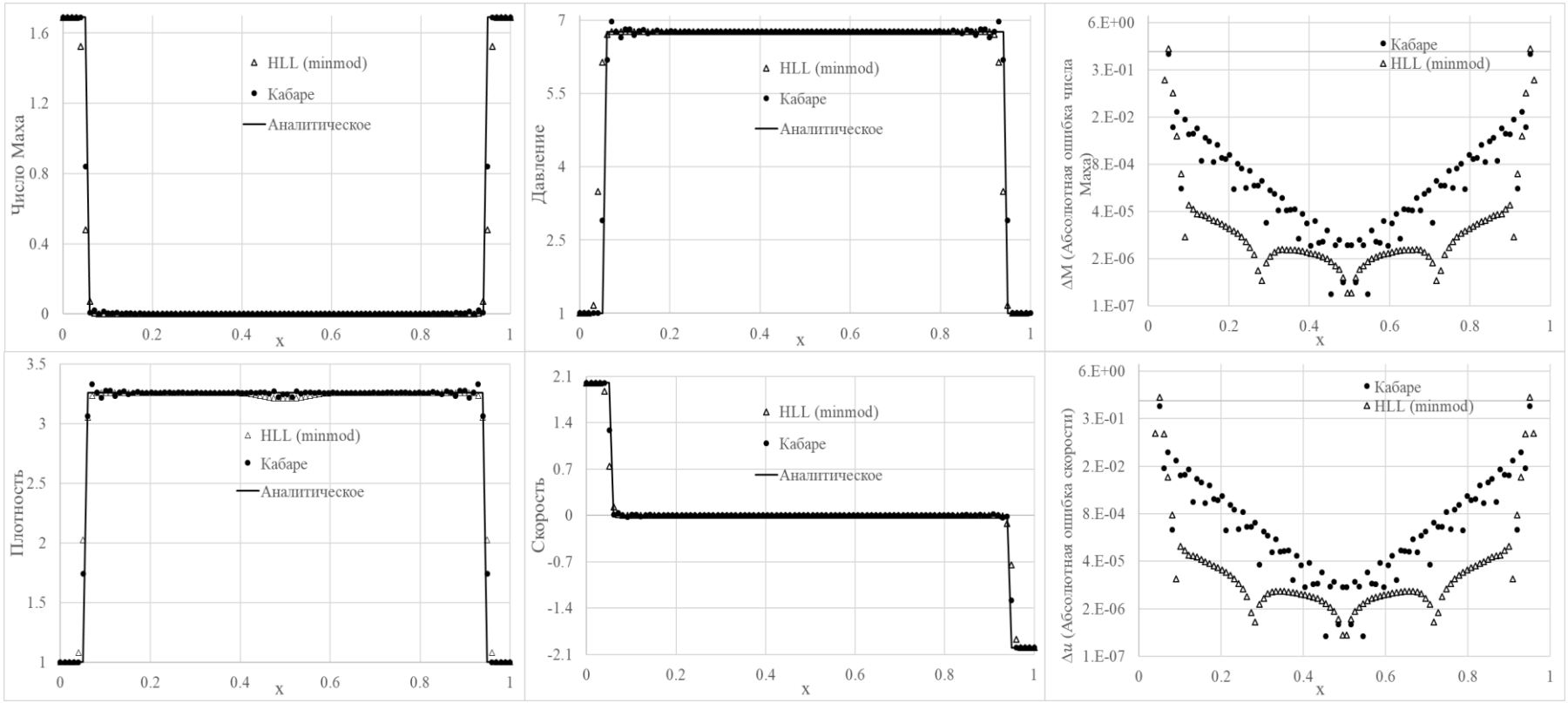


Solution profile at $CFL = 0.1$, number of time steps of the CABARET scheme $N_t = 5057$

Transonic collision problem

Error table for Test No. 2

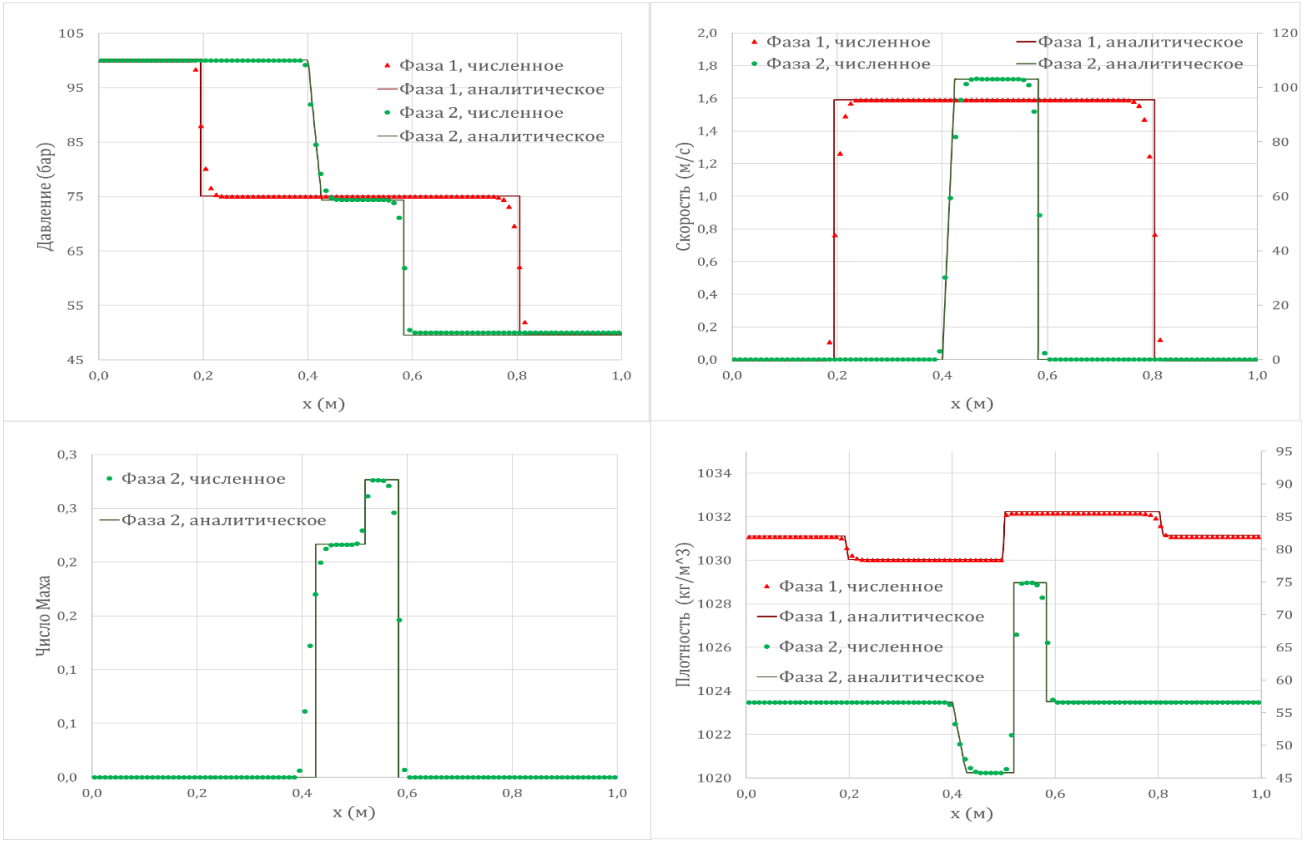
Метод	ρ	u	P	e	C	Mach number
CABARET	0.02533	0.017125	0.063558	0.041879	0.008291	0.018632
minmod	0.032015	0.030319	0.067274	0.052226	0.010312	0.02916



Solution profile at $CFL = 0.1$, number of time steps of the CABARET scheme N_t =1593

Two-phase Riemann problem (shock tube)

Phase	a_{Left}	ρ_{Left}	u_{Left}	P_{Left}	a_{Right}	ρ_{Right}	u_{Right}	P_{Right}	T, K
Gas	0.5	$\rho_g(P,T)$	0	10000000	0.5	$\rho_g(P,T)$	0	5000000	308.15
Liquid	0.5	$\rho_l(P,T)$	0	10000000	0.5	$\rho_l(P,T)$	0	5000000	308.15



Water-air Riemann problem with uniform volume fractions. In the absence of relaxation terms, each phase develops as a separate single-phase system.
The stiffened gas equation of state is used.

$t = 0.002$

$N = 100$ cells

$N_t = 101$

$CFL = 0.3$

Instantaneous pressure relaxation algorithm

$$\frac{\partial a_g}{\partial t} = \frac{P_g - P_l}{\mu}$$

$$\frac{\partial a_g \rho_g E_g}{\partial t} = \bar{P}\frac{P_{l,g} - P_{g,l}}{\mu}$$

$$\frac{\partial a_g \rho_g E_g}{\partial t} = -\bar{P}\frac{\partial a_g}{\partial t}$$

$$\left(a_g \rho_g E_g\right)^* - \left(a_g \rho_g E_g\right)^0 = -\int_{a_g^0}^{a_g^*} \bar{P}\, da_g$$

$$E_g^* = E_g^0 - \frac{1}{a_g \rho_g}\bar{P}(a_g^* - a_g^0)$$

$$E_l^* = E_l^0 + \frac{1}{a_l \rho_l}\bar{P}(a_g^* - a_g^0)$$

Relaxation algorithm:

$$a_g^i = \frac{(a_g^{max} + a_g^{min})}{2}$$

$$\rho_g^i = \frac{a_g^0 \rho_g^0}{a_g^i} \quad \rho_l^i = \frac{a_l^0 \rho_l^0}{1 - a_g^i}$$

$$E_g^i = E_g^0 - \frac{1}{a_g^i \rho_g^i}\bar{P}^i\left(a_g^i - a_g^0\right), \qquad E_l^i$$

$$P_g^i = P\left(\rho_g^i, e_g^i\right), \qquad P_l^i = P(\rho_l^i, e_l^i)$$

$$\begin{cases} a_g^{max} = a_g^i, & if\ P_g^i < P_l^i \\ a_g^{min} = a_g^i, & else \end{cases}$$

Temperature relaxation procedure

$$\begin{cases} F_1 = \dfrac{[a_g\rho_g]_0}{\rho_g(P,T)} + \dfrac{[a_l\rho_l]_0}{\rho_l(P,T)} - 1 = 0 \\ F_2 = [a_g\rho_g]_0 e_g(P,T) + [a_l\rho_l]_0 e_g(P,T) - const = 0 \end{cases}$$

Newton's method is used for the solution

Jacobian

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial P} & \frac{\partial F_1}{\partial T} \\ \frac{\partial F_2}{\partial P} & \frac{\partial F_2}{\partial T} \end{pmatrix} \quad J^{-1} = \frac{1}{|J|}\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$$

Initial approximation:

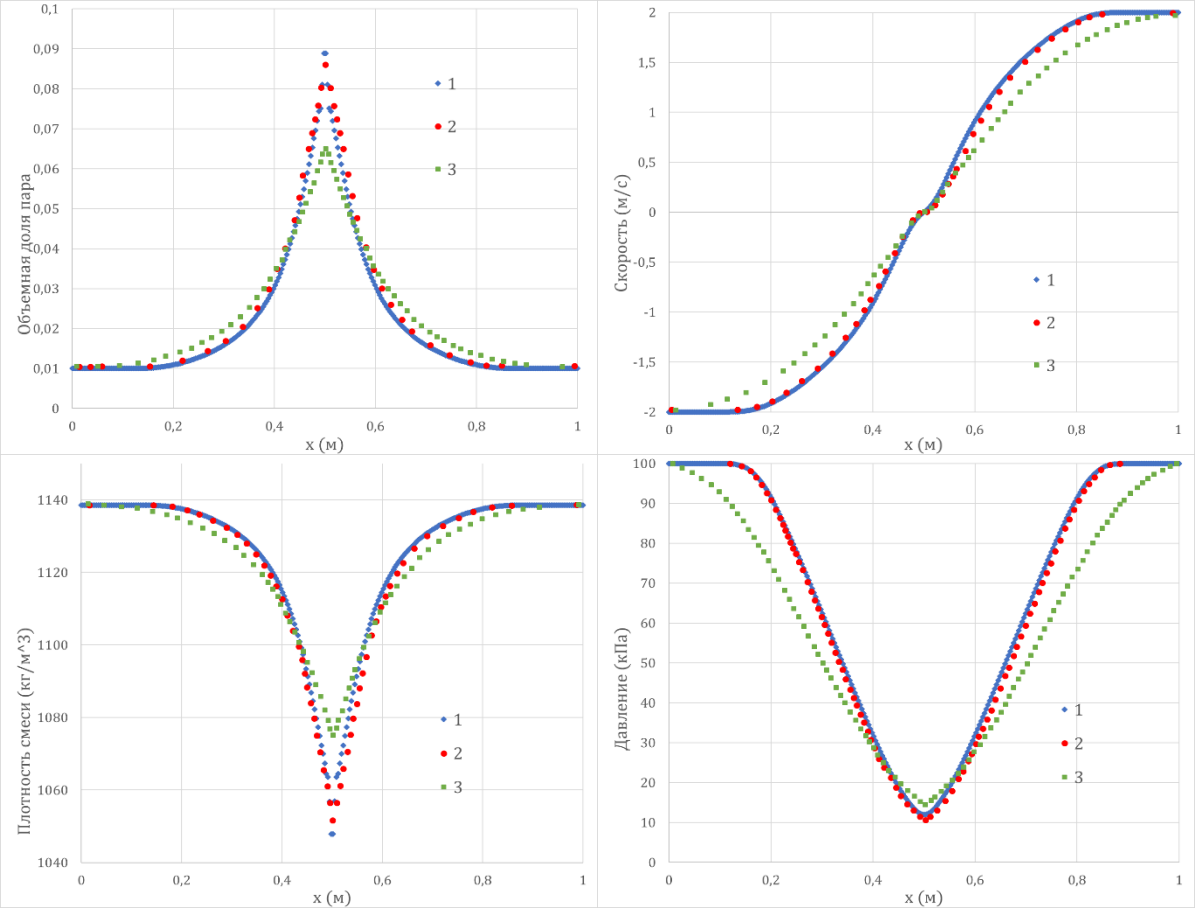
$$T_0 = 0.5\left(T_g^0 + T_l^0\right)$$
$$P_0 = a_g^0 P_g^0 + a_l^0 P_l^0$$

Values at the new iteration

$$P_{k+1} = P_k - \frac{1}{|J^k|}\left[a_{11}^k F_1(P_k,T_k) + a_{12}^k F_2(P_k,T_k)\right]$$
$$T_{k+1} = T_k - \frac{1}{|J^k|}\left[a_{21}^k F_1(P_k,T_k) + a_{22}^k F_2(P_k,T_k)\right]$$

Einfeldt problem

Phase	a_{Left}	ρ_{Left}	u_{Left}	P_{Left}	a_{Right}	ρ_{Right}	u_{Right}	P_{Right}
Vapor	0.01	1	-2	100000	0.01	1	2	100000
Liquid	0.99	1150	-2	100000	0.99	1150	2	100000



The Einfeldt problem consists of two rarefaction waves on the left and right. Between them is a low-pressure region.

$t = 0.0032s$

$N = 300$ cells

$N_t = 4364$

$CFL = 0.01$

label «1» - CABARET scheme;

label «2» - MUSCL scheme ($N = 2500$);

label «3» - MUSCL scheme ($N = 300$);

Conclusion

- The CABARET scheme has been investigated for modeling unsteady two-phase flows using a hyperbolic two-pressure, two-velocity, and two-temperature model.
- The characteristic forms of the equations have been obtained, allowing the scheme to be implemented in the characteristic form.
- Solutions to the single-phase problems of transonic rarefaction and collision have been obtained using a new technique for treating sonic points.
- Interfacial relaxation procedures for pressure, velocity, and temperature have been implemented.
- The solution to the Einfeldt problem for the case of a small gas volume fraction in the mixture has been obtained. The results were compared with the solution obtained using the Godunov-type HLLC + MUSCL scheme.