II. Fermi-liquid theory and Green's functions

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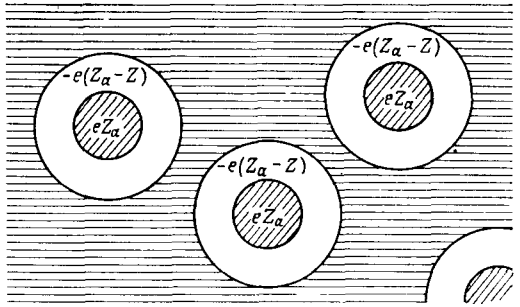
Fermionic systems

electron $m_e = 0.5\,\mathrm{MeV}, \quad s = \frac{1}{2}$

electron gas in metals



$$n = Z\frac{\rho}{M_m}N_A \qquad \text{for Fe} \qquad \rho_{\mathrm{Fe}} = 7.874\,\frac{\mathrm{g}}{\mathrm{cm}^3}$$

$$N_A = 6.022\times 10^{23}\,\frac{\mathrm{atom}}{\mathrm{mole}} \qquad M_m = 55.845\,\frac{\mathrm{g}}{\mathrm{mole}} \qquad Z = 2\ (4s^2\ \text{electrons}) \qquad n = 17.0\times 10^{22}\,\frac{\mathrm{atoms}}{\mathrm{cm}^3}$$

typical number densities $n \sim (1\text{–}20)\times 10^{22}\,\mathrm{cm}^{-3}$

nucleons: protons and neutrons $m_N = 940\,\mathrm{MeV}, \quad s = \frac{1}{2}$ **nuclear matter** (Coulomb interaction is neglected)

nucleus size $R(A) = r_0 A^{1/3}$, $r_0 = 1.1\text{–}1.2\,\mathrm{fm}$

$$n_{\mathrm{nucl}} \sim \left(\frac{4\pi}{3}r_0^3\right)^{-1} = 0.14\text{–}0.18\,\mathrm{fm}^{-3}$$

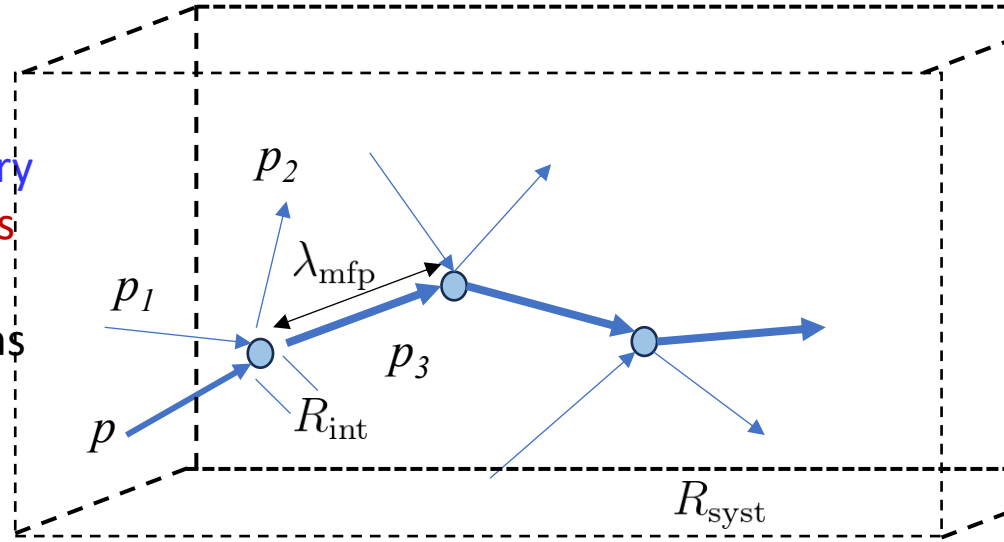
Helium-3 (n p p + e +e) 5 particles with spin ½ → total spin ½ **liquid helium 3**

Dilute fermion gas

Let us consider a gas of neutral atoms interacting through a short-range binary potential. At high enough temperatures and low enough pressures, the gas is dilute.

Each atom moves as if it were essentially free, apart from infrequent collisions with other atoms or with the container walls.

The system is well described by the elementary kinetic theory of gases.



Equation for the particle distribution $n_f(\boldsymbol{x},\boldsymbol{p},t)$ $\qquad \dfrac{\mathrm{d}n_f}{\mathrm{d}t} = I[n_f]$

$$\frac{\mathrm{d}n_f}{\mathrm{d}t} = \frac{\partial n_f}{\partial t} + \dot{\boldsymbol{x}}\frac{\partial n_f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\frac{\partial n_f}{\partial \boldsymbol{p}} = I[n_f]$$

$$I[n_{\boldsymbol{p}}] = \frac{1}{V^2}\sum_{p_1}\sum_{p_2} W(p,p_1;p_2,p_3)\delta(\epsilon_{\boldsymbol{p}}+\epsilon_{\boldsymbol{p}_1}-\epsilon_{\boldsymbol{p}_2}-\epsilon_{\boldsymbol{p}_3}) \times \delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{p}_3)\left[n_{f,\boldsymbol{p}_2}n_{f,\boldsymbol{p}_3}-n_{f,\boldsymbol{p}}n_{f,\boldsymbol{p}_1}\right]$$

transition probability

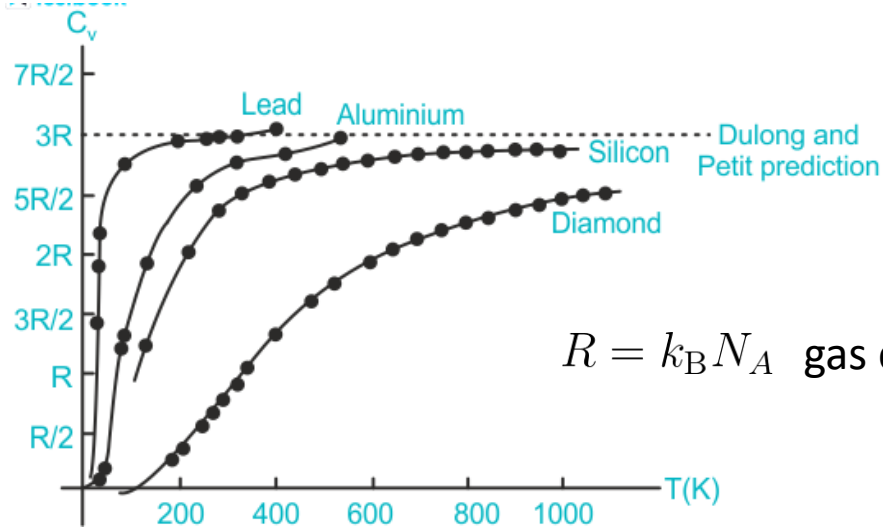
$$R_{\rm syst} \gg \lambda_{\rm mfp} \sim \frac{1}{n\sigma} \gg R_{\rm int} \qquad \sigma \sim 4\pi R_{\rm int}^2 \qquad nR_{\rm int}^3 \ll 1$$

It displays the usual properties of a classical gas;
the specific heat C_V is temperature independent

$$C_V = \left(\frac{\delta Q}{\delta T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$$

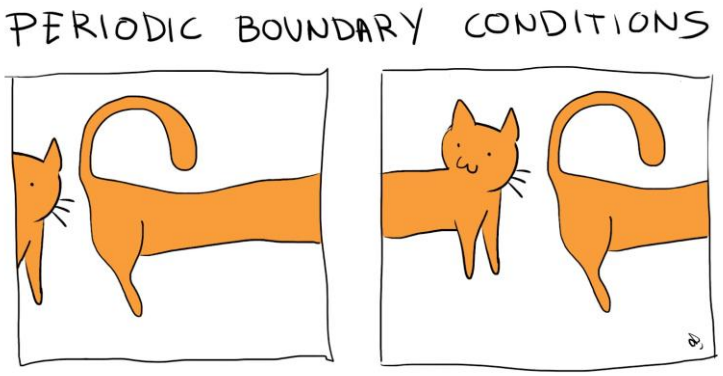
$$n_f(p) = \frac{N}{V}\left(\frac{2\pi}{mT}\right)^{3/2} e^{-\frac{p^2}{2mT}} \qquad E = V\int n_f(p)\frac{p^2}{2m}\frac{2\mathrm{d}^3p}{(2\pi)^3} = \frac{3}{2}T2N$$

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V$$



$R = k_{\rm B}N_A$ gas constant

Thermodynamics of a Fermi gas



one type of fermions

single particle Hamiltonian $\hat{H}_1 = \dfrac{\hat{p}^2}{2m}$ box $L_x \times L_y \times L_z$

Box with periodic boundary conditions

$$\Psi_{\boldsymbol{p}}(\boldsymbol{r}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{p}\boldsymbol{r}}$$

$$\boldsymbol{p_l} = \hbar \boldsymbol{k_l} = \hbar\left(2\pi \frac{l_x}{L_x}\boldsymbol{i} + 2\pi \frac{l_y}{L_y}\boldsymbol{j} + 2\pi \frac{l_z}{L_z}\boldsymbol{k}\right) \qquad (l_x, l_y, l_z)$$

integers

The total wave function is a Slater determinant made up of N such plane waves. All the eigenstates of the system can be characterized by the distribution function $n_{\boldsymbol{p}s}$, which is equal to 1 if the state $\boldsymbol{p}s$, is occupied, to zero otherwise.

Probability that the system has the particular distribution of particles on various levels

$$\{n_p\} = (n_{p_1+}, n_{p_1-}, n_{p_2+}, n_{p_2-}, n_{p_3+}, n_{p_3-}, \ldots, n_{p_n} \ \ldots)$$

$$P(\{n_p\}) = \frac{1}{Z_\mu^{(\mathrm{f})}} \exp\left\{ -\frac{1}{T}\left(\sum_p (\epsilon_p - \mu) n_{p+} + (\epsilon_p - \mu) n_{p-} \right) \right\}$$

here μ is the Legandre coefficients, which fix the averaged number of particles. Quantity $Z_\mu^{(\mathrm{f})}$ is the statistical sum, which is the normalization factor for the probability.



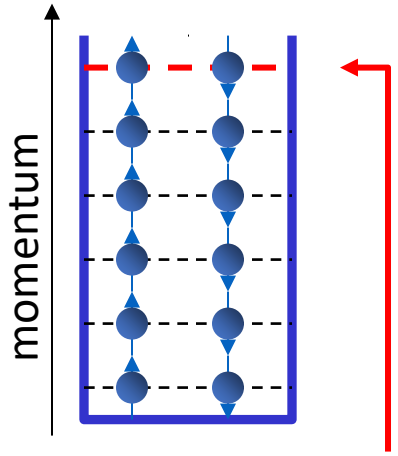
Pauli exclusion principle

Two identical particles with uneven spin cannot occupy simultaneously the same quantum states, i.e. have the same quantum numbers

$T = 0$

$$Z_\mu^{(\mathrm{f})} = \sum_{n_{1,+}=0}^{1}\sum_{n_{1,-}=0}^{1}\sum_{n_{2,+}=0}^{1}\sum_{n_{2,-}=0}^{1}\cdots\sum_{n_{i,+}=0}^{1}\sum_{n_{i,-}=0}^{1}\ldots\exp\left\{-\frac{1}{T}\left(\sum_p(\epsilon_p-\mu)n_{p,+}(\epsilon_p-\mu)n_{p,-}\right)\right\})$$

spin ½ particles in a box



Fermi momentum p_F
Fermi energy $E_F=e(p_F)$

For one sum of the particle numbers on the ith (enegy/momentum) level with spin $s = \pm$ we have

$$\sum_{n_{i,s}=0}^{1} e^{-\frac{1}{T}(\epsilon_{p_i}-\mu)n_{i,s}} = z_{p_i,s}^{(\mathrm{f})} \equiv 1 + e^{-\frac{1}{T}(\epsilon_{p_i}-\mu)}$$

Then the statistical sum for the fermion system is equal to $Z_\mu^{(\mathrm{f})} = \prod_p z_{p,+}^{(\mathrm{f})} z_{p,-}^{(\mathrm{f})}$

and the thermodynamical potential for fermions is

$$\Omega^{(\mathrm{f})} = -T\log Z_\mu^{(\mathrm{f})} = -T\sum_p\sum_{s=\pm}\log z_{p,s}^{(\mathrm{f})} = -TV\int\frac{2\mathrm{d}^3p}{(2\pi)^3}\log\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right)$$

$$\Omega^{(\mathrm{f})} = -TV\int\frac{2\mathrm{d}^3p}{(2\pi)^3}\log\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right)$$

Thermodynamic limit $L\to\infty, V\to\infty, \langle N\rangle\to\infty$, but $\langle N\rangle/V = n = \text{const.}$ $\sum_p \to V\int\frac{\mathrm{d}^3k}{(2\pi)^3} = V\int\frac{\mathrm{d}^3p}{(2\pi\hbar)^3}$ $(\hbar = 1)$

$$\Omega^{(\mathrm{f})} = -TV \int \frac{2\mathrm{d}^3 p}{(2\pi)^3} \log\left(1 + e^{-\frac{1}{T}(\epsilon_p - \mu)}\right)$$

$$N = -\frac{\partial \Omega}{\partial \mu}\Big|_T = V \int \frac{2\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{e^{(\epsilon_p - \mu)/T} + 1}$$

Fermi-Dirac distribution function

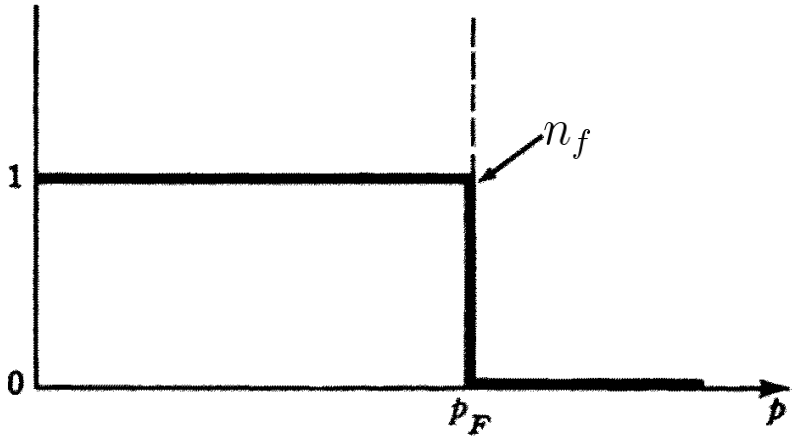
For particles with specific spin projection

$$n_f(p, s) = \frac{1}{e^{\frac{1}{T}(\epsilon_p - \mu)} + 1} \qquad s = \pm$$

Fermi gas at T=0

$$n_f(p,s) = \frac{1}{e^{\frac{1}{T}(\epsilon_p - \mu)} + 1} \qquad s = \pm \qquad T \to 0 \qquad n_f(p,s) = \begin{cases} \frac{1}{0+1} = 1 & \epsilon_p < \mu \\ \frac{1}{\infty + 1} = 0 & \epsilon_p > \mu \end{cases} = \theta(\mu - \epsilon_p) = \theta(p_{\mathrm{F}} - p)$$

Fully degenerate fermion system



p_F is the limiting momentum occupied by particles (Fermi momentum)

$\mu = \epsilon_{\mathrm{F}}$ limiting energy (Fermi energy) $\epsilon_{\mathrm{F}} = \frac{p_{\mathrm{F}}^2}{2m}$

Fermi temperature $T_{\mathrm{F}} = k_{\mathrm{B}}\epsilon_{\mathrm{F}}$

$$n = \frac{N}{V} = \sum_{s=\pm} \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} n_f(p,s) = 2 \int_0^{p_{\mathrm{F}}} \frac{4\pi \mathrm{d}p p^2}{(2\pi\hbar)^3} = \frac{p_{\mathrm{F}}^3}{3\pi^2\hbar^3}$$

Relation between the Fermi momentum and density $p_{\mathrm{F}} = \hbar(3\pi^2 n)^{1/3}$ $\epsilon_{\mathrm{F}} = \frac{\hbar^2(3\pi^2 n)^{2/3}}{2m}$

electron gas in metals (Fe) $n_e = 17.0 \times 10^{22}\mathrm{cm}^{-3}$ $k_{\mathrm{F}} = \frac{p_{\mathrm{F}}}{\hbar} = (3\pi^2 \cdot 17 \times 10^{22})^{1/3}\,\mathrm{cm}^{-1} = 1.71 \times 10^8\,\mathrm{cm}^{-1} = 17.1\,\mathrm{nm}^{-1}$

$$v_{\mathrm{F}} = \frac{\hbar k_{\mathrm{F}}}{m_e} = \frac{\hbar c k_{\mathrm{F}}}{m_e c^2} c = \frac{197.33\,\mathrm{eV \cdot nm} \cdot 17.1\,\mathrm{nm}^{-1}}{0.511 \times 10^6\,\mathrm{eV}} c \approx 0.6 \times 10^{-2} c \qquad \epsilon_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2m_e} = 11\,\mathrm{eV} \qquad T_{\mathrm{F}} = k_{\mathrm{B}}\epsilon_{\mathrm{F}} \simeq 10^5\,\mathrm{K}$$

$\epsilon_{\mathrm{F}} \ll m_e$

neutron matter $n_{\rm nucl} = 0.16\,{\rm fm}^{-3} \approx 0.45 m_\pi^3$

$$p_{\rm F} = (3\pi^2 n)^{1/3} = 2.4\, m_\pi \left(\frac{n}{n_{\rm nucl}}\right)^{1/3} = 331\,{\rm MeV}\left(\frac{n}{n_{\rm nucl}}\right)^{1/3} \qquad \frac{p_{\rm F}}{m_N} \simeq 0.35$$

relativistic effects could be important

$$\epsilon_{\rm F} = \frac{p_{\rm F}^2}{2m_N} = 0.43\, m_\pi \left(\frac{n}{n_0}\right)^{2/3} = 60\,{\rm MeV}\left(\frac{n}{n_0}\right)^{2/3}$$

Energy density $$\frac{E}{V} = 2\int \frac{{\rm d}^3p}{(2\pi\hbar)^3}\theta(p_{\rm F}-p)\frac{p^2}{2m} = \frac{1}{\pi^2\hbar^3}\int_0^{p_{\rm F}} \frac{{\rm d}p p^4}{2m} = \frac{p_{\rm F}^5}{10m\pi^2\hbar^3} = \frac{3}{5}n\frac{p_{\rm F}^2}{2m} = \frac{3}{5}n\epsilon_{\rm F}$$

Pressure $$P = -\frac{\Omega^{({\rm f})}}{V} = \lim_{T\to 0} TV\int \frac{2{\rm d}^3p}{(2\pi)^3}\log\left(1+e^{-\frac{1}{T}(\epsilon_p-\mu)}\right) = -\int\frac{2{\rm d}^3p}{(2\pi)^3}(\epsilon_p-\mu) = \mu n - \frac{E}{V} = n\,\epsilon_{\rm F} - \frac{3}{5}n\epsilon_{\rm F} = \frac{2}{5}n\epsilon_{\rm F}$$

$$PV = \frac{2}{3}E$$

for the electron gas in metals (Fe)

$$n_e = 17.0\times 10^{22}{\rm cm}^{-3} = 170{\rm nm}^{-3} \qquad \epsilon_{\rm F} = 11\,{\rm eV} \qquad E/V = \frac{3}{5}n\epsilon_{\rm F} \simeq \frac{\rm keV}{\rm nm^3}$$

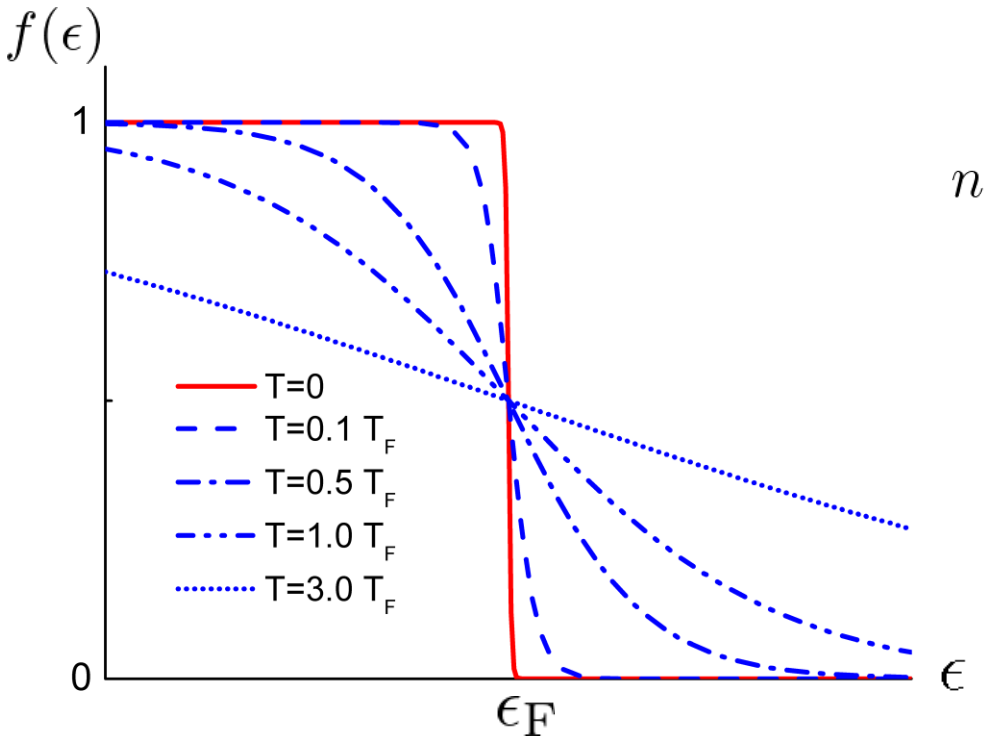
$$P = \frac{2}{5}(17\times 10^{22}\,{\rm cm}^{-3}\cdot 11\,{\rm eV}\cdot 1.6^{-19}\,{\rm J/eV}) \approx 1.2\times 10^5\frac{\rm J}{\rm cm^3} = 1.2\times 10^{11}\frac{\rm J}{\rm m^3} = 1.2\times 10^{11}\,{\rm Pa}$$

Fermi gas at T=/=0

$n_f(p) = f(\epsilon_p)$ where $f(\epsilon) = \dfrac{1}{e^{(\epsilon-\mu)/T}+1}$

at finite temperature $\mu \neq \epsilon_{\mathrm{F}}$

μ fixes the total number of particles (particle density)



$$n = \frac{N}{V} = \sum_{s=\pm} \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} n_f(p,s) = 2\int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} f(\epsilon_p)$$

$$= \int_0^{+\infty} \mathrm{d}\epsilon \int \frac{2\mathrm{d}^3 p}{(2\pi\hbar)^3} \delta(\epsilon - \epsilon_p) f(\epsilon) = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) f(\epsilon)$$

$D(\epsilon)$ is the density of states $\quad D(\epsilon) = \dfrac{\mathrm{d}n}{\mathrm{d}\epsilon}$

$$D(\epsilon) = \int \frac{2\mathrm{d}^3 p}{(2\pi\hbar)^3} \delta(\epsilon - \epsilon_p) = \frac{1}{\pi^2\hbar^3} \int_0^{\infty} \mathrm{d}p p^2 \delta\left(\epsilon - \frac{p^2}{2m}\right) = \frac{2m}{\pi^2\hbar^3} \int_0^{\infty} \frac{1}{2} \mathrm{d}u u^{1/2} \delta(2m\epsilon - u) = \frac{m}{\pi^2\hbar^3}\sqrt{2m\epsilon}$$

Density of states on the Fermi surface $\quad D(\epsilon_{\mathrm{F}}) = \left.\dfrac{\mathrm{d}n}{\mathrm{d}\epsilon}\right|_{\mathrm{F}} = \dfrac{m p_{\mathrm{F}}}{\pi^2\hbar^3}$

particle density

$$n = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) f(\epsilon) \quad \Rightarrow \quad \mu = \mu(n, T)$$

energy density

$$u = \frac{E}{V} = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) \epsilon f(\epsilon) \quad \Rightarrow \quad u = u(\mu, T) = u(\mu(n, T), T) = u(n, T)$$

entropy density

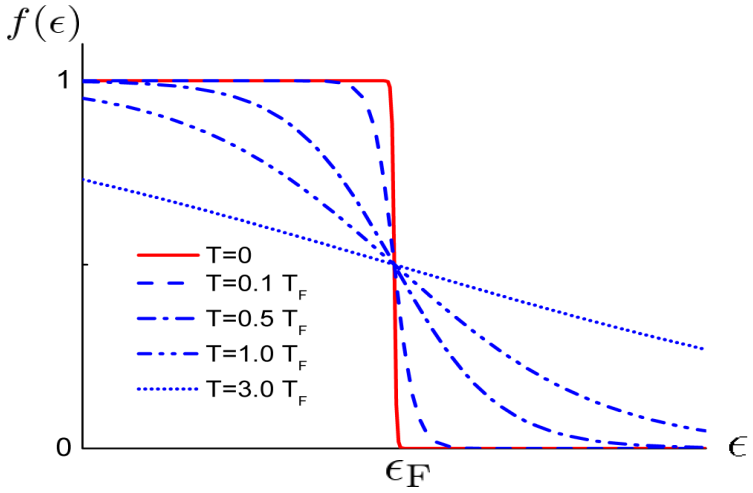
$$s = \frac{S}{V} = \int_0^{+\infty} \mathrm{d}\epsilon D(\epsilon) \sigma_f(\epsilon) \qquad \sigma_f(\epsilon) = -(1 - f(\epsilon)) \ln(1 - f(\epsilon)) - f(\epsilon) \ln f(\epsilon) \,,$$

pressure

$$P = u - Ts - \mu n = \int_0^{\infty} \mathrm{d}\epsilon D(\epsilon) \left[\epsilon - T\sigma_f(\epsilon) - \mu\right]$$

Low temperature expansion. The Sommerfeld's integrals

$$\int_0^\infty \frac{H(\epsilon)}{e^{\frac{\epsilon-\mu}{T}}+1}\mathrm{d}\epsilon = \int_0^\mu H(\epsilon)\mathrm{d}\epsilon + \frac{\pi^2}{6}T^2H'(\mu) + \frac{7(\pi T)^4}{360}H'''(\mu)$$



Chemical potential

$$\mu \approx \epsilon_\mathrm{F}\left(1-\frac{\pi^2}{12}\frac{T^2}{\epsilon_\mathrm{F}^2}\right) + O(T^4/\epsilon_\mathrm{F}^4)$$

Energy density

$$u \approx \frac{3}{5}n\epsilon_\mathrm{F}\left[1+5\frac{\pi^2}{12}\frac{T^2}{\epsilon_\mathrm{F}^2}\right]$$

Pressure

$$p = \frac{2}{3}u$$

Heat capacity of the fermion gas

$$C_V = \left.\frac{\partial u}{\partial T}\right|_V = \frac{\pi^2}{2}\frac{T}{\epsilon_\mathrm{F}}n = \frac{mp_\mathrm{F}}{3}T$$

Heat capacity of the fermion gas $$C_V = \left.\frac{\partial u}{\partial T}\right|_V = \frac{\pi^2}{2}\frac{T}{\epsilon_F}n = \frac{mp_F}{3}T$$

Heat capacity in metals has also a phonon contribution (oscillations of the ion lattice) $$C_V = \gamma T + AT^3$$

$$\frac{C_V}{T} = \gamma + AT^2$$

In order to separate out these two contributions it has become the practice to plot C_V/T against T^2

One finds γ by extrapolating the C_V/T curve linearly down to $T^2 = 0$, and noting where it intercepts the C_V/T-*axis*

SOME ROUGH EXPERIMENTAL VALUES FOR THE COEFFICIENT OF THE LINEAR TERM IN T OF THE MOLAR SPECIFIC HEATS OF METALS, AND THE VALUES GIVEN BY SIMPLE FREE ELECTRON THEORY

ELEMENT	FREE ELECTRON γ (in 10^{-4} cal-mole^{-1}-K^{-2})	MEASURED γ	RATIO* (m^*/m)
Li	1.8	4.2	2.3
Na	2.6	3.5	1.3
K	4.0	4.7	1.2
Rb	4.6	5.8	1.3
Cs	5.3	7.7	1.5
Cu	1.2	1.6	1.3
Ag	1.5	1.6	1.1
Au	1.5	1.6	1.1
Be	1.2	0.5	0.42
Mg	2.4	3.2	1.3
Ca	3.6	6.5	1.8
Sr	4.3	8.7	2.0
Ba	4.7	6.5	1.4
Nb	1.6	20	12
Fe	1.5	12	8.0
Mn	1.5	40	27
Zn	1.8	1.4	0.78
Cd	2.3	1.7	0.74
Hg	2.4	5.0	2.1
Al	2.2	3.0	1.4
Ga	2.4	1.5	0.62
In	2.9	4.3	1.5
Tl	3.1	3.5	1.1
Sn	3.3	4.4	1.3

In-medium “effective mass” of the electron can strongly deviate from the vacuum one!

Fermi liquids. The role of interactions

For real fermion systems, the particle interaction and the exclusion principle act simultaneously.

We consider degenerate Fermi liquids in which both effects are important.

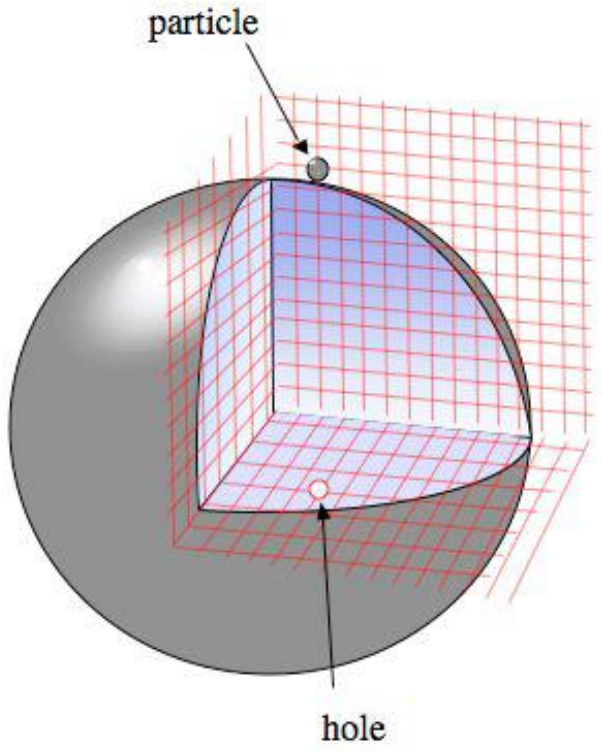
In some systems, the nature of the degenerate gas is drastically modified by the particle interactions. Such is the case, for instance, in a superconducting electron gas.

Frequently, the interacting liquid retains many properties of the gas: it is then said to be **normal.**

A normal Fermi liquid at T = 0 has a sharply defined Fermi surface S_F

Its elementary excitations may be pictured as *quasiparticles* outside S_F and *quasiholes* inside S_F in close analogy with the single-particle excitations of a noninteracting Fermi gas.

Such a resemblance explains why so many properties of the liquid can be interpreted in terms of a "one-particle approximation."



free fermion system

Excitation in Fermi liquids

Let us now turn to the case of an interacting Fermi liquid. We are interested in the nature of its elementary excitations.

A "frontal“ attack on the problem involves the introduction of Green's functions, and the mathematical apparatus of many-body perturbation theory.

We start with an alternative approach, which consists in comparing the interacting "real" liquid with the noninteracting "ideal" gas; we establish a one-to-one correspondence between the eigenstates of the two systems.

Consider an eigenstate of the ideal system, characterized by a distribution function n_p. In order to establish a connection with the real system, we imagine that the interaction between the particles is switched on infinitely slowly. Under such "adiabatic" conditions, the ideal eigenstates will progressively transform into certain eigenstates of the real interacting system.

However, there is no a priori reason why such a procedure should generate ***all*** real eigenstates. For instance, it may well happen that the real ground state may not be obtained in that way (superconductors!)
We ***assume*** that the real ground state may be adiabatically generated starting from some ideal eigenstate with a distribution $n^0_{\boldsymbol{p}}$.

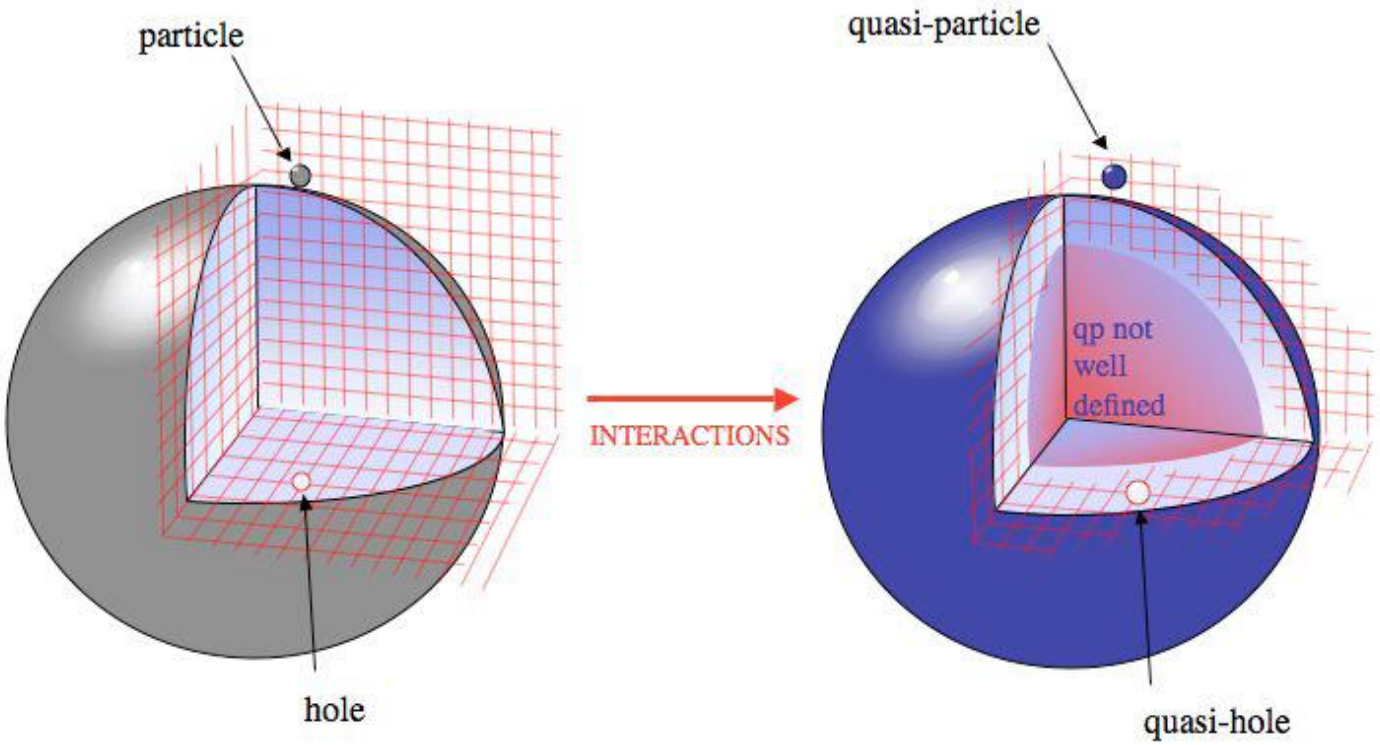
This is the definition of a *normal* fermion system.

For reasons of symmetry, the distribution $n^0_{\boldsymbol{p}}$ of an isotropic system is ***spherical***. As a result, the spherical Fermi surface is not changed when the interaction between particles is switched on: the real ground state is generated adiabatically from the ideal ground state.

Let us now *add* a particle with momentum $\boldsymbol{p}$ to the ideal distribution $n^0_{\boldsymbol{p}}$ and, again, turn on the interaction between the particles adiabatically. We generate an *excited state* of the real liquid, which likewise has momentum $\boldsymbol{p}$, since momentum is conserved in particle collisions.

As the interaction is increased, we may picture the bare particle as slowly perturbing the particles in its vicinity; if the change in interaction proceeds sufficiently slowly, the entire system of N + 1 particles will remain in equilibrium.

Once the interaction is completely turned on, we find that our particle moves together with the surrounding particle distortion brought about by the interaction. In the language of field theory, we would say that the particle is *"dressed" with a self-energy cloud*. We shall consider the "dressed" particle as an independent entity, which we call a quasipartide.



The above excited state corresponds to the real ground state plus a quasiparticle of momentum $\boldsymbol{p}$.

Let S_F be the Fermi surface characterizing the unperturbed distribution $n^0_{\boldsymbol{p}}$ from which the real ground state is built up. Because of the exclusion principle, quasiparticle excitations can be generated only if their momentum $\boldsymbol{p}$ lies outside S_F. The quasiparticle distribution in $\boldsymbol{p}$ space is sharply bounded by the Fermi surface S_F.

Using the same adiabatic switching procedure, we can define a quasihole, with a momentum **p** lying inside the Fermi surface SF; we may do likewise for higher configurations involving several excited quasiparticles and quasiholes.

The quasiparticles and quasiholes thus appear as elementary excitations of the real system which, when combined, give rise to a large class of excited states. We have established our desired one-to-one correspondence between ideal and real eigenstates.

Landau's Fermi-liquid approach



interacting fermions **system of quasiparticles**

quantized excitations in the system

quasi-particles =/= original "bare" fermions [constituents of the system]

Landau wrote the Boltzmann equation for q.p distribution function: $n(\boldsymbol{x}, \boldsymbol{p}, t)$

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \dot{\boldsymbol{x}} \frac{\partial n}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}} \frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

equations of motion for q.p.

$$\dot{\boldsymbol{x}} = \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{x}}$$

"generalized" velocity Newton's law

$$\frac{\partial n}{\partial t} + \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{p}} \frac{\partial n}{\partial \boldsymbol{x}} - \frac{\partial \epsilon(\boldsymbol{p}, \boldsymbol{x})}{\partial \boldsymbol{x}} \frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

[G.E. Brown, RMP 43, 1]

"I am indebted to Professor I. M. Khalatniltov for explaining this (Landau's reasoning) to me."

$$\boldsymbol{\mathcal{F}}(\boldsymbol{x},t) = \int \boldsymbol{p}\, n(t,\boldsymbol{x},\boldsymbol{p})\, \frac{d^3p}{(2\pi)^3}$$

momentum flux density

Aim is to obtain conservation of total quasiparticle momentum

$$\frac{\partial n}{\partial t} + \frac{\partial \epsilon(\boldsymbol{p},\boldsymbol{x})}{\partial \boldsymbol{p}}\frac{\partial n}{\partial \boldsymbol{x}} - \frac{\partial \epsilon(\boldsymbol{p},\boldsymbol{x})}{\partial \boldsymbol{x}}\frac{\partial n}{\partial \boldsymbol{p}} = I(n)$$

$$\int \boldsymbol{p}_i\, I(n)\, \frac{d^3p}{(2\pi)^3} = 0$$ momentum conservation in collisions

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial t} &= \int \boldsymbol{p}_i \frac{\partial n}{\partial t}\frac{d^3p}{(2\pi)^3} = \int \boldsymbol{p}_i\, I(n)\, \frac{d^3p}{(2\pi)^3} - \int \boldsymbol{p}_i \left[\frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{\partial n}{\partial \boldsymbol{x}_j} - \frac{\partial \epsilon}{\partial \boldsymbol{x}_j}\frac{\partial n}{\partial \boldsymbol{p}_j}\right]\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n\, \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} + \int \boldsymbol{p}_i\, n\, \frac{\partial^2 \epsilon}{\partial \boldsymbol{x}_j \partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \int n\, \frac{\partial \boldsymbol{p}_i}{\partial \boldsymbol{p}_j}\frac{\partial \epsilon}{\partial \boldsymbol{x}_j}\frac{d^3p}{(2\pi)^3} - \int n\, \boldsymbol{p}_i \frac{\partial \epsilon}{\partial^2 \boldsymbol{p}_j \partial \boldsymbol{x}_j}\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n\, \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \int n\, \frac{\partial \epsilon}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\int \boldsymbol{p}_i\, n\, \frac{\partial \epsilon}{\partial \boldsymbol{p}_j}\frac{d^3p}{(2\pi)^3} - \frac{\partial}{\partial \boldsymbol{x}_i}\int n\, \epsilon \frac{d^3p}{(2\pi)^3} + \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3} \\
&= -\frac{\partial}{\partial \boldsymbol{x}_j}\boldsymbol{\Pi}^{ij} + \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3}
\end{aligned}$$

momentum flux tensor $\boldsymbol{\Pi}^{ij} = \int n \left(\boldsymbol{p}_i \frac{\partial \epsilon}{\partial \boldsymbol{p}_j} + \delta_{ij}\epsilon\right)\frac{d^3p}{(2\pi)^3}$

[G.E. Brown, RMP 43, 1]

momentum conservation

$$0 = \frac{\partial}{\partial t}\int \boldsymbol{\mathcal{F}}_i d^3x = -\int \frac{\partial}{\partial x_j}\boldsymbol{\Pi}^{ij}\, d^3x + \int d^3x \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}_i}\frac{d^3p}{(2\pi)^3}$$

Assuming $\boldsymbol{\Pi}$ to be zero on the surfaces of the spatial volume integrated over

$$\Rightarrow \quad \int d^3x \int \epsilon \frac{\partial n}{\partial \boldsymbol{x}} \frac{d^3p}{(2\pi)^3} = 0$$

$$\int \epsilon \frac{\partial n}{\partial \boldsymbol{x}} \frac{d^3p}{(2\pi)^3} = \frac{\partial}{\partial \boldsymbol{x}} E$$

$$\delta E = \int \epsilon\, \delta n \frac{d^3p}{(2\pi)^3}$$

$$\frac{\delta E}{\delta n(\boldsymbol{p})} = \epsilon(\boldsymbol{p})$$

if the $\partial/\partial \boldsymbol{x}_j$ can be taken outside the integral, in other words, if $e(p)\delta n(p)$ is the differential of some quantity E

The quasiparticle energy $\bar{e}(p)$ is obtained by varying the energy with respect to quasiparticle number. We see that Landau was led to this assumption by his construction of conservation laws for the quasiparticle momentum.

[G.E. Brown, RMP 43, 1]

E = energy density of the system

single particle mechanism of excitation!

Now the energy of the system is a **functional of occupation number** of all of the quasiparticles. $E = E\{n_{\boldsymbol{p}_1,s_1}, n_{\boldsymbol{p}_2,s_2}, \dots\}$

(states are quantified by their momentum $\boldsymbol{p}$ and spin s)

If one varies many of the *n(p)* away from their equilibrium value by, e.g., exciting a collective excitation, then the resulting energy is a functional $E' = E'\{\delta n_{\boldsymbol{p}_1),s_1}, \delta n_{\boldsymbol{p}_2,s_2}, \dots\}$

The quasiparticle energy $\epsilon_{\boldsymbol{p},s} = \epsilon_{\boldsymbol{p},s}\{n_{\boldsymbol{p},s}\}$, which is a functional of the quasiparticle distributions, is defined as the variaton of the system energy with respect to $n_{\boldsymbol{p},s}$.

$$\delta E = E' - E = \frac{1}{V}\sum_{\boldsymbol{p},s} \epsilon_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s}$$

A variation of the distribution function produces a variation of the quasiparticle energy given by

$$\delta\epsilon_{\boldsymbol{p},s} = \frac{1}{V}\sum_{\boldsymbol{p}',s'} f_{\boldsymbol{p}s,\boldsymbol{p}'s'}\delta n_{\boldsymbol{p}',s'}$$

$f_{\boldsymbol{p}s,\boldsymbol{p}'s'}$ quantifies the interaction between quasiparticles. It can be viewd as the second variation of the total energy $f_{\boldsymbol{p}s,\boldsymbol{p}'s'} = V^2\dfrac{\delta^2 E}{\delta n_{\boldsymbol{p},s}\delta n_{\boldsymbol{p}',s'}}$. Of course, the quasiparticle interaction energy f is itself a functional of the distribution function. Usually one assumes that f is evaluated for the equilibrium distributions $f\{n^{(0)}_{\boldsymbol{p}'',s''}\}$.

The variation of the energy due to a variation, $\delta n_{\boldsymbol{p},s}$, of the distribution function from its ground state form can be written as

$$E = E_0 + \frac{1}{V}\sum_{\boldsymbol{p}s} \epsilon^{(0)}_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} + \frac{1}{2}\frac{1}{V^2}\sum_{\boldsymbol{p}s,\boldsymbol{p}'s'} f_{\boldsymbol{p}s,\boldsymbol{p}'s'}\delta n_{\boldsymbol{p},s}\delta n_{\boldsymbol{p}',s'} \qquad \epsilon^{(0)}_{\boldsymbol{p}s} = \epsilon_{\boldsymbol{p}s}\{n^{(0)}_{\boldsymbol{p}'s'}\}$$

Distribution function for quasiparticles $n_{\boldsymbol{p},s}$

For any variation about thermodynamic equilibrium at finite temperature $\delta E = T\delta s + \mu\delta n$ V is constant

Particle density $n = \frac{1}{V}\sum_{\boldsymbol{p},s} n_{\boldsymbol{p},s}$ Entropy density $s = -\frac{1}{V}\sum_{\boldsymbol{p}s}\left\{n_{\boldsymbol{p}s}\ln n_{\boldsymbol{p}s} + (1-n_{\boldsymbol{p}s})\ln(1-n_{\boldsymbol{p}s})\right\}$

$$\delta n = \frac{1}{V}\sum_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} \qquad \delta s = -\frac{1}{V}\sum_{\boldsymbol{p}s}\delta n_{\boldsymbol{p}s}\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}}$$

$$\delta E = T\delta s + \mu\delta n \Rightarrow \frac{1}{V}\sum_{\boldsymbol{p},s}\epsilon_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s} = -\frac{T}{V}\sum_{\boldsymbol{p}s}\delta n_{\boldsymbol{p}s}\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} + \frac{\mu}{V}\sum_{\boldsymbol{p},s}\delta n_{\boldsymbol{p},s}$$

$$\Rightarrow \frac{1}{V}\sum_{\boldsymbol{p},s}\left[\epsilon_{\boldsymbol{p},s} + T\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} - \mu\right]\delta n_{\boldsymbol{p},s} = 0 \Rightarrow \epsilon_{\boldsymbol{p},s} + T\ln\frac{n_{\boldsymbol{p}s}}{1-n_{\boldsymbol{p}s}} - \mu = 0$$

$$\Rightarrow n_{\boldsymbol{p}s} = \frac{1}{e^{\frac{\epsilon_{\boldsymbol{p}s}-\mu}{T}}+1}$$

This is a complicated implicit equation for $n_{\boldsymbol{p}s}$

For $T = 0$ $\quad n^{(0)}_{\boldsymbol{p}s} = \begin{cases} 1 & \epsilon^{(0)}_{\boldsymbol{p}s} < \mu \\ 0 & \epsilon^{(0)}_{\boldsymbol{p}s} > \mu \end{cases}$

The Fermi momentum p_{F} at which $\epsilon^{(0)}_{\boldsymbol{p}_{\mathrm{F}}s} = \mu$

Quasiparticle energy $\epsilon^{(0)}_{\boldsymbol{p}s} = \epsilon_{\boldsymbol{p}s}\{n^{(0)}_{\boldsymbol{p}'s'}\} \neq \dfrac{\boldsymbol{p}^2}{2m}$ Some complicated function of p

Close to the Fermi surface $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \left(\dfrac{\partial \epsilon_{\boldsymbol{p}s}}{\partial p}\right)_{p=p_\mathrm{F}} (p - p_\mathrm{F})$ Fermi momentum is related to the density $\dfrac{p_\mathrm{F}^3}{3\pi^2} = n$

Fermi velocity $\left(\dfrac{\partial \epsilon_{\boldsymbol{p}s}}{\partial p}\right)_{p=p_\mathrm{F}} = v_\mathrm{F} = \dfrac{p_\mathrm{F}}{m^*}$ Effective mass of the quasiparticle $\dfrac{1}{m^*} = \left(\dfrac{\partial^2 \epsilon_{\boldsymbol{p}s}}{\partial p^2}\right)_{p=p_\mathrm{F}}$

➡ $p \sim p_\mathrm{F}$ $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \dfrac{p_\mathrm{F}}{m^*}(p - p_\mathrm{F})$ with the same precision O(p-p_F) $\epsilon^{(0)}_{\boldsymbol{p}s} \approx \mu + \dfrac{p^2 - p_\mathrm{F}^2}{2m^*}$

Density of states $D(\epsilon) = \dfrac{1}{V}\sum_{\boldsymbol{p}s} \delta\left(\epsilon_{\boldsymbol{p},s} - \epsilon\right)$

at the Fermi surface $N_0 = D(\mu) = \dfrac{1}{V}\sum_{\boldsymbol{p}s} \delta\left(\epsilon_{\boldsymbol{p},s} - \mu\right) = \dfrac{m^* p_\mathrm{F}}{\pi^2}$ ➡ Heat capacity $C_V = \dfrac{m^* p_\mathrm{F}}{3} T$

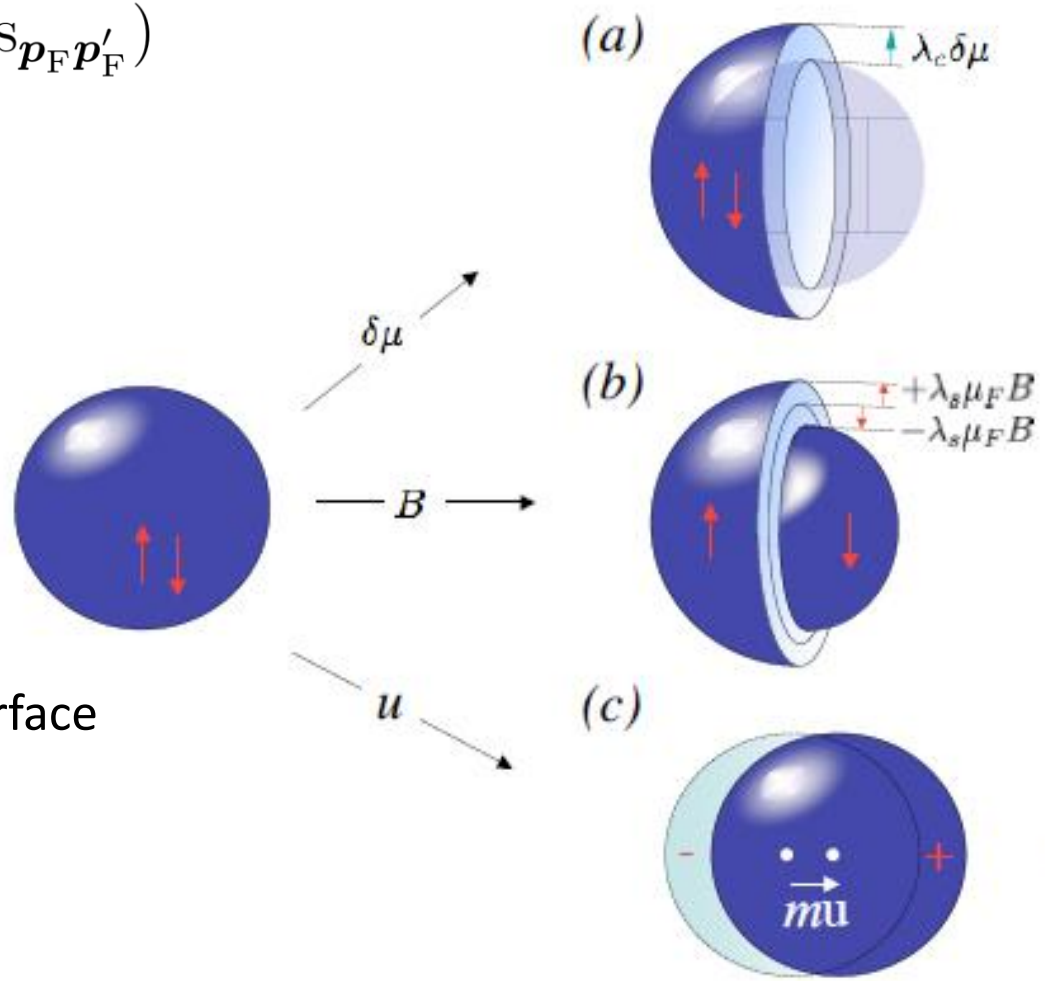
Landau parameters

$$f_{\boldsymbol{p}s,\boldsymbol{p}'s'} = f^s(\boldsymbol{p}_\mathrm{F},\boldsymbol{p}'_\mathrm{F})\sigma_0\sigma'_0 + f^a(\boldsymbol{p}_\mathrm{F},\boldsymbol{p}'_\mathrm{F})(\boldsymbol{\sigma}\boldsymbol{\sigma}')$$

There are relations between some Landau parameters and bulk properties of the system

$$f^{s,a} = \sum_l f_l^{s,a} P_l(\cos_{\boldsymbol{p}_\mathrm{F}\boldsymbol{p}'_\mathrm{F}})$$

$$F_l^{s,a} = N_0 f_l^{s,a}$$



F_0^s compressibility $\dfrac{\partial^2 E}{\partial n^2}$

F_0^a Spin susceptibility

F_1^s Effective mass

Stability of the Fermi surface (Pomeranchuk criterion)

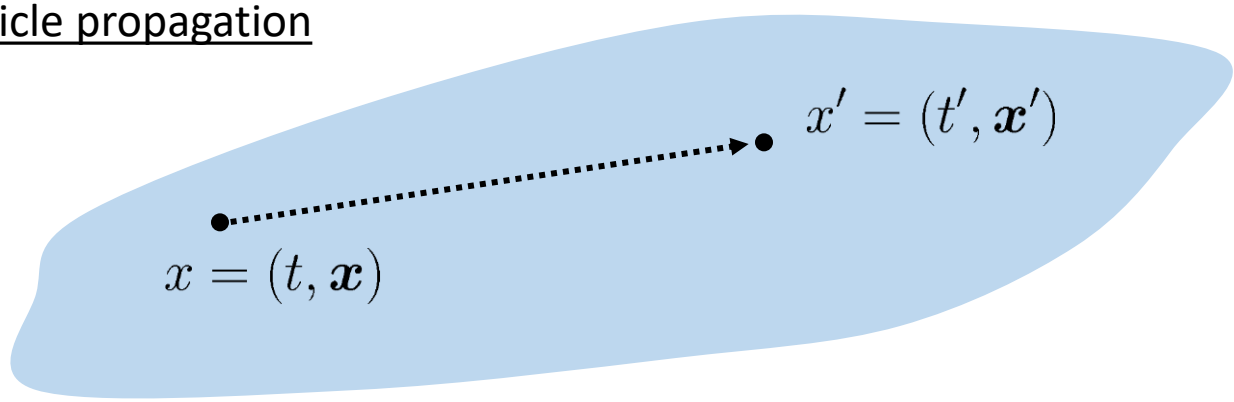
$$F_l^{s,a} > -2l - 1$$

Green's functions

N-body system: wave function of the whole system $\Psi(x_1,x_2,\ldots,x_N)$
encodes the dynamics of all particles and is very complicated

Introduce the object which describes the dynamics of the reduced number of particles of interest

<u>one-particle propagation</u>



Amplitude of particle transition from a point (x,t) to a point (x',t')

$$\Psi(\boldsymbol{x}', t') = \int d\boldsymbol{x}\, G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t)\Psi(\boldsymbol{x}, t) \qquad t' > t$$

$$\text{for } t' = t + 0 \qquad \Psi(\boldsymbol{x}', t+0) = \int d\boldsymbol{x}\, G^{(+)}(\boldsymbol{x}', t+0; \boldsymbol{x}, t)\Psi(\boldsymbol{x}, t)$$

$$G^{(+)}(\boldsymbol{x}', t+0; \boldsymbol{x}, t) = \delta(\boldsymbol{x}' - \boldsymbol{x})$$

If $\Psi(\boldsymbol{x},t)$ obeys the Schrödinger equation $\quad [i\partial_t - H(\boldsymbol{x})]\,\Psi(\boldsymbol{x}, t) = 0$

$$[i\partial_t - H(\boldsymbol{x})]\, G^{(+)}(\boldsymbol{x}, t; \boldsymbol{x}', t') = i\,\delta(t - t')\,\delta(\boldsymbol{x} - \boldsymbol{x}')$$

for homogeneous system : $\quad G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t) = G^{(+)}((\boldsymbol{x}' - \boldsymbol{x})^2, t' - t > 0)$

eigenfunctions: $\quad H\,\varphi_\lambda(\boldsymbol{x}) = \epsilon_\lambda(\boldsymbol{x})\,\varphi(\boldsymbol{x})$

$$G^{(+)}(\boldsymbol{x}', \boldsymbol{x}, \tau = t' - t) = -\sum_\lambda \int_{-\infty}^{\infty} \frac{d\epsilon}{2\,\pi\,i}\, e^{-i\epsilon\,\tau} \frac{\varphi_\lambda(\boldsymbol{x}')\,\varphi_\lambda^*(\boldsymbol{x})}{\epsilon - \epsilon_\lambda + i\,0}$$

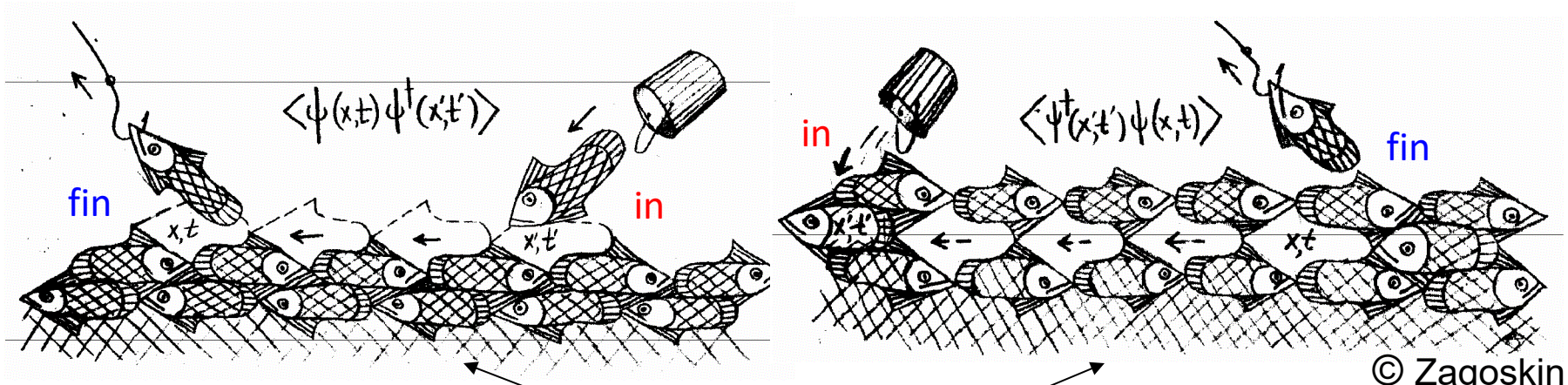
$$G^{(+)}(\boldsymbol{x}', t'; \boldsymbol{x}, t) = < N|\hat{\Psi}(\boldsymbol{x}', t')\,\hat{\Psi}^\dagger(\boldsymbol{x}, t)|N >$$

$$\hat{\Psi}(\boldsymbol{x}, t) = \sum_\lambda \varphi_\lambda(x)\,\hat{a}_\lambda\, e^{-i\,\epsilon_\lambda\, t} \qquad |N >= a_1^\dagger\, a_2^\dagger\, a_3^\dagger \ldots\, a_N^\dagger\, |0>$$

$a_i,\ a_i^\dagger$ annihilation and creation operator

Green's function of non-interacting fermions

$$i\,G(\boldsymbol{x},t;\boldsymbol{x}',t') = <N|T\{\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^\dagger(\boldsymbol{x}',t')\}|N> \qquad \text{fin} \quad \text{in}$$
$$= <N|\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^\dagger(\boldsymbol{x}',t')|N>\,\theta_{t-t'} - <N|\hat{\Psi}^\dagger(\boldsymbol{x}',t')\,\hat{\Psi}(\boldsymbol{x},t)|N>\,\theta_{t'-t}$$



© Zagoskin

no interaction

$$G(\boldsymbol{x},t;\boldsymbol{x}',t') = G(t-t',\boldsymbol{x}-\boldsymbol{x}')$$

$$G_0(\epsilon,p) = \frac{1-n_p}{\epsilon-\epsilon_p+i\,0} + \frac{n_p}{\epsilon+\epsilon_p^h-i\,0}$$

$$T=0$$

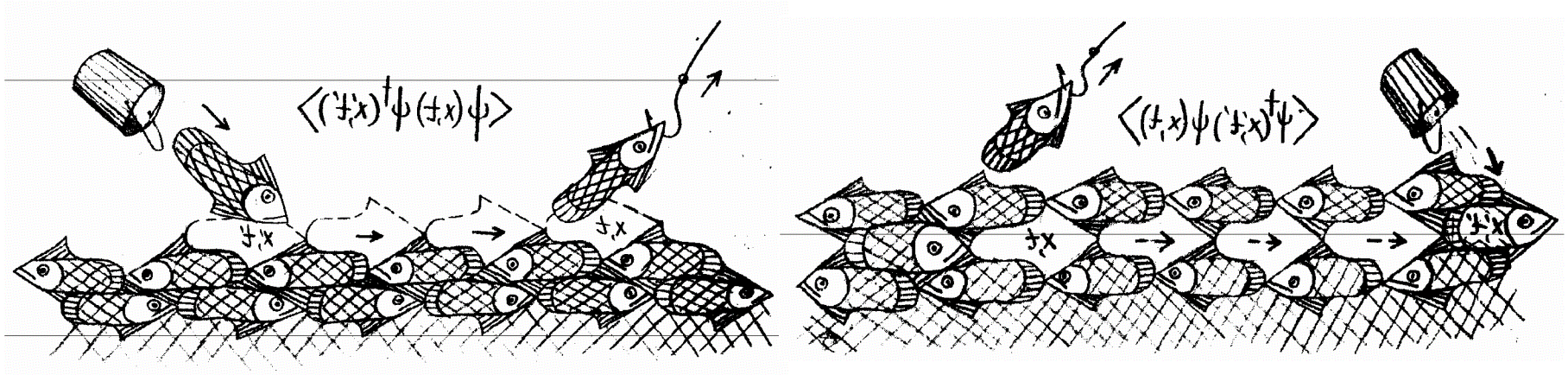
$$n_p = \theta(p_\mathrm{F}-p)$$
$$\epsilon_p^h = -\epsilon_p$$
$$\epsilon_p = \frac{p^2-p_\mathrm{F}^2}{2\,m}$$

$$G(t,\boldsymbol{x}) = \int \frac{\mathrm{d}\epsilon\mathrm{d}^2p}{(2\pi)^4} G(\epsilon,\boldsymbol{p}) e^{-i\epsilon t+i\boldsymbol{p}\boldsymbol{x}}$$

$$G_0(\epsilon,\boldsymbol{p}) = \frac{1}{\epsilon-\epsilon_p+i0\,\mathrm{sign}(\epsilon)}$$

particle

$G(\varepsilon,p)$



hole

$G^h(\varepsilon,p)=G(-\varepsilon,p)$

Momentum distribution of particles

$$G(\boldsymbol{x},t;\boldsymbol{x}',t') = -i < N|T\{\hat{\Psi}(\boldsymbol{x},t)\,\hat{\Psi}^{\dagger}(\boldsymbol{x}',t')\}|N >= G(t-t',\boldsymbol{x}-\boldsymbol{x}')$$

$t' = t + 0$ $\qquad G(-0,\boldsymbol{x}-\boldsymbol{x}') = i < N|\hat{\Psi}^{\dagger}(\boldsymbol{x}',t+0)\hat{\Psi}(\boldsymbol{x},t)\,|N >$

$$G(0,-0) = i < N|\hat{\Psi}^{\dagger}(\boldsymbol{x},t+0)\hat{\Psi}(\boldsymbol{x},t)\,|N >= i\frac{N}{V}$$

particle density in the homogenous system

Particle momentum distribution $\qquad n(\boldsymbol{p}) = \int (-i)G(t\to -0,\boldsymbol{x})e^{-i\boldsymbol{x}\boldsymbol{p}}\mathrm{d}^3\boldsymbol{x}$

$$n(\boldsymbol{p}) = \int (-i)G(t\to -0,\boldsymbol{x})e^{-i\boldsymbol{x}\boldsymbol{p}}\mathrm{d}^3\boldsymbol{x} = \lim_{t\to -0}\int G(\epsilon,\boldsymbol{p})e^{-i\epsilon t}\frac{\mathrm{d}\epsilon}{2\pi i}$$

Free Green's function $\qquad n(\boldsymbol{p}) = \lim_{t\to -0}\int \frac{1}{\epsilon - \frac{p^2-p_{\mathrm{F}}^2}{2\,m} + i\,0\mathrm{sign}(\epsilon)}e^{-i\epsilon t}\frac{\mathrm{d}\epsilon}{2\pi i}$

$$n(\boldsymbol{p}) = \theta(p_{\mathrm{F}} - p)$$

for $t < 0$ we have $-i\epsilon t = i\epsilon|t|$

we have to close the contour for $\mathrm{Im}\,\epsilon > 0$
positive direction of the contour

$\mathrm{Im}\,\epsilon = 0(-)\mathrm{sign}(p^2 - p_{\mathrm{F}}^2) > 0$ if $p < p_{\mathrm{F}}$

Diagram technique

Ground state:

$$iG(x,y) = < N|\widehat{T}\{\widehat{\Psi}(x)\widehat{\Psi}^{\dagger}(y)\}|N > =< N|\widehat{S}^{-1}\widehat{T}\{\widehat{\Psi}_I(x)\widehat{\Psi}_I^{\dagger}(y)\}\widehat{S}|N >$$

in interaction picture: $$iG =< N|\widehat{T}\{\widehat{\Psi}_I(x)\widehat{\Psi}_I^{\dagger}(y)\}\widehat{S}|N >< \widehat{S}^{-1} >$$

transition from the ground state to the ground state under action of evolution operator

$$\widehat{S} = \widehat{T}\exp\left\{-i\int_{-\infty}^{\infty}\widehat{V}_I(t)dt\right\}$$

time ordering

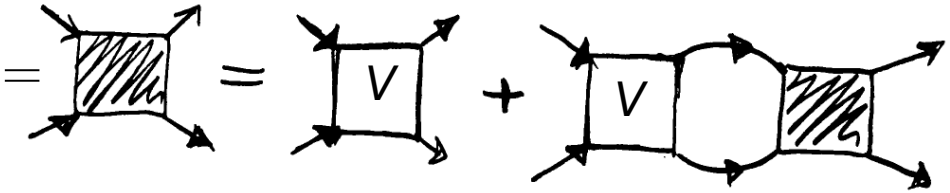
$$\widehat{V}_I(t) = e^{i\widehat{H}_0(\mu)\,t}\widehat{V}\,e^{-i\widehat{H}_0(\mu)t}$$

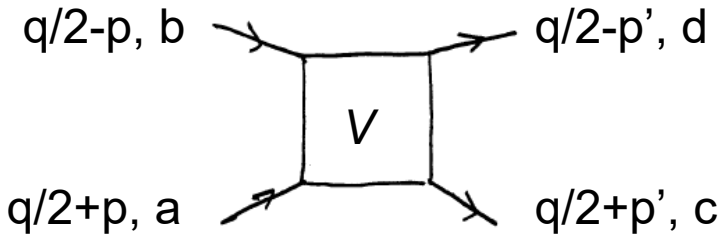
$$\widehat{H}_0(\mu) = H_0 - \sum_a \mu_a \widehat{N}_a$$

Only one type of Green's functions

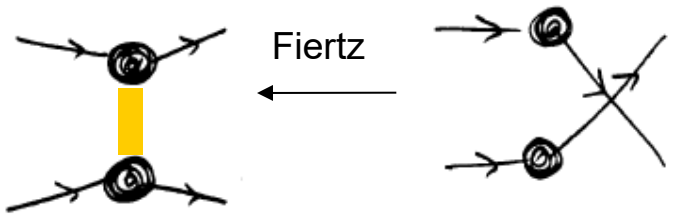


Particle-particle interaction

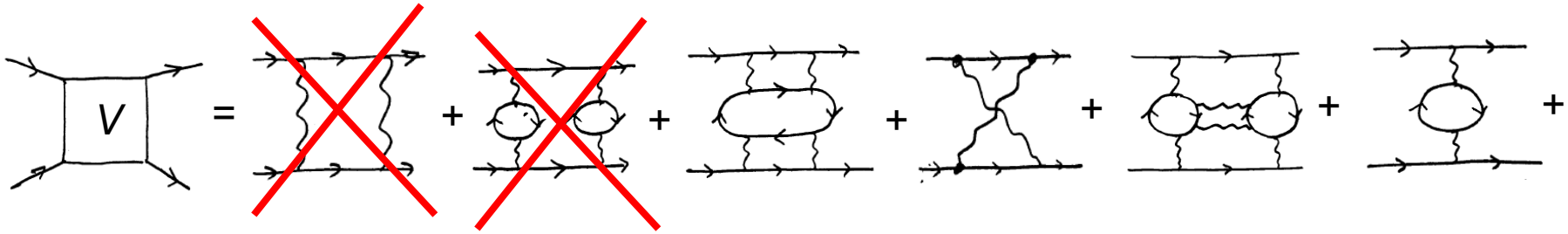
$$-i\,T_{pp}(p,p';q) =$$  $$=$$



$$\left[\widehat{V}(p,p',q)\right]_{cd,ab} = V_0(p,p',q)(i\sigma_2)_{dc}(i\sigma_2)_{ab} + V_1(p,p',q)(\boldsymbol{\sigma} i\sigma_2)_{dc}(i\sigma_2\boldsymbol{\sigma})_{ab}$$



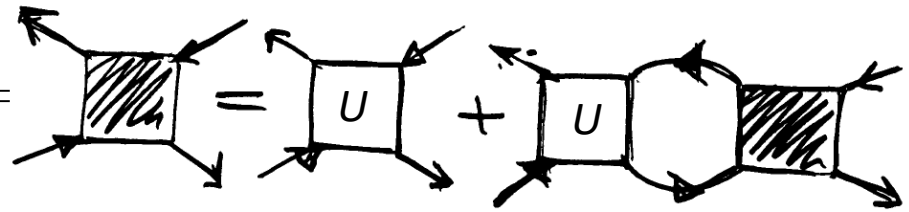
two-particle irreducible interaction

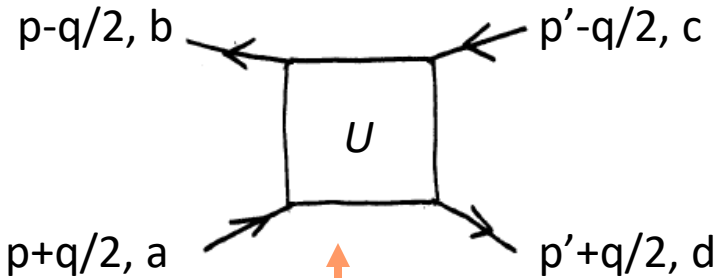


$$\widehat{T}_{\rm pp}(p,p',q) = \widehat{V}(p,p',q) + \int \frac{{\rm d}^4p''}{(2\pi)^4\,i}\widehat{V}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}(q/2-p'')\,\widehat{T}_{\rm pp}(p'',p',q)$$

Particle-particle interaction

$$-i\,T_{\rm ph}(p,p';q) =$$

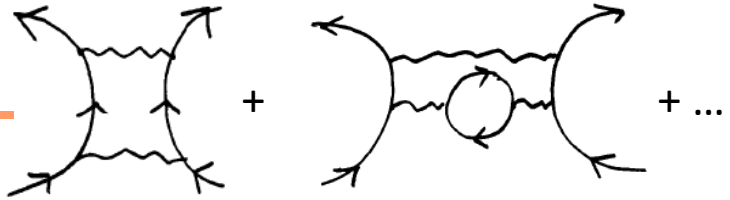




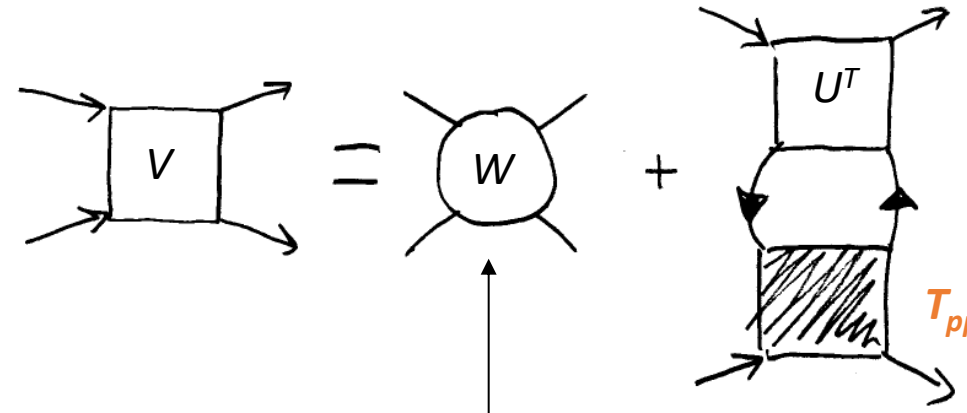
particle-hole irreducible interaction

$$\left[\widehat{U}(p,p',q)\right]^{ac}_{bd} = U_0(p,p',q)\,\delta^a_b\,\delta^c_d + U_1(p,p',q)\,(\boldsymbol{\sigma})^a_b\,(\boldsymbol{\sigma})^c_d$$



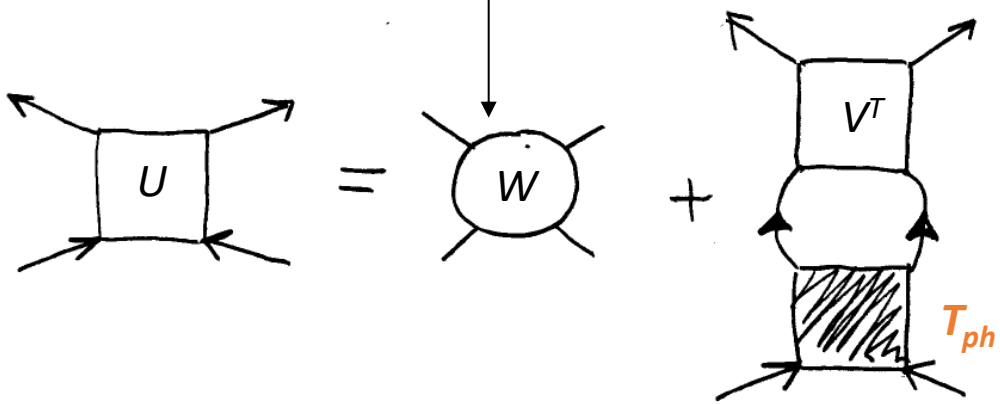


$$\widehat{T}_{\rm ph}(p,p',q) = \widehat{U}(p,p',q) + \int \frac{{\rm d}^4p''}{(2\pi)^4\,i}\widehat{U}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}^h(q/2-p'')\,\widehat{T}_{\rm ph}(p'',p',q)$$



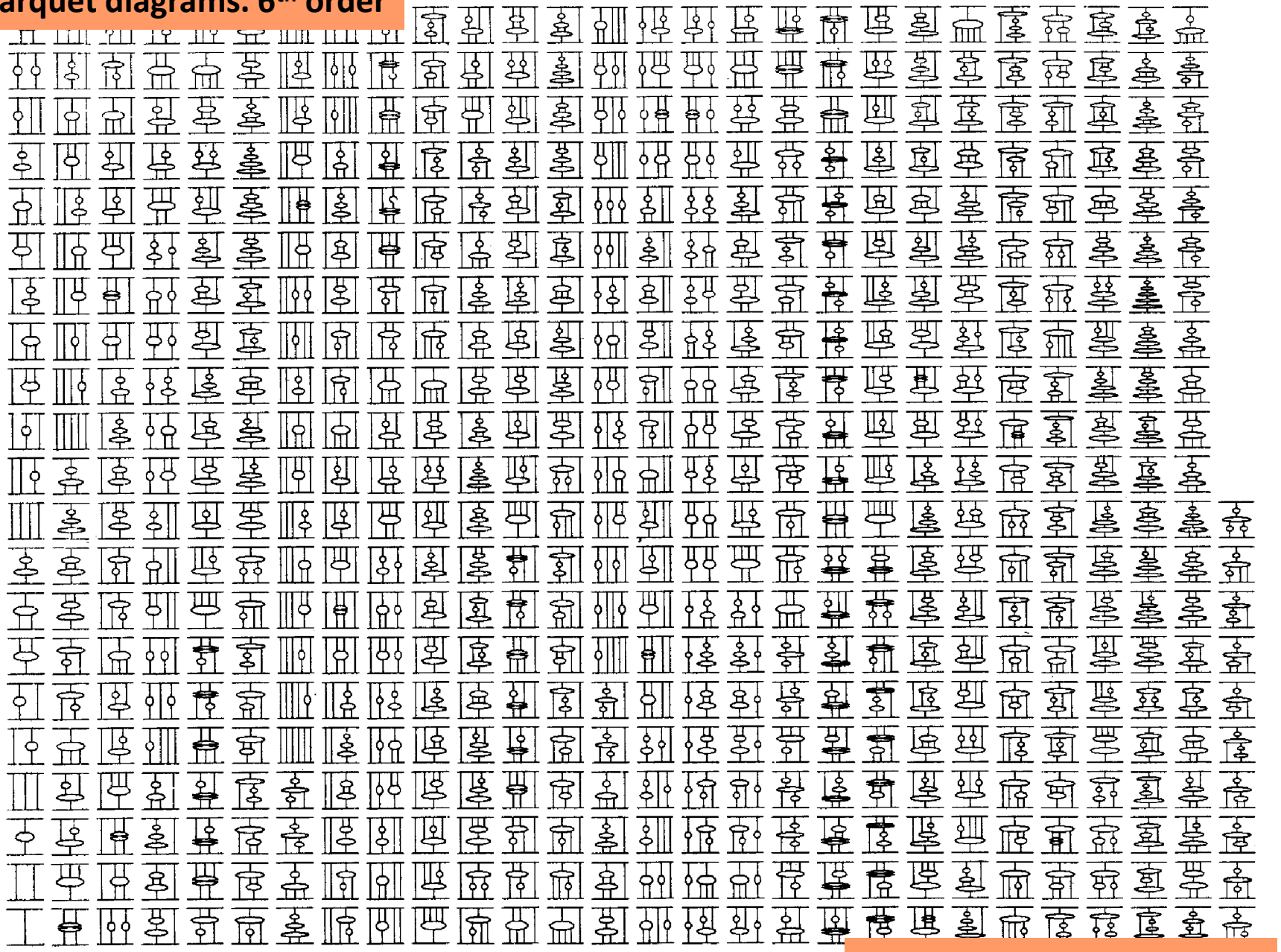
2p-, 2h-irreducible interaction

vacuum NN potential



[Wambach, Ainsworth,Pines NPA555]

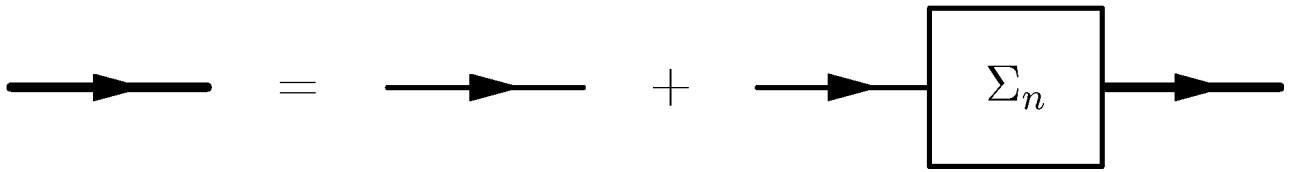
Parquet diagrams. 6th order



Jackson, Lande, Smith PR86

Full Green's function

particle-line

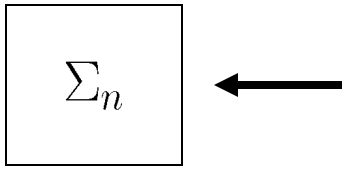


$$\widehat{G}_{\text{n.s.}} = \widehat{G}_0 + \widehat{G}_0\,\widehat{\Sigma}_{\text{n.s.}}\,\widehat{G}_{\text{n.s.}} = \left[[\widehat{G}_0]^{-1} - \widehat{\Sigma}_{\text{n.s.}}\right]^{-1}$$

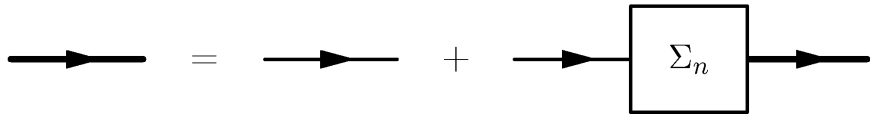
diagonal in spin-space

$$\hat{G}_0(\epsilon, \boldsymbol{p}\,) = \frac{\hat{1}}{\epsilon - \boldsymbol{p}^{\,2}/2\,m_N + i\,0\,\text{sign}\,(\epsilon - \epsilon_{\text{F}})}$$

analogously for the hole-line



particle-particle, particle-hole hole-hole interactions



$$\widehat{G} = \widehat{G}_0 + \widehat{G}_0\,\widehat{\Sigma}\,\widehat{G} = \left[[\widehat{G}_0]^{-1} - \widehat{\Sigma}\right]^{-1}$$

$T \ll \varepsilon_{\mathrm{F},n}, \varepsilon_{\mathrm{F,p}}$ and $\epsilon \sim \epsilon_{\mathrm{F}}\,,\ p \sim p_{\mathrm{F}}$

pole residue $\quad a^{-1} = 1 - \dfrac{\partial}{\partial\epsilon}\Sigma(\epsilon, 0, T)\Big|_{\epsilon\simeq\epsilon_{\mathrm{F}}}$

$$G(\epsilon, \boldsymbol{p}) = \frac{a}{\epsilon - \epsilon_p + i\,\gamma\,\epsilon^2\,\mathrm{sign}\epsilon} + G_{\mathrm{reg}}(\epsilon, \boldsymbol{p})$$

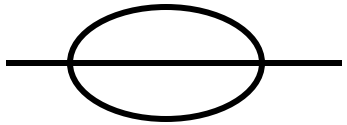
q.p. energy

q.p. width

$$\epsilon_p = \frac{p^2 - 2m_N\mu_N}{2\,m_N^*} \approx \frac{p^2 - p_{\mathrm{F}}^2}{2\,m_N^*} \approx v_{\mathrm{F}}\,(p - p_{\mathrm{F}})$$

$$\gamma = -\lim_{\epsilon\to 0} \mathrm{Im}\,\Sigma^R(\epsilon, p^2 = 2m_N\mu_N, T)/\epsilon^2$$

small for $T \ll T_F$



q.p. effective mass

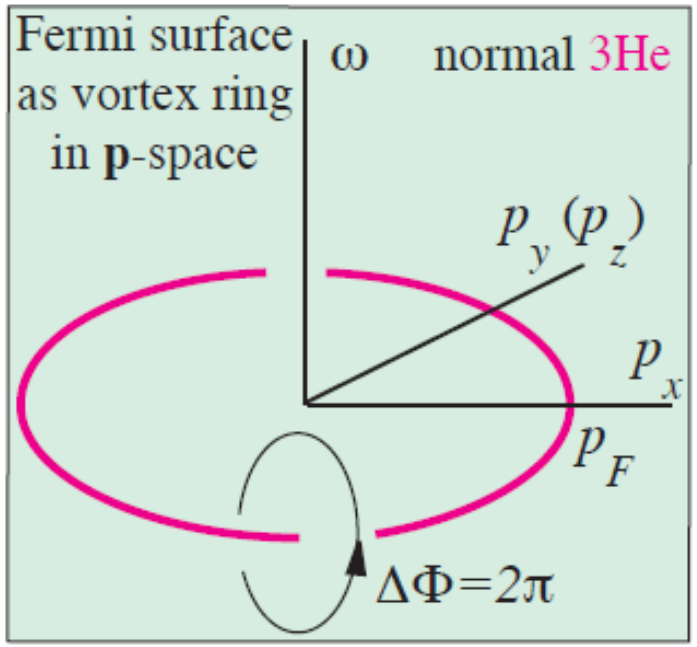
$$\frac{1}{m_N^*} = \frac{a}{m_N} + 2\,a\,\frac{\partial}{\partial p^2}\Sigma(\epsilon, \boldsymbol{p}, T)\Big|_{p=0,\epsilon\simeq\epsilon_{\mathrm{F}}}$$

$G_{\mathrm{reg}}(\epsilon, \boldsymbol{p})$ complicated background part

Fermi surface is a topological object.

ideal gas $$G_0(z = i\omega, p) = \frac{1}{i\omega - v_{\rm F}(p - p_{\rm F})} \qquad v_{\rm F} = p_{\rm F}/m_N$$

In 4D space (ω,p) there is a singularity at (ω=0,p=p_F) [singular hyperline] where this function is not defined!



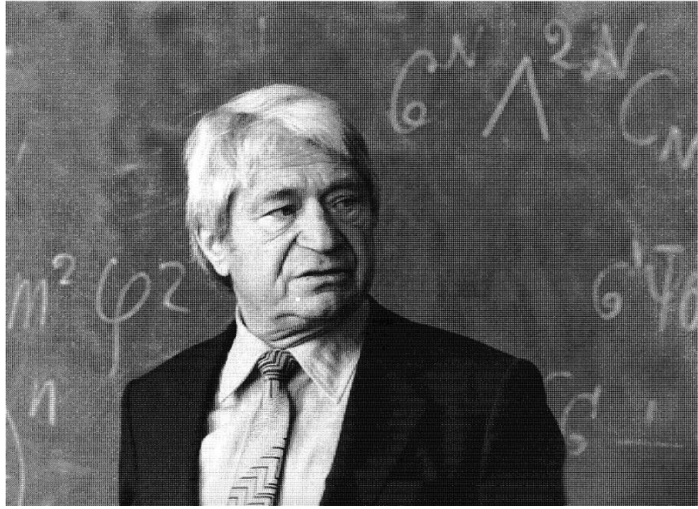
The phase of the Green's function changes by 2p when one goes along a contour encircling this singular line. One can define a topological invariant [see book y G.E. Volovik, The Universe in a helium droplet] The singular-line is topologically protected and thus robust against perturbations

(normal) Fermi liquid $$G(z = i\omega, p) = \frac{a}{i\omega - v_{\rm F}(p - p_{\rm F})} \qquad v_{\rm F} = p_{\rm F}/m^*_N$$

Migdal jump

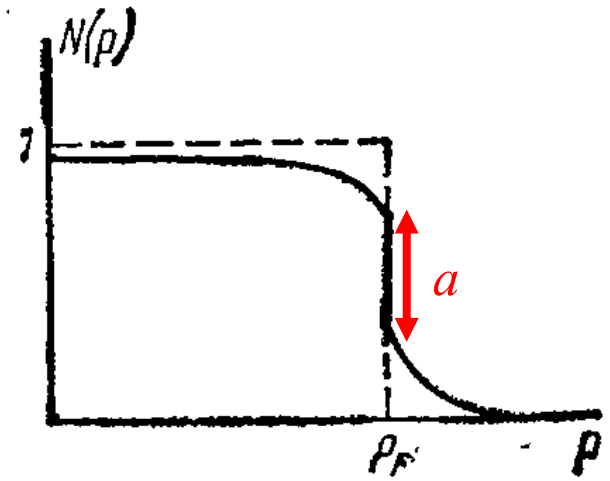
$$G(\epsilon, \boldsymbol{p}) = \frac{a}{\epsilon - v_{\mathrm{F}}\,(p - p_{\mathrm{F}}) + i\,\gamma\,\epsilon^2\,\mathrm{sign}\epsilon} + G_{\mathrm{reg}}(\epsilon, \boldsymbol{p})$$

$$n(\boldsymbol{p}) = \lim_{t\to -0} \int G(\epsilon, \boldsymbol{p}) e^{-i\epsilon t} \frac{\mathrm{d}\epsilon}{2\pi i}$$

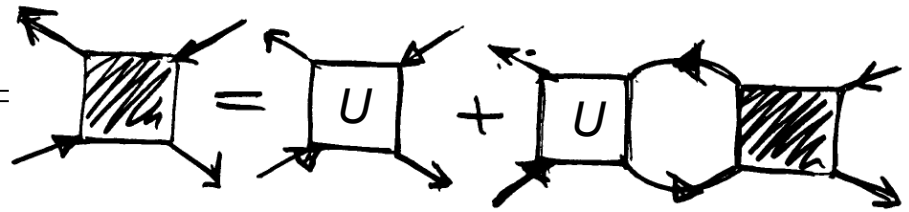


$$\lim_{q\to 0}\left[n(p_{\mathrm{F}} - q) - n(p_{\mathrm{F}} + q)\right] = \lim_{t\to -0} \int \left\{ \frac{a}{\epsilon + v_{\mathrm{F}}\,q - i\,\gamma\,\epsilon^2} - \underbrace{\frac{a}{\epsilon - v_{\mathrm{F}}\,q + i\,\gamma\,\epsilon^2}}_{=0} + \underbrace{G_{\mathrm{reg}}(\epsilon, p - q) - G_{\mathrm{reg}}(\epsilon, p + q)}_{\to 0} \right\} e^{-i\epsilon t} \frac{\mathrm{d}\epsilon}{2\pi i}$$

$$= a$$



Fermi surface exists even for the strongly interacting systems!

$$-i\,T_{\rm ph}(p,p';q) =$$ 

$$\widehat{T}_{\rm ph}(p,p',q) = \widehat{U}(p,p',q) + \int \frac{{\rm d}^4p''}{(2\pi)^4\,i}\widehat{U}(p,p'',q)\,\widehat{G}(q/2+p'')\,\widehat{G}^h(q/2-p'')\,\widehat{T}_{\rm ph}(p'',p',q)$$

$$G(q/2+p)\,G^h(q/2-p) = G(q/2+p)\,G(p-q/2)$$

$$= \frac{a}{[\epsilon+\omega/2-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+i\,0\,{\rm sign}\,(\epsilon+\omega/2)]}\,\frac{a}{[\epsilon-\omega/2-\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0\,{\rm sign}\,(\epsilon-\omega/2)]} + \tilde{B}(p,q)$$

$$\simeq a^2\,\delta(\epsilon)\int d\epsilon\,\frac{1}{[\epsilon+\omega/2-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+i\,0\,{\rm sign}\,(\epsilon+\omega/2)]}\,\frac{1}{[\epsilon-\omega/2-\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0\,{\rm sign}\,(\epsilon-\omega/2)]} + B(p,q)$$

$$= -2\pi i\,a^2\,\delta(\epsilon)\frac{f(\boldsymbol{p}+\boldsymbol{q}/2)-f(\boldsymbol{p}-\boldsymbol{q}/2)}{\omega-\epsilon_{\boldsymbol{p}+\boldsymbol{q}/2}+\epsilon_{\boldsymbol{p}-\boldsymbol{q}/2}+i\,0} + B(p,q)$$

$$p\sim p_{\rm F}$$

Fermi liquid approximation

particle-hole propagator .... for $q \to 0$

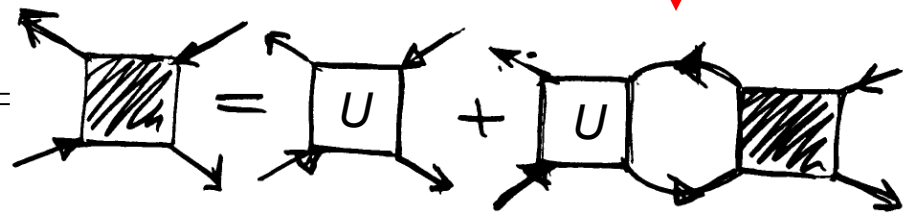
$\boldsymbol{n} = \boldsymbol{p}/p$

$$G(q/2+p)\, G^h(q/2-p) \simeq 2\pi i\, a^2\, \delta(\epsilon) \frac{v_{\rm F}\, \boldsymbol{qn}}{\omega - v_{\rm F}\boldsymbol{qn} + i\,0}\, \delta(p - p_{\rm F}) + B(p,q)$$

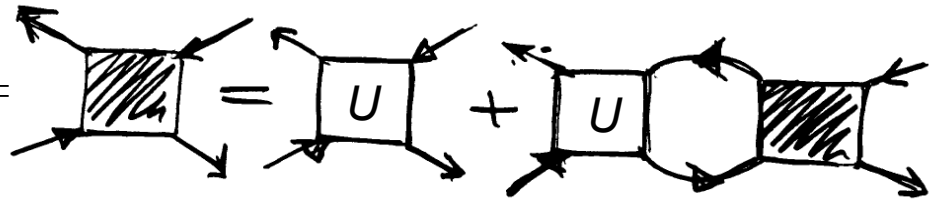
singular pole term

complicated background

$$-i\, T_{\rm ph}(p,p';q) =$$



$$-i\,T_{\rm ph}(p,p';q) =$$



for $|\boldsymbol{p}| \simeq p_{\rm F} \simeq |\boldsymbol{p}'|$ and $|\boldsymbol{qp}| << \omega << \epsilon_{\rm F}$

$$\widehat{T}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}',q) = \widehat{\Gamma}^{\omega}(\boldsymbol{n},\boldsymbol{n}') - \int \frac{d\Omega_{p''}}{4\,\pi}\,\widehat{\Gamma}^{\omega}(\boldsymbol{n},\boldsymbol{n}')\,A(\boldsymbol{n},q)\,\widehat{T}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}',q)$$

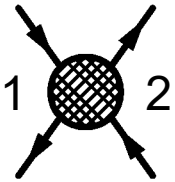
$$A(\boldsymbol{n},q) = a^2\,\frac{m^*\,p_{\rm F}}{\pi^2}\,\frac{v_{\rm F}\,\boldsymbol{qn}}{\omega - v_{\rm F}\boldsymbol{qn} + i\,0}$$

complicated dynamics is here:

$$\widehat{\Gamma}^{\omega}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}') = \widehat{U}(\boldsymbol{n},\boldsymbol{n}') - \int \frac{d^4p''}{(2\pi)^4\,i}\widehat{U}(\boldsymbol{n},\boldsymbol{n}')\,B(p,\omega\to 0,\frac{\boldsymbol{q}}{\omega}\to 0)\,\widehat{\Gamma}^{\omega}_{\rm ph}(\boldsymbol{n},\boldsymbol{n}')$$

parameterize

Landau-Migdal parameters



$$= f_{12}(\boldsymbol{n},\boldsymbol{n}') + g_{12}(\boldsymbol{n},\boldsymbol{n}')\,\boldsymbol{\sigma_1\sigma_2}$$

extracted from experiment

$$a^2 N\,\Gamma_0^\omega(\theta) = f(\theta) = \sum_l f_l P_l(\cos\theta) \qquad N = \nu m^* p_{\rm F}/\pi^2$$

$$a^2 N\,\Gamma_1^\omega(\theta) = g(\theta) = \sum_l g_l P_l(\cos\theta) \qquad \textit{density of states at the Fermi surface}$$

$\theta = \angle(\boldsymbol{n}, \boldsymbol{n}')$ $\qquad \nu = 1, 2$ *number of fermion types* $\qquad n = \nu p_{\rm F}^3/3\pi^2$

neutron matter: $f = f_{nn}$ $\quad g = g_{nn}$ (1 parameter in each channel)

nuclear matter: $f_{nn},\ f_{np},\ f_{pp}$ $\quad g_{nn},\ g_{np},\ g_{pp}$ (3 parameters in each channel)

In matter of arbitrary isospin composition these parameters are independent. Fermi-liquid renormalization is different for these parameters.

small isospin disballance $\quad f_{nn} = f_{pp} = f + f' \qquad f_{np} = f - f'$

$$a^2 N\Gamma^\omega = f + f' \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$$ (2 parameters in each channel)

In nuclear physics one uses also the normalization on the nuclear Fermi surface

$$\tilde{f}(\boldsymbol{n}', \boldsymbol{n}) = a^2\, N_0 \Gamma_0^\omega(\boldsymbol{n}', \boldsymbol{n}) \qquad \tilde{g}(\boldsymbol{n}', \boldsymbol{n}) = a^2\, N_0 \Gamma_1^\omega(\boldsymbol{n}', \boldsymbol{n})$$

$N_0 = N(n = n_0)$ $\quad$ constant, independent of density $\quad (N_0^{-1} = 300\,\text{MeV fm}^3)$

Density dependence? Residual momentum dependence $\Gamma(\boldsymbol{n}', \boldsymbol{n}; q)$?

Sounds in a Fermi liquid

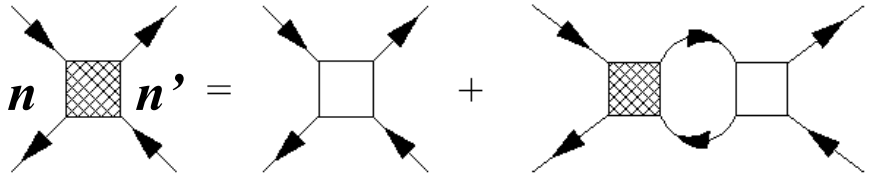
system of strongly interacting fermions (no pairing)

single-particle excitation mechanism

$$\mathbf{G}(\epsilon, \boldsymbol{p}) = \frac{a}{\epsilon - v_{\mathrm{F}}(p - p_{\mathrm{F}})} + G_{\mathrm{reg}}$$

quasiparticle

particle-hole interaction
on Fermi surface



short-range ***long-range***

$$\widehat{T}_{\mathrm{ph}}(\boldsymbol{n}', \boldsymbol{n}; q) = \widehat{\Gamma}^{\omega}(\boldsymbol{n}', \boldsymbol{n}) + \langle \widehat{\Gamma}^{\omega}(\boldsymbol{n}', \boldsymbol{n}'')\, \mathcal{L}_{\mathrm{ph}}(\boldsymbol{n}''; q)\, \widehat{T}_{\mathrm{ph}}(\boldsymbol{n}'', \boldsymbol{n}; q) \rangle_{\boldsymbol{n}''}$$

particle-hole propagator

$$\mathcal{L}_{\mathrm{ph}}(\boldsymbol{n}; q) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\epsilon}{2\pi i} \int_{0}^{+\infty} \frac{\mathrm{d}p\, p^2}{\pi^2}\, G(p_{\mathrm{F}+})\, G(p_{\mathrm{F}-})$$

pole parts

$$\langle \dots \rangle_{\boldsymbol{n}} = \int \frac{\mathrm{d}\Omega_{\boldsymbol{n}}}{4\pi} (\dots)$$

$$p_{\mathrm{F}\pm} = (\epsilon \pm \omega/2, p_{\mathrm{F}}\, \boldsymbol{n} \pm \boldsymbol{k}/2)$$

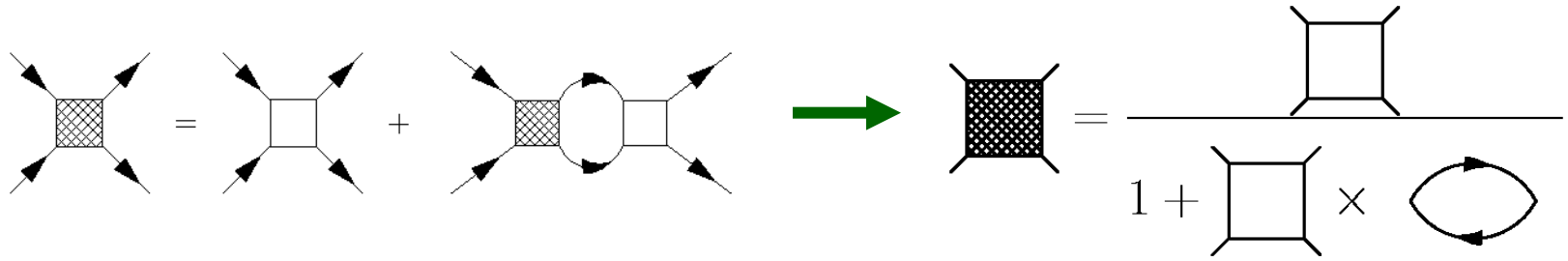
Landau parameters

$$\hat{\Gamma}^{\omega}(\boldsymbol{n}', \boldsymbol{n}) = \Gamma_0^{\omega}(\boldsymbol{n}'\boldsymbol{n})\, \sigma_0' \sigma_0 + \Gamma_1^{\omega}(\boldsymbol{n}'\boldsymbol{n})\, (\boldsymbol{\sigma}'\boldsymbol{\sigma})$$

Solutions for zeroth harmonics

$$\hat{\Gamma}^{\omega}(\boldsymbol{n}', \boldsymbol{n}) = \Gamma_0^{\omega}\, \sigma_0' \sigma_0 + \Gamma_1^{\omega}\, (\boldsymbol{\sigma}' \boldsymbol{\sigma})$$

keep only zeroth harmonics



$$\widehat{T}_{\mathrm{ph}}(\boldsymbol{n}', \boldsymbol{n}; q) = T_{\mathrm{ph},0}(q)\, \sigma_0' \sigma_0 + T_{\mathrm{ph},1}(q)\, (\boldsymbol{\sigma}' \boldsymbol{\sigma})$$

$$T_{\mathrm{ph},0(1)}(q) = \frac{1}{1/\Gamma^{\omega}_{0(1)} - \langle \mathcal{L}_{\mathrm{ph}}(\boldsymbol{n}; q) \rangle_{\boldsymbol{n}}}$$

$$\langle \mathcal{L}_{\mathrm{ph}}(\boldsymbol{n}; q) \rangle_{\boldsymbol{n}} = -a^2 N \Phi\left(\frac{\omega}{v_{\mathrm{F}}\, k}, \frac{k}{p_{\mathrm{F}}}\right)$$

Lindhard function

Lindhard function

$$\Phi(s,x) = \frac{z_-^2 - 1}{4(z_+ - z_-)} \ln \frac{z_- + 1}{z_- - 1} - \frac{z_+^2 - 1}{4(z_+ - z_-)} \ln \frac{z_+ + 1}{z_+ - 1} + \frac{1}{2} \qquad (z_\pm = s \pm x/2)$$

$s = \omega / k v_\mathrm{F}$

$x = k / p_\mathrm{F}$

Imaginary part

$$\Im\Phi(s,x) = \begin{cases} \frac{\pi}{2}\, s & , \quad 0 \le s \le 1 - \frac{x}{2} \\ \frac{\pi}{4x}(1 - z_-^2) & , \quad 1 - \frac{x}{2} \le s \le 1 + \frac{x}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Results of expansions depends on the expansion order:

$$\Phi(s,x) \approx -\frac{1}{3\, z_+\, z_-} = -\frac{1}{3}\frac{1}{s^2 - x^2/4} \qquad \text{for } s \gg 1$$

$$\Phi(s,x) \approx 1 + \frac{s}{2} \log \frac{s-1}{s+1} - \frac{x^2}{12\,(s^2-1)^2} \qquad \text{for } x \ll 1$$

Temperature corrections

$$\Phi_T(s,x,T) = \Phi(s,x)\left(1 - \frac{\pi^2}{12}\frac{T^2}{\epsilon_\mathrm{F}^2}\right)$$

Particle-hole interaction in the scalar channel

scalar channel zeroth harmonics

$$T_{\mathrm{ph},0} = \frac{1}{\left[\Gamma^{\omega}_{00}\right]^{-1} + N\,\Phi(\omega, \boldsymbol{q})} = \frac{N^{-1}}{f_0^{-1} + \Phi(\omega, \boldsymbol{q})}$$

$\Phi(\omega, k)$ Lindhard function

solutions of equation: $f_0^{-1} + \Phi(\omega, \boldsymbol{q}) = 0$ ⟶ spectrum of excitations in the scalar channel $\omega(k)$

(zero-sound modes)

for $\omega \sim \omega(k)$

$$T_{\mathrm{ph},0} \approx \frac{V^2(k)}{\omega - \omega(k)} \quad \text{with} \quad V^{-2} = N \left.\frac{\partial \mathrm{Re}\,\Phi}{\partial \omega}\right|_{\omega(k)}$$

$f_0^{-1} + \Phi(\omega, \boldsymbol{q}) = 0$ **Scalar modes in Fermi liquid**

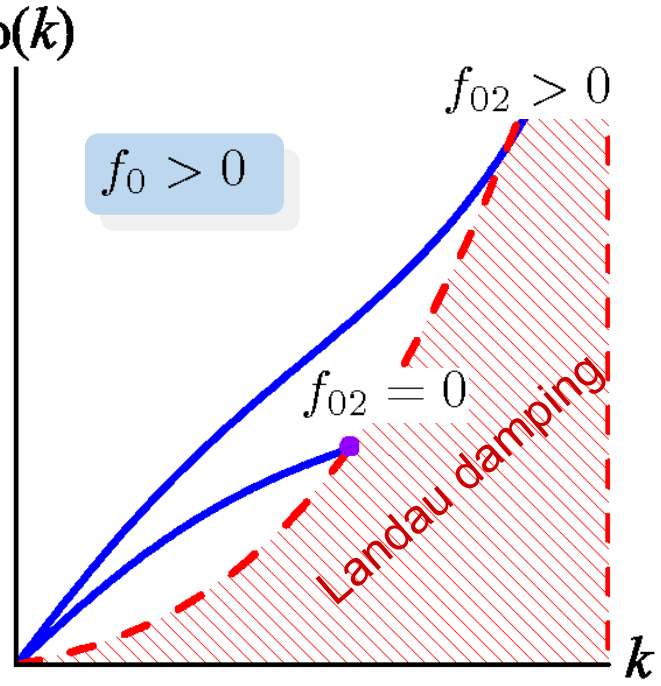
I. $f_0(k) = f_{00} + f_{02}\, k^2/p_{\rm F}^2 > 0$ **zero-sound mode**

residual momentum dependence

$\omega_s(k) = k\, v_{\rm F}\, s(x)$ $\quad s(x) \approx s_0 + s_2\, x^2 + s_4\, x^4$

$$\frac{1+f_{00}}{f_{00}} = \frac{s_0}{2} \ln \frac{s_0+1}{s_0-1}$$

$$s_2 = \frac{s_0(\alpha + f_{02})(s_0^2-1)}{f_{00}\,(1+f_{00}-s_0^2)} \qquad \alpha = f_{00}^2/[12(s_0^2-1)^2]$$

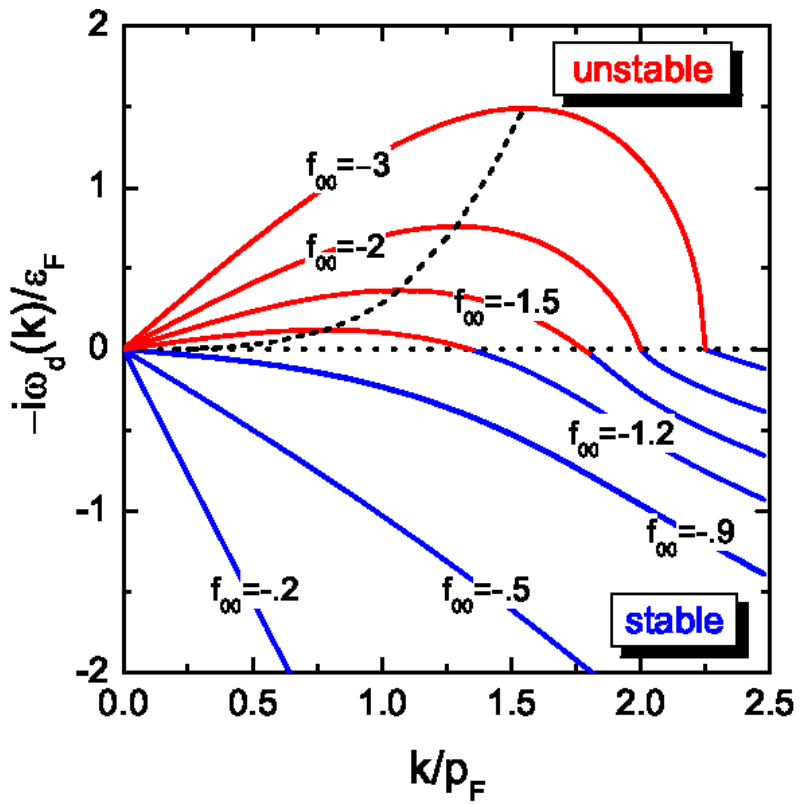


$s_4 > 0$

For $f_{02} < -\alpha$ the ratio $\omega_s(k)/k$ has a minimum at k_0 in which the group velocity of the excitation $v_{\rm gr} = {\rm d}\omega_s/{\rm d}k$ coincides with the phase one $v_{\rm ph} = \omega_s/k$.

II. $-1 < f_0 < 0$ **"diffuson"** $\omega_{\rm d} \approx -i\frac{2}{\pi}kv_{\rm F}\left[\frac{(1-|f_0|)}{|f_0|}+\frac{k^2}{12p_{\rm F}^2}\right]$

$\varphi \sim e^{-i\omega t} \to 0 \qquad t \to \infty$ stable mode $k/p << 1_F$ *expansion*



III. $f_0 < -1$

$\omega_{\rm d} \approx +i\frac{2}{\pi}kv_{\rm F}\left[1-\frac{1}{|f_0|}-\frac{k^2}{12p_{\rm F}^2}\right]$

$\varphi \sim e^{-i\omega t} \to \infty$

$t \to \infty$

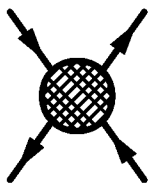
the mode growth rate is determined by f_0

$$-i\omega_m = \frac{8}{3\pi}v_{\rm F}p_{\rm F}\left(1-\frac{1}{|f_{00}|}\right)^{3/2}$$

Nuclear Fermi liquid. Approximations

- Quasiparticle approximation for nucleons, $T \ll \epsilon_{\rm FN}$.
  Only then diagrams with open nucleon lines make sence.
  Otherwise closed diagram technique

- Reduction of the more local interaction to the point-like interaction

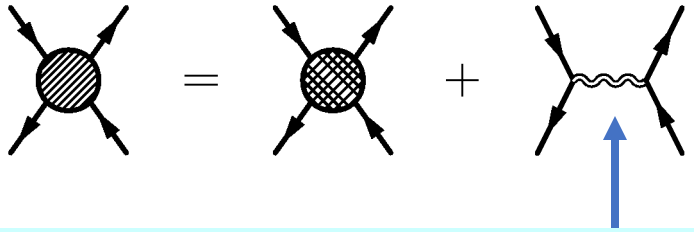


$$= C_0 \left(f_{12} + g_{12} \boldsymbol{\sigma_1} \boldsymbol{\sigma_2} \right),$$

f_{12}, g_{12} are Landau-Migdal parameters.

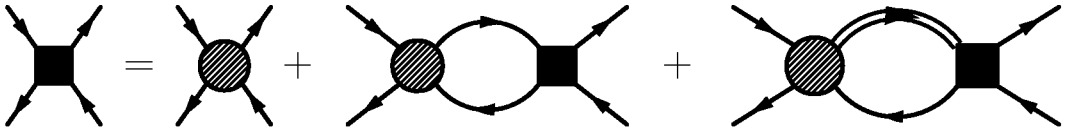
Constants!= rough approximation! But then eqs become algebraic!

- **explicit pionic degrees of freedom**

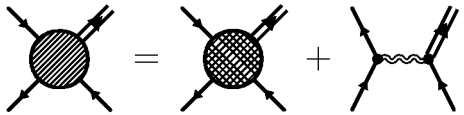


pion with residual (irreducible in NN^{-1} and ΔN^{-1}) s-wave π N interaction and $\pi\pi$ scattering``

- **explicit Δ degrees of freedom**

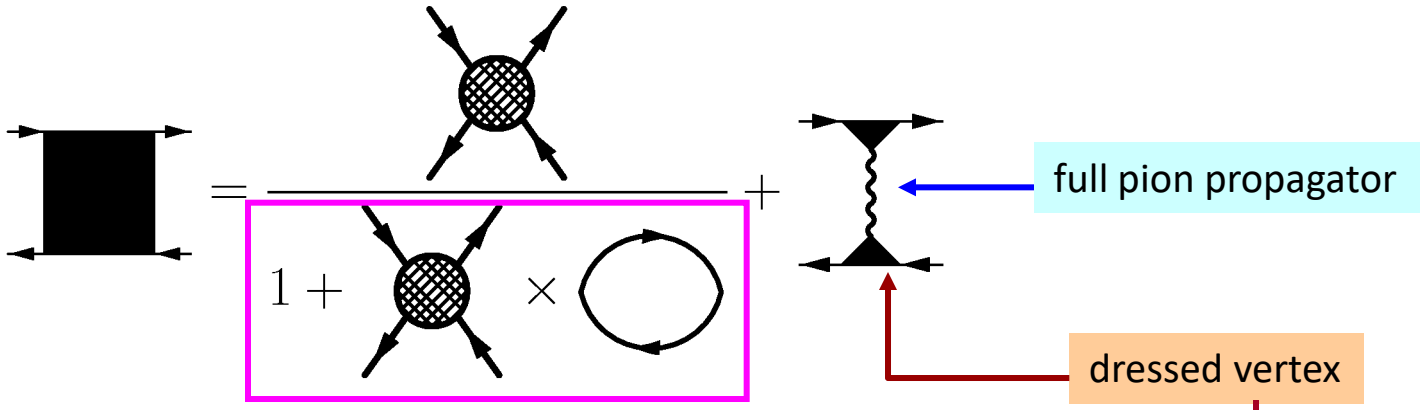


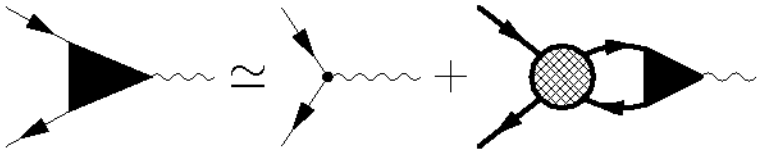
Part of the interaction involving Δ isobar is analogously constructed:



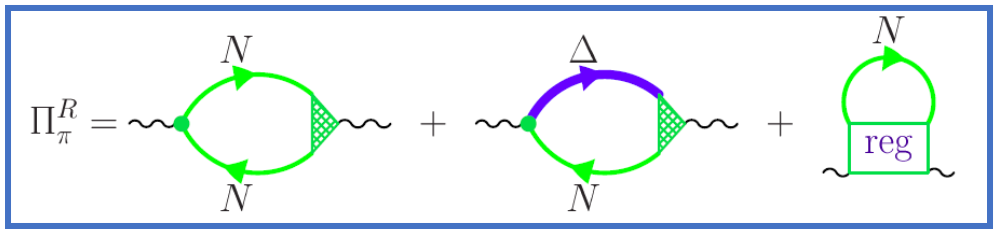
Resummed NN interaction

Graphically, the resummation is straightforward and yields:



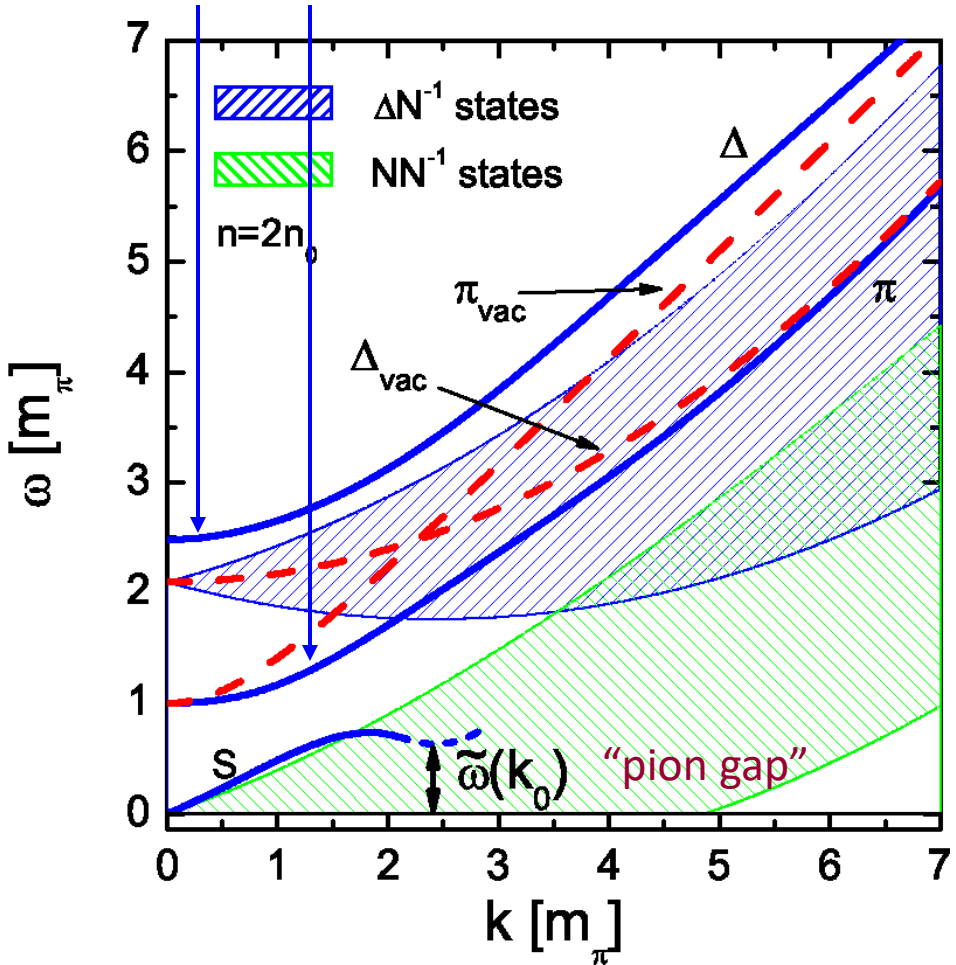


Pion modes in nuclear medium



$$A_\pi(\omega, \boldsymbol{k}) \approx \sum_{i=\pi,\Delta} \frac{2\,\pi\,\delta(\omega - \omega_i(\boldsymbol{k}))}{\left(2\omega - \dfrac{\partial \Pi^R}{\partial \omega}\right)\Big|_{\omega=\omega_i(\boldsymbol{k})}} + \frac{2\,\beta\,k\,\omega}{\widetilde{\omega}^4(k) + \beta^2\,k^2\,\omega^2}\,\theta(\omega < v_F\,k)$$

quasi-particle modes



pion propagator has a complex pole

$$D^{-1}(\omega, k) \simeq D^{-1}(0, k) + i\beta\,\omega$$

$$\beta = m_N^{*2}\,k\,f_{\pi NN}^2/\pi$$

$$\widetilde{\omega}^2(k) = -D^{-1}(0, k)$$

$$\omega \propto -i\widetilde{\omega}^2(k_{\min})/\beta$$

when $\widetilde{\omega}^2(k_{\min} \simeq p_F) < 0$

→ instability → pion condensation

[A.B.Migdal et al, Phys. Rept. 192 (1990) 179]