



Pion off-shell form factors

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Introduction

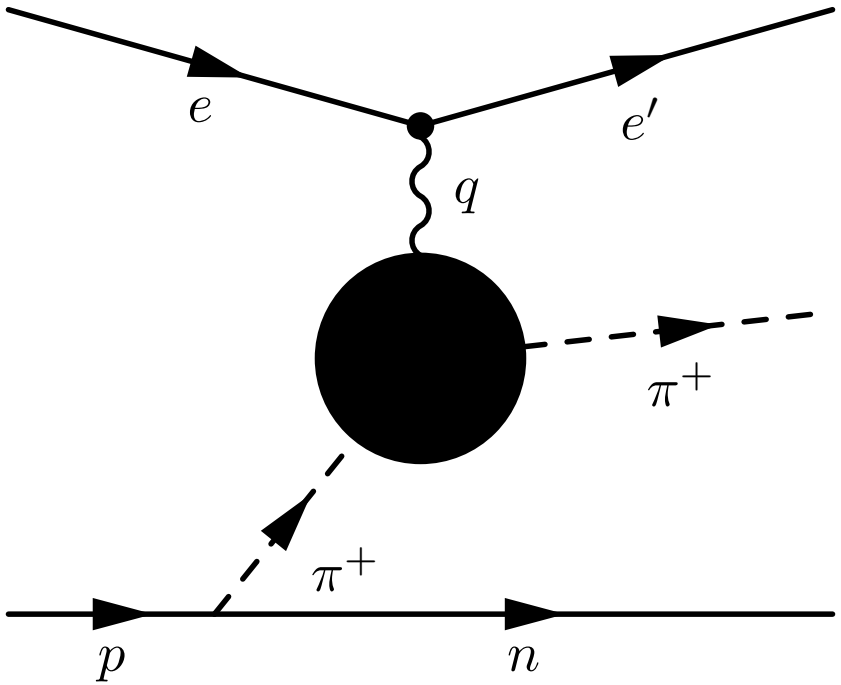
There are many models to describe pion:

- QCD sum rules;
- a relativistic model using the light-front formalism;
- the Nambu-Jona-Lasinio models;
- lattice calculations;
- and others...
- the relativistic Bethe-Salpeter equation with a separable kernel;

The electromagnetic (EM) form factor of the pion $F_\pi(Q^2)$ is determined on the mass shell and characterizes the spatial distribution of charge inside the pion as a function of the square of the transferred 4-momentum $Q^2(=-q^2)$. In regions where Q^2 is small, this form factor can be measured directly in experiments on elastic scattering of pions on electrons. However, direct determination of the form factor is difficult for intermediate and high values of Q^2 due to the short lifetime of the pion, which makes it unsuitable for such studies.

Motivation

An alternative approach for studying the form factor in these regions experimentally is based on the exclusive Sullivan process. In particular, it involves the analysis of the cross sections of the ${}^1H(e, e'\pi^+)n$ EM reaction. In this case, however, the virtual pion in the intermediate state means that the measured form factor appears to be half-off-shell.



Bethe-Salpeter approach with a separable kernel

The Bethe-Salpeter equation for the vertex function of the pion is written as follows

$$\Gamma_{\alpha\beta}(k;p) = i\int \frac{d^4k''}{(2\pi)^4} V_{\alpha\beta:\epsilon\lambda}(k,k'';p) S_{\lambda\eta}(k''+p/2)\Gamma_{\eta\zeta}(k'';p) S_{\zeta\epsilon}(k''-p/2). \quad (1)$$

A separable interaction kernel of rank one is considered in the following form

$$V_{\alpha\beta:\delta\gamma}(k',k;p) = \Delta_{\alpha\beta}(k';p)\overline{\Delta}_{\delta\gamma}(k;p).$$

$$\Delta_{\alpha\beta}(k;p) = f(k^2, k\cdot p)\,\Omega_{\alpha\beta},$$

The following types of a kernel for a pion are considered:

$$\Omega_{\alpha\beta} = \gamma^5_{\alpha\beta};\;\; f(k^2) = 1/(k^2-\Lambda^2+i\epsilon) - \text{monopole},$$

$$f(k^2) = 1/((k^2-\Lambda^2)^2+i\epsilon) - \text{dipole}$$

$$V_{\alpha\beta:\delta\gamma}(k',k) = g\,\gamma^5_{\alpha\beta}\,f(k'^2)\,\gamma^5_{\delta\gamma}\,f(k^2), \quad (2)$$

the solution for the vertex function $\pi q\bar{q}$ is:

$$\Gamma(k;p) \equiv \Gamma(k) = N\gamma^5 f(k^2), \quad (3)$$

Pion constants

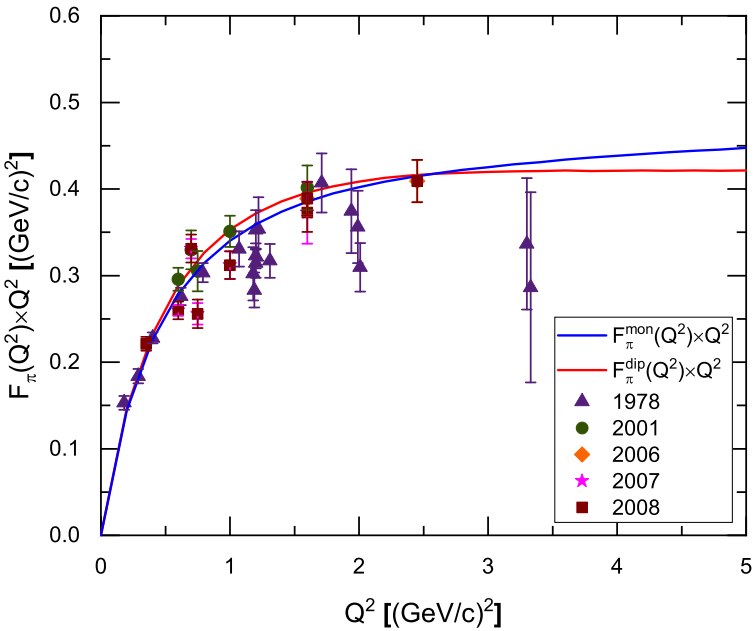
Model parameters: $\Lambda,\ m_q,\ m_\pi = 0.14$ GeV.

model	m_q (MeV)	Λ (MeV)	f_π (MeV)	$r_{\pi\gamma}$ (fm)	$< r_\pi^2 > (\mathrm{fm}^2)$	$\Gamma_{\pi^0 \to \gamma\gamma}$ (eV)
monopole	260	550	143.363(1)	0.640(1)	0.459(1)	7.25(1)
dipole	245	1100	135.962(1)	0.652(1)	0.438(1)	7.74(1)
[1]	267	400	132	0.655	0.533	—
[2]	300	500	153.7	—	0.41	—
[3]	220	600	130.67	—	0.542	—
Exp. [4]			130.410(2)	0.659(4)	0.434(8)	7.82(1)

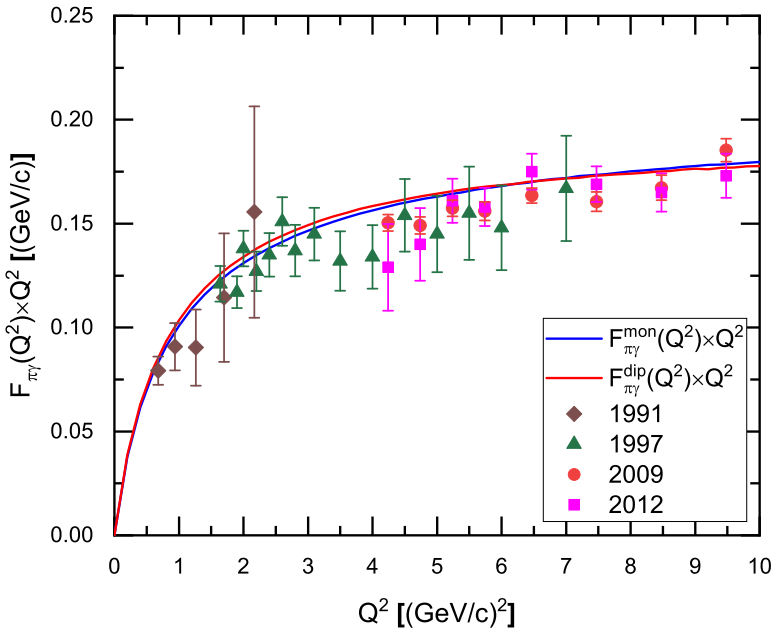
[1]—Anikin, I.V., Ivanov, M.A., Kulimanova, N.B., Lyubovitskij, V.E. Z. Phys. C 65, 681–690 (1995)
[2]—Ito, H., Buck, W.W., Gross, F. Phys. Rev. C 45, 1918–1934 (1992)
[3]—Leao, J., Melo, J.P.B.C., Frederico, T., Choi, H.-M., Ji, C.-R. Phys. Rev. D 110(7), 074035 (2024)
[4]—Review of Particle Physics: Particle Data Group / M Tanabashi, PD Grp, K Hagiwara et al. // Physical Review D. — 2018. — 08. — Vol. 98.

Fig. 1 Pion form factors on the on-shell surface

The pion elastic EM form factor:



The pion transition form factor:



The Ward-Takahashi identity

The photon-pion vertex can be represented in the following form:

$$G_\mu(p,p') = (p'+p)_\mu F_1(q^2,p^2,p'^2) + q_\mu F_2(q^2,p^2,p'^2), \tag{4}$$

where $p, p', q = p' - p$ are 4-momenta.
The Ward-Takahashi identity (WTI) is satisfied by this vertex on the off-shell

$$q^\mu G_\mu(p,p') = \Delta^{-1}(p') - \Delta^{-1}(p), \tag{5}$$

where

$$\Delta(p) = \frac{1}{p^2 - m_\pi^2 + i\varepsilon} \tag{6}$$

In the case when the initial pion is off-shell, and the final pion is on the mass shell ($p^2 = t; p'^2 = m_\pi^2$), it follows from equations (4),(5) that

$$F_2(Q^2,t) = \frac{(t-m_\pi^2)[F_1(0,t) - F_1(Q^2,t)]}{Q^2} \tag{7}$$

The g form factor and pion constant

Since $F_2(Q^2, t = m_\pi^2) = 0$, then

$$g(Q^2, t) = \frac{F_2(Q^2, t)}{t - m_\pi^2};$$

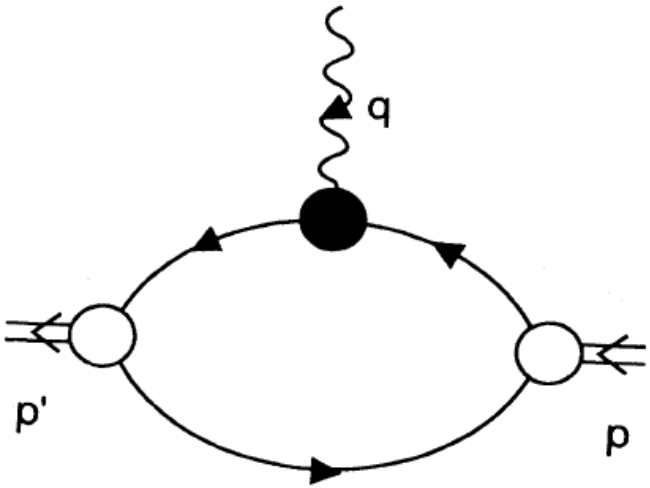
$$g(Q^2 = 0, m_\pi) = -\frac{\partial}{\partial Q^2} F_1(Q^2 = 0, m_\pi) = \frac{1}{6}\langle r_\pi^2 \rangle.$$

Model	$g(0, m_\pi^2)$ $[\mathrm{GeV}^2]$
monopole	1.97(2)
dipole	1.79(3)
[2]	2.32
Exp.	1.86(5)

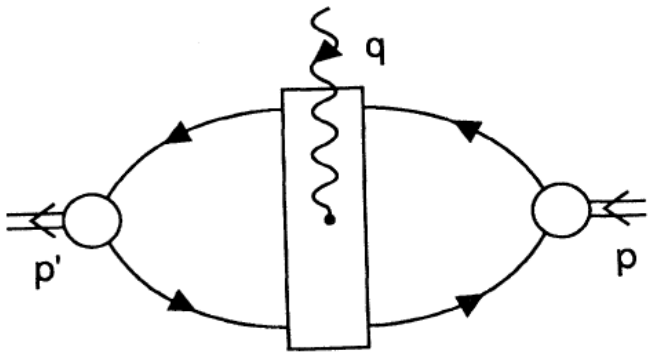
Expressions for pion form factors

$$F_{(1,2)}(Q^2,t) = F^{\rm RIA}_{(1,2)}(Q^2,t) + F^{\rm int}_{(1,2)}(Q^2,t)$$

$$\int d^4k\ k^\mu\ \tilde{F}(k,Q^2,t) \to (p'+p)^\mu \int d^4k\ \tilde{F}_1(k,Q^2,t)\ +\ q^\mu \int d^4k\ \tilde{F}_2(k,Q^2,t).$$



relativistic impulse approximation (RIA)
$(a) + (\bar{b})$

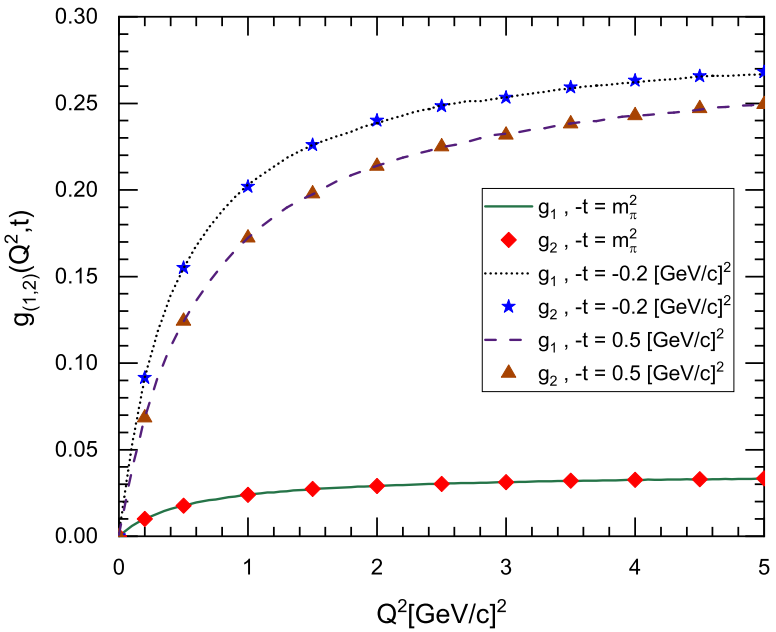


interaction curent (INT)

$i\partial_i^\mu \to i\partial_i^\mu + ie_i A^\mu(x_i)$

Fig. 2

Comparison of the directly calculated $g_2 = |Q^2 F_2(Q^2, t)|$ and calculated using WTI $g_1 = |(t - m_\pi^2)[F_1(0, t) - F_1(Q^2, t)])|$:



Comparison of the RIA and INT contributions of the $F_\pi(Q^2)$:

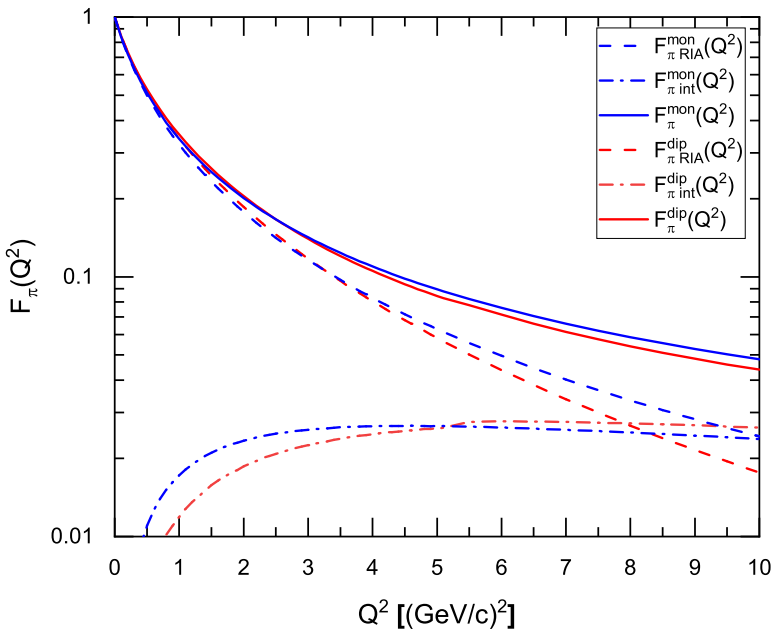
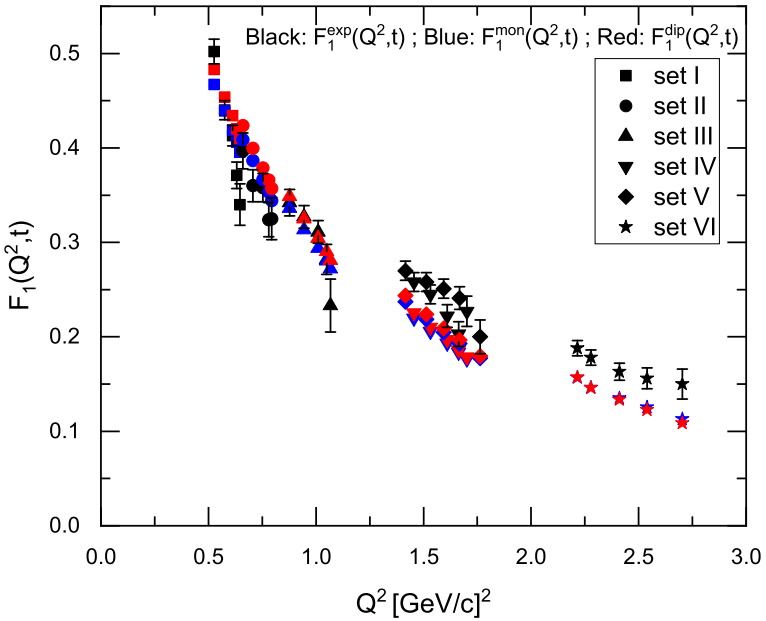
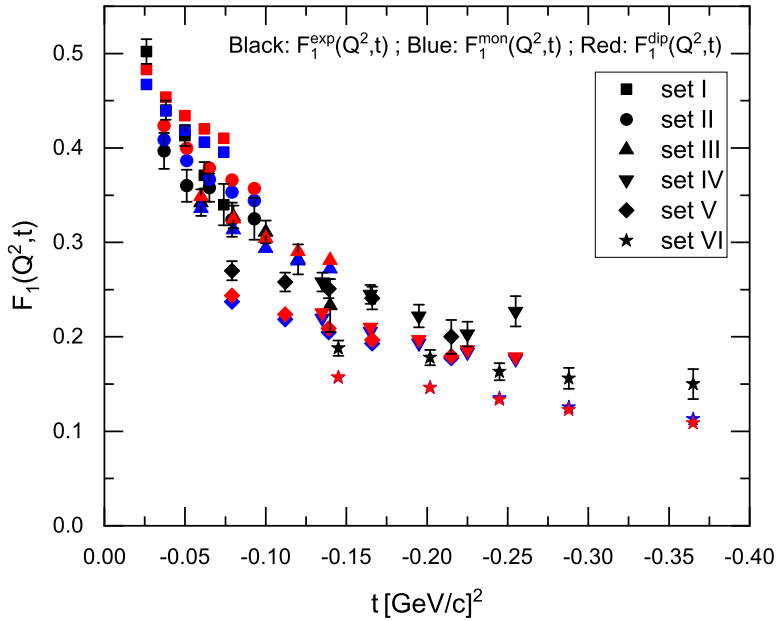


Fig. 3 Pion off-shell form factor $F_1(Q^2, t)$

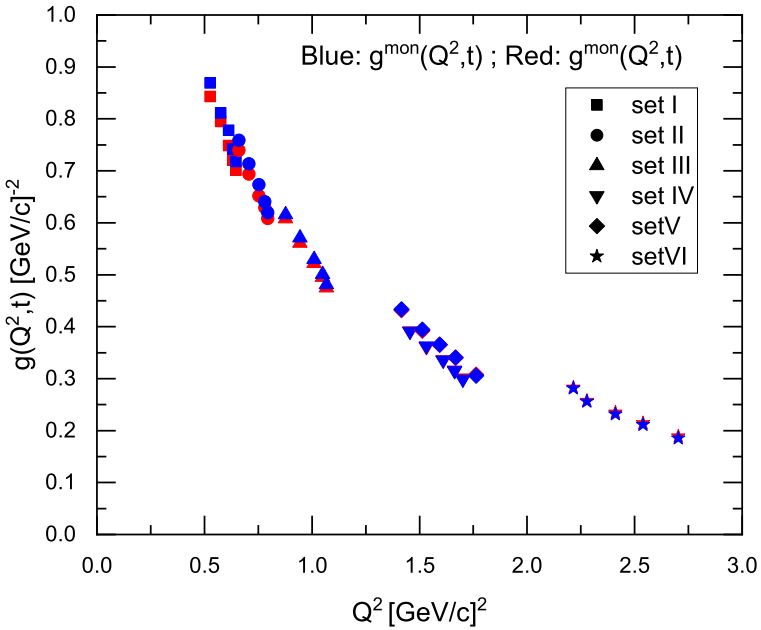




Set I: $< Q^2 >= 0.60\ \mathrm{GeV}^2/c^2, W = 1.95\ \mathrm{GeV}$; Set II: $< Q^2 >= 0.75\ \mathrm{GeV}^2/c^2, W = 1.95\ \mathrm{GeV}$;
Set III: $< Q^2 >= 1.00\ \mathrm{GeV}^2/c^2, W = 1.95\ \mathrm{GeV}$;
Set IV: $< Q^2 >= 1.60\ \mathrm{GeV}^2/c^2, W = 1.95\ \mathrm{GeV}$;
Set V: $< Q^2 >= 1.60\ \mathrm{GeV}^2/c^2, W = 2.22\ \mathrm{GeV}$; Set VI:
$< Q^2 >= 2.45\ \mathrm{GeV}^2/c^2, W = 2.22\ \mathrm{GeV}$

Exp. — Phys. Rev. C. — 2008. — Vol. 78. — P. 045202.

Fig. 4 Pion off-shell form factor $g(Q^2, t)$



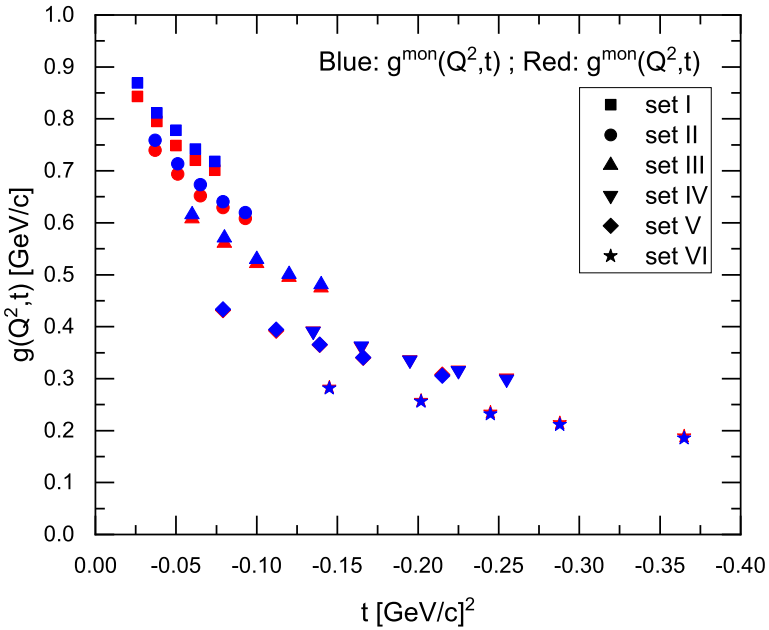
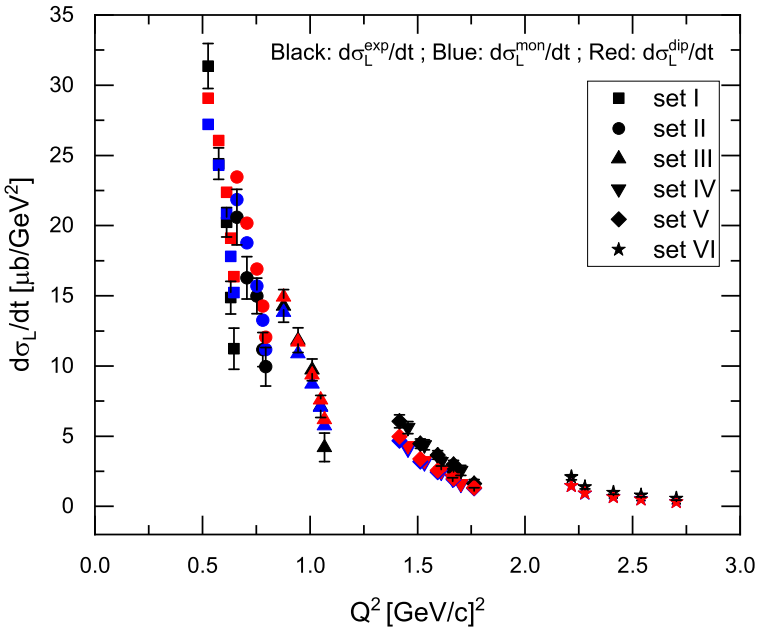
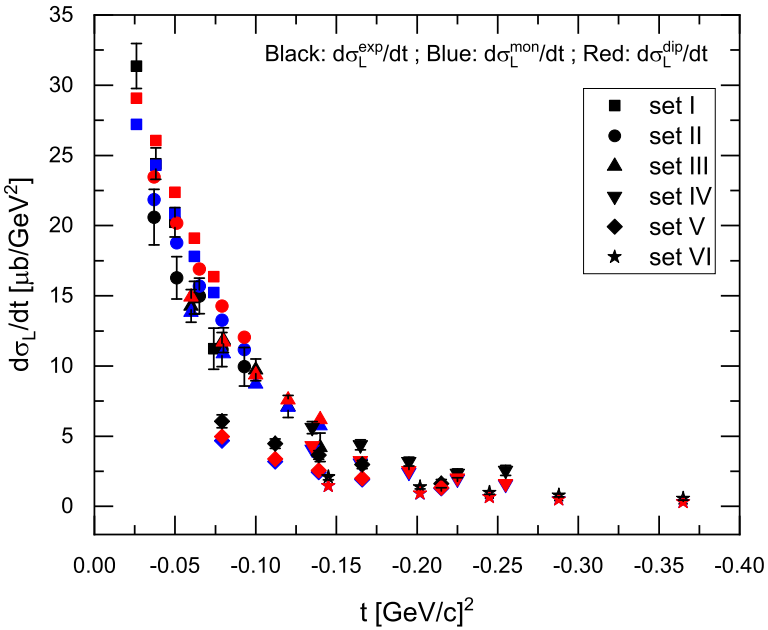


Fig. 5 Longitudinal differential cross section $d\sigma_{\rm L}/dt$





Conclusions

- Using the Bethe-Salpeter model with a separable kernel, the pion constants and form factors on the on-shell and off-shell surfaces for two types of kernel functions are calculated and compared with experimental data.
- The Ward-Takahashi identity for a pion off-shell form factor has been numerically verified. In addition, in order for the identity to be fulfilled, it was necessary to take into account two contributions: the relativistic impulse approximation and the interaction current.
- It turned out that the extra-mass form factors $F_1(Q^2, t)$ and $g(Q^2, t)$ are insensitive to the choice of a vertex function at low Q^2, but differ at high Q^2.
- The longitudinal differential cross section $d\sigma_L/dt$ of the exclusive Sullivan process was calculated and compared with experimental data.