Physics of a neutron star crust
Lecture 3.

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Ioffe Institute



Nuclear Dynamics: from Stars to Colliders

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BLTR, JINR, Dubna, Russia

Physics of a neutron star crust

Selection of topics and plan

"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein

According to Robinson [*Nature* **557**, 30 (2018)], it can be a compressed version of lines from a 1933 lecture by Einstein:
"*It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience*."

My preference in these lectures: *models, which can be solved analytically*

Lecture 1: Introduction and basic models of outer crust at T=0

Lecture 2: Outer crust: thermodynamics and elasticity

Lecture 3: Inner crust

+ Mass-radius relation (without crust EOS)

Inner crust

- Nuclear physics

$$\epsilon = \epsilon(n_p(\boldsymbol{r}), n_n(\boldsymbol{r}), n_e(\boldsymbol{r}))$$

Atomic mass tables are unavailable for nuclei beyond the drip line: we need to calculate nuclei properties specially for inner crust conditions

- Compressible liquid drop model CLDM
- Exteded Thomas-Fermi (ETF)
- ETF+Strutinsky integral (ETFSI)
- HFB
- …

- Thermodynamics

To calculate required thermodynamic functions

- Finite temperature effects:
  - Relatively low temperature: application of outer crust thermodynamics of solid phase (lattice of clusters with Coulomb interaction among them)
  - High temperature: Finite temperature CLDM, ETF,…

- **Composition**

Common introduction:
Inner crust is composed of degenerate electrons, unbound neutrons + nuclei (clusters)

Not enough for quantitative description!!!

That are the nuclei types???

- Equilibrium

Optimal energy density

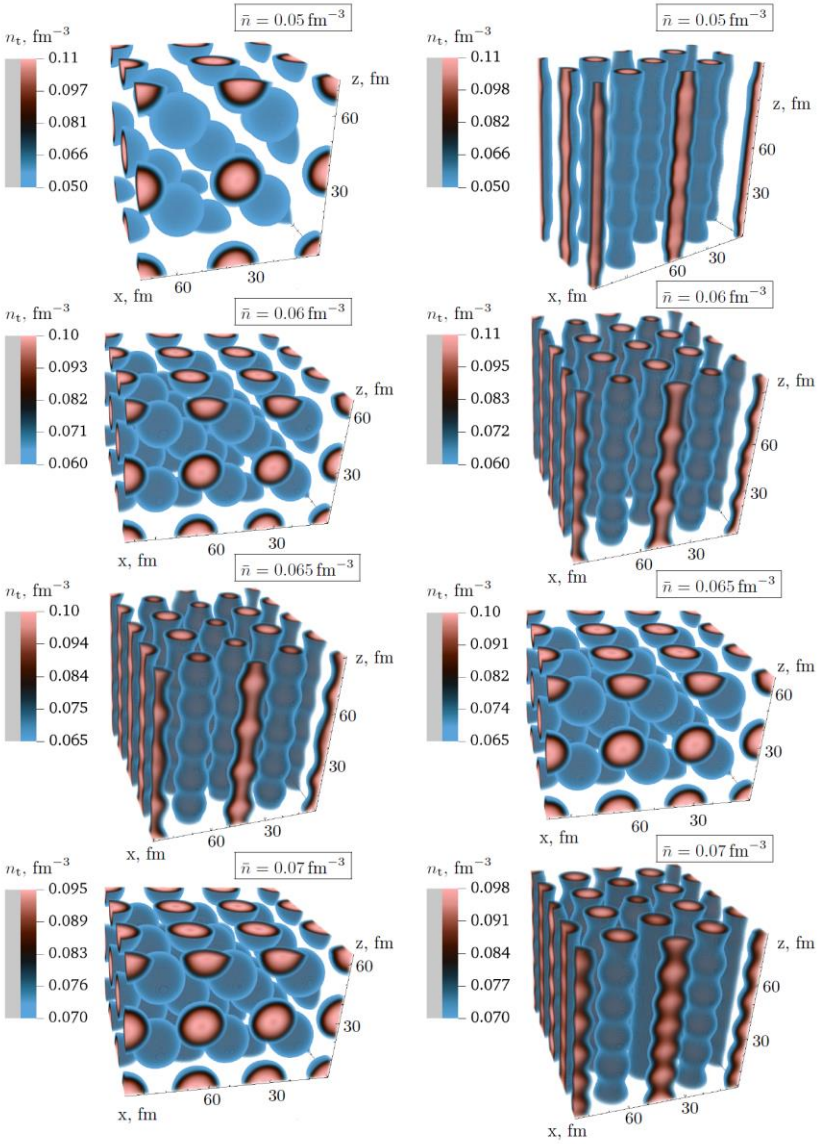
- Nonequilibrium

Formation history + nuclei reaction

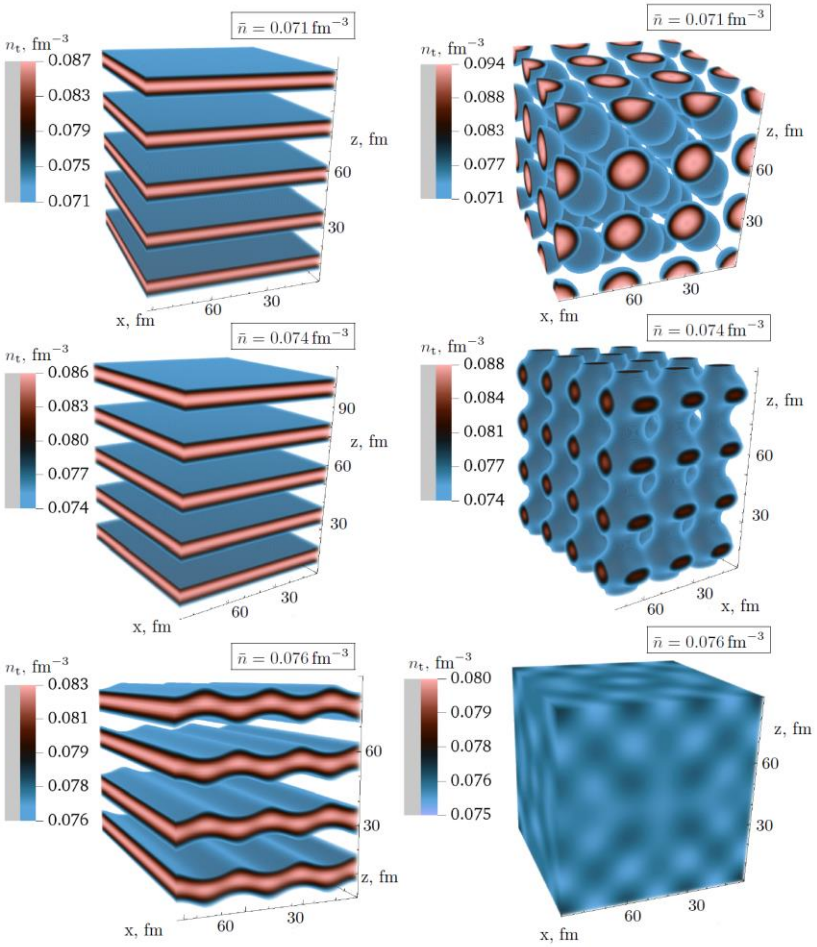
Inner crust

- Nuclear physics

$$\epsilon = \epsilon(n_p(\boldsymbol{r}), n_n(\boldsymbol{r}), n_e(\boldsymbol{r}))$$



The best of 2026: 3D HF+BCS
(very intensive numerical simulations)



Figures from PhD thesis by
N.N. Shchechilin (2026)

Extended Thomas-Fermi approach

Density functional (4th order Wigner-Kirkwood expansion)

$$E_{\rm ETF} = \underset{\text{nuclear}}{E_{\rm nuc}} + \underset{\text{Coulomb}}{E_{\rm C}} + \underset{\text{electron}}{E_{\rm e}}$$

$$E_{\rm nuc} = \frac{1}{A}\int \mathcal{E}(\rho_q(\mathbf{r}), \nabla\rho_q(\mathbf{r}), \triangle\rho_q(\mathbf{r}))\mathrm{d}\mathbf{r}$$

Nucleon profile optimization: (accurate) Euler–Lagrange equations

$$\frac{\partial\delta e_{\rm ETF}}{\partial\delta\rho_p} = \mu_p; \quad \frac{\partial\delta e_{\rm ETF}}{\partial\delta\rho_n} = \mu_n; \quad \mu_n - \mu_p = \mu_e$$

Real life nucleon profile optimization: minimization over parametrized profiles is spherically symmetric WS cell

$$\rho_{\rm q}(r) = \rho_{{\rm B}q} + \frac{\rho_{\Lambda q}}{1 + \exp\left[\left(\frac{C_q - R}{r - R}\right)^2 - 1\right]\exp\left(\frac{r - C_q}{a_q}\right)}$$

Pearson et al., Mon. Not. R Astron. Soc. **2018**, 481, 2994

Extended Thomas-Fermi approach

Density functional (4th order Wigner-Kirkwood expansion)

$$E_{\rm ETF} = E_{\rm nuc} + E_{\rm C} + E_{\rm e}$$

nuclear Coulomb electron

$$E_{\rm nuc} = \frac{1}{A}\int \mathcal{E}(\rho_q(\mathbf{r}), \nabla\rho_q(\mathbf{r}), \triangle\rho_q(\mathbf{r}))\mathrm{d}\mathbf{r}$$

Nucleon profile optimization: (accurate) Euler–Lagrange equations

$$\frac{\partial\delta e_{\rm ETF}}{\partial\delta\rho_p} = \mu_p; \quad \frac{\partial\delta e_{\rm ETF}}{\partial\delta\rho_n} = \mu_n; \quad \mu_n - \mu_p = \mu_e$$

Real life nucleon profile optimization: minimization over parametrized profiles

- Optimization over a few parameters – not too complicated
- PC computation to got the tables (integrals should be taken numerically)
- Choose of the functional form of the profiles:

  Conclusion 'results are not sensitive to the functional form of profiles' depends on the required accuracy… Even predictions for structure of the mantle can be affected!

Extended Thomas-Fermi approach

Conclusion ‘results are not sensitive to the functional form of profiles’ depends on the required accuracy…

Shchechilin et al. (PRC, 2024):

Three parametrizations: $\rho_{\rm q}^i(r) = \rho_{{\rm B}q} + \rho_{\Lambda q} f^i(r)$

Strong damping $$f_{\rm q}^{\rm StrD}(r) = \frac{1}{1 + \exp\left[\left(\frac{C_q - R}{r - R}\right)^2 - 1\right] \exp\left(\frac{r - C_q}{a_q}\right)}$$

Soft damping $$f_{\rm q}^{\rm SoftD}(r) = \frac{1}{1 + \left(\frac{C_q - R}{r - R}\right)^2 \left(\frac{r}{C_q}\right)^2 \exp\left(\frac{r - C_q}{a_q}\right)}$$

3FD damping

$$f_{\rm q}^{\rm 3FD}(r) = f_q^{FD}(-r) + f_q^{FD}(r) + f_q^{FD}(2R - r) + f_q^{FD}(-R) + 2\, f_q^{FD}(R)$$

$$f_{\rm q}^{\rm FD}(r) = \frac{1}{1 + \exp\left(\frac{r - C_q}{a_q}\right)}$$

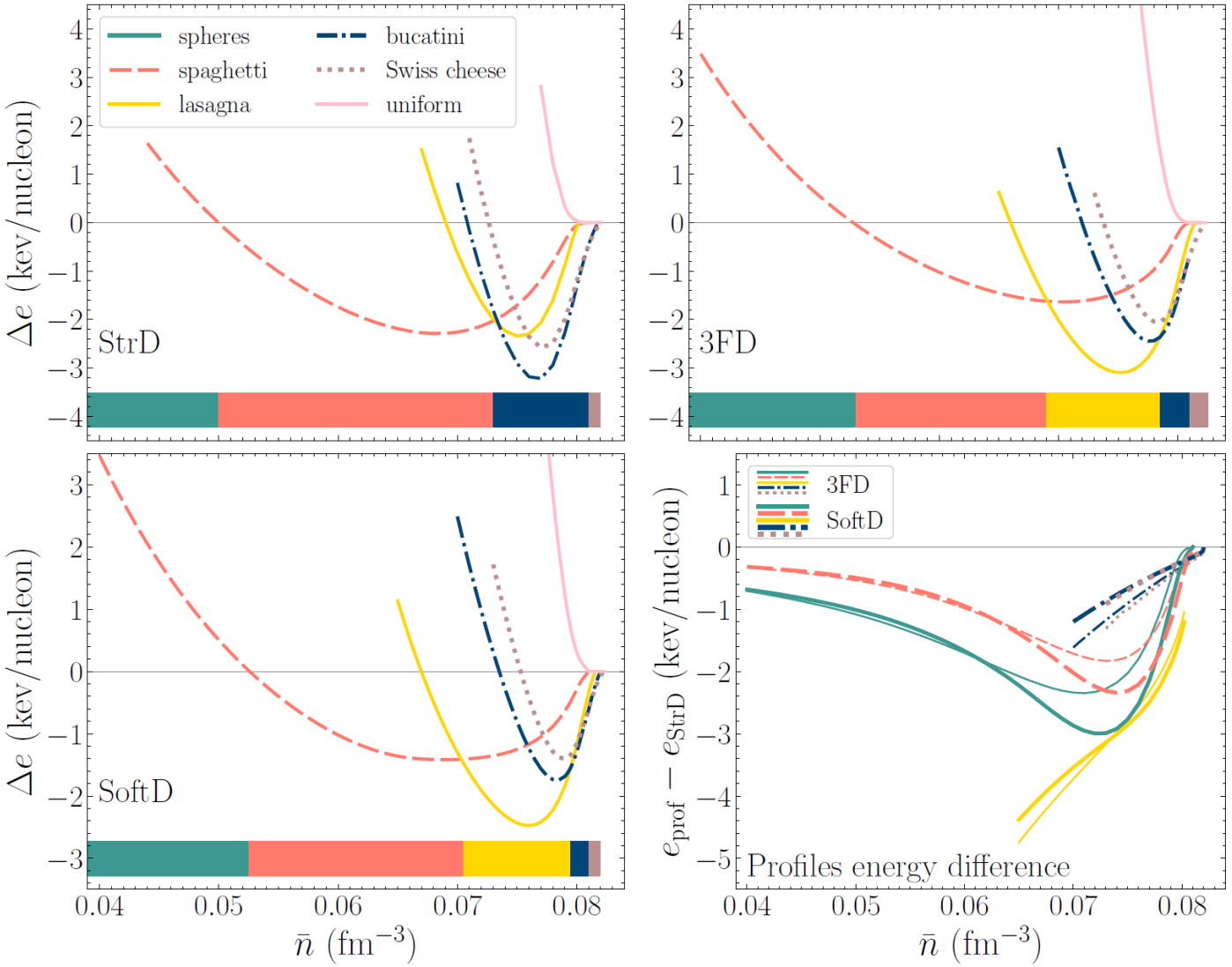
The differences are ~1 keV/nucleon; it is large enough to alter the pasta sequence

Extended Thomas-Fermi approach

Conclusion ‘results are not sensitive to the functional form of profiles’ depends on the required accuracy.

Shchechilin et al. (under review):

BSK24

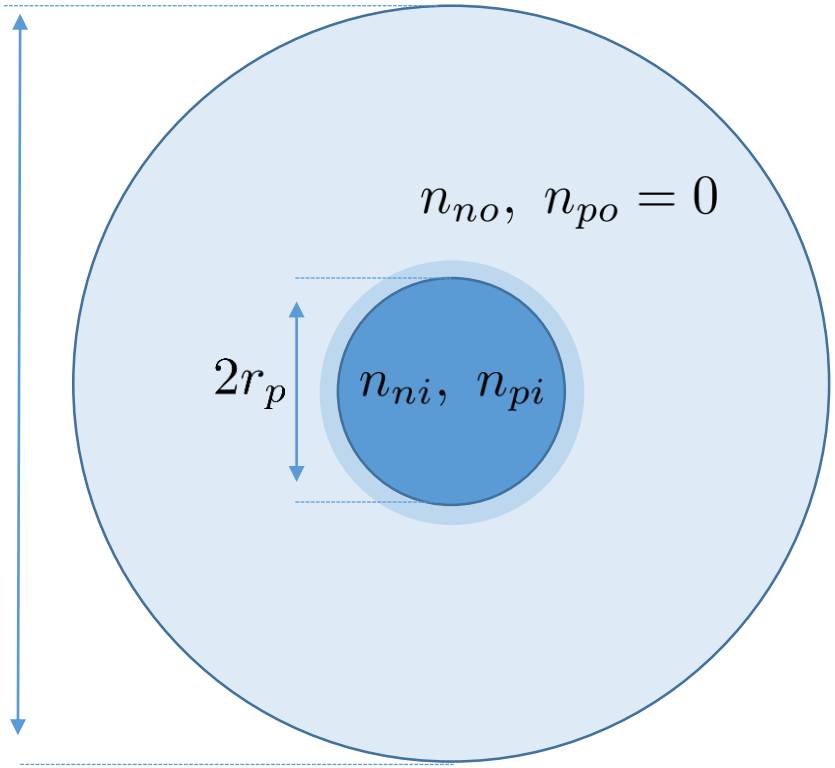


The differences are large enough to alter the pasta sequence

Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- CLDM does not assume that step-like profile for proton and neutron density is real. Rather, general feature of CLDMs is that they start from explicit analytical expression (for energy) and adjust available parameters to make this expression as simple as possible, but (reasonable) accurate

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$



- Explicit expressions for all thermodynamic quantities can be obtained analytically. It guaranties absolute (up to numerical accuracy) thermodynamic consistency of the model (if properly applied)

"Everything should be made as simple as possible, but no simpler"

Attributed to Albert Einstein

Compressible liquid drop model for inner crust

"Everything should be made as simple as possible, but no simpler"

Attributed to Albert Einstein

(Overconfident) physicist of nowadays : Liquid drop model is obsolete! You are not in 1960s!!! It is simpler, than possible!!!



Why CLDM in 2020-s is still applicable? It is physically motivated approach to fit ETF calculations

Reminder: Extended Thomas-Fermi approach

Density functional (4th order Wigner-Kirkwood expansion)

$$E_{\mathrm{ETF}} = E_{\mathrm{nuc}} + E_{\mathrm{C}} + E_{\mathrm{e}}$$

nuclear Coulomb electron

$$E_{\mathrm{nuc}} = \frac{1}{A}\int \mathcal{E}(\rho_q(\mathbf{r}), \nabla\rho_q(\mathbf{r}), \triangle\rho_q(\mathbf{r}))\mathrm{d}\mathbf{r}$$

Restricted ETF (real life ETF): minimization over parametrized profiles

$$\rho_{\mathrm{q}}(r) = \rho_{\mathrm{B}q} + \rho_{\Lambda q}\frac{1}{1+\left(\frac{C_q-R}{r-R}\right)^2\left(\frac{r}{C_q}\right)^2\exp\left(\frac{r-C_q}{a_q}\right)}$$

- Optimization over a few parameters – not too complicated
- PC computation to got the tables (integrals should be taken numerically)

$$E_{\mathrm{ETF}} = E_{\mathrm{ETF}}(C_q, a_q, \rho_{B,q}, \rho_{\Lambda,q})$$

CLDM: Energy -- explicit function (without integrals => fast computation)

Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- General feature of these models is that they start from explicit analytical expression (for energy)

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$

Nuclear energy inside cluster:

$$E_{\rm in} = \epsilon^{\rm bulk}(n_{ni}, n_{pi}) V_i$$

Nuclear energy outside cluster:

$$E_{\rm out} = \epsilon^{\rm bulk}(n_{no}, n_{po}) V_o$$

Surface energy

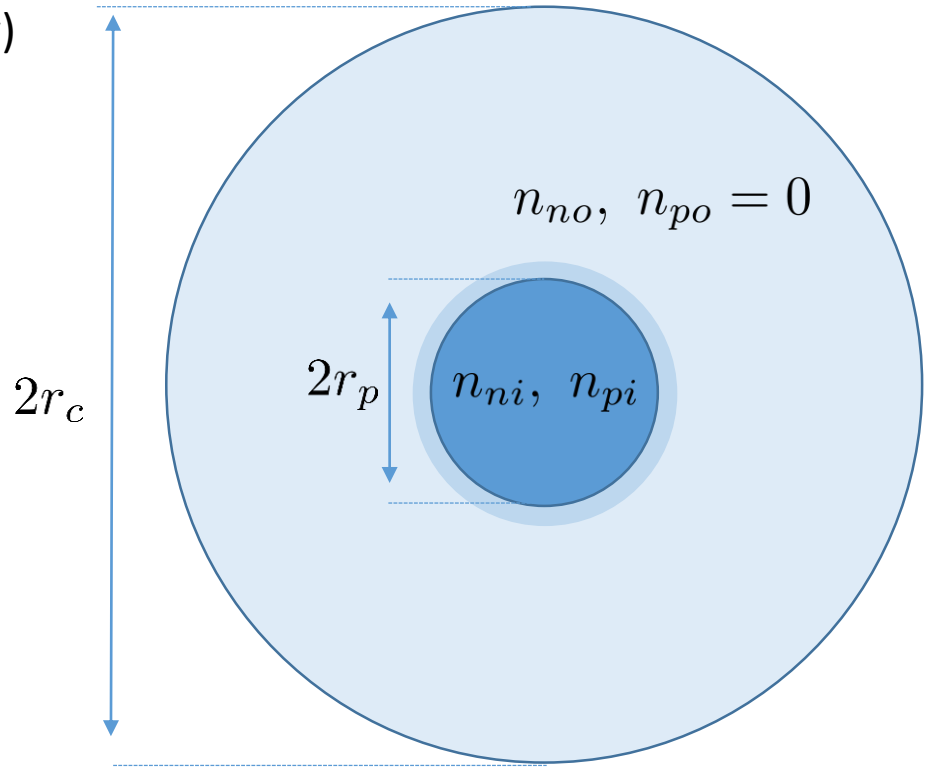
$$E_{\rm surf} = \sigma(\ldots) S + \ldots$$

Coulomb energy: two uniformly charged balls

$$E_C = \frac{3}{5}\frac{Z^2 e^2}{R_N}\left(1 - \frac{3}{2}w^{1/3} + \frac{1}{2}w\right) + \ldots$$

Energy of electrons (uniform degenerate ideal gas)

$$E_e = \epsilon_e(n_e)(V_o + V_i)$$



Surface energy? What is it?
Something artificial???

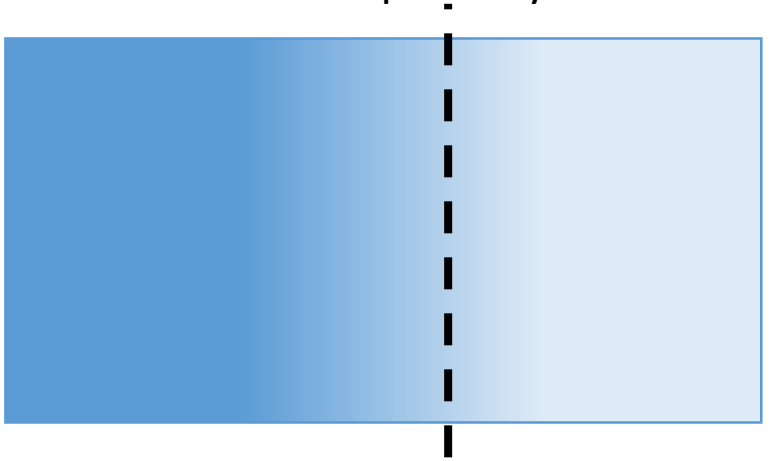
Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

Real two-phase system



Reference two-phase system



$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

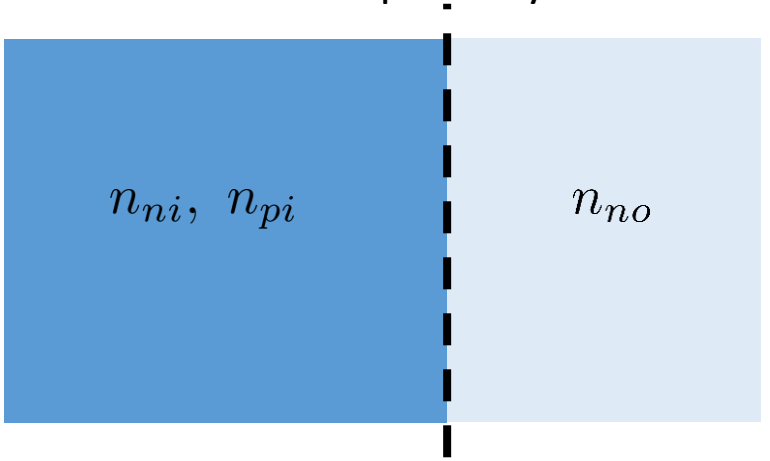
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Real two-phase system



Reference two-phase system



$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

$$\begin{aligned} E_{\text{ref}} &= \epsilon_i V_i + \epsilon_o V_o, \\ N_{\text{ref}} &= n_{ni} V_i + n_{no} V_o, \\ Z_{\text{ref}} &= n_{pi} V_i \end{aligned}$$

$$\begin{aligned} E_{\text{surf}} &= E_{\text{two phase}} - E_{\text{ref}} \\ N_{\text{surf}} &= N_{\text{two phase}} - N_{\text{ref}} \\ Z_{\text{surf}} &= Z_{\text{two phase}} - Z_{\text{ref}} \end{aligned}$$

Can be nulled by choose of the reference system

- Surface energy is a correction, required to reproduce two phase system energy
- (Neutron) adsorption is required for thermodynamically consistent description of two phase boundary
- Surface energy describe two-phase thermodynamics precisely

Thermodynamically consistent CLDM

Thermodynamics

Consider a cell thermodynamically:

- Write a general expression for cell energy
- Present the surface terms as

$$E_{\rm surf} = E_{\rm two\ phase} - E_{\rm ref}$$
$$N_{\rm surf} = N_{\rm two\ phase} - N_{\rm ref}$$
$$Z_{\rm surf} = Z_{\rm two\ phase} - Z_{\rm ref}$$

- Consider thermodynamic derivatives
- Obtain relations for surface terms

Restricted perturbative ETF

- Apply plane-boundary nucleon distribution profiles as a zero-approximation
- Consider small corrections perturbatively, threating surface curvature as perturbation parameter.
- ***Up to the second order in surface curvature all parameters of the theory can be determined from the plane-boundary nucleon profiles***

(it is due to optimal choice of plane boundary nucleon profiles: small perturbations of these profile should not affect energy for plane boundary)

Surface energy

$$E_s = \sigma S + \mu_{\mathrm{ns}} N_{\mathrm{s}}$$

Thermodynamic relation: $$\nu_s \equiv \frac{N_s}{S} = -\frac{\partial \sigma}{\partial \mu_{\mathrm{ns}}}$$



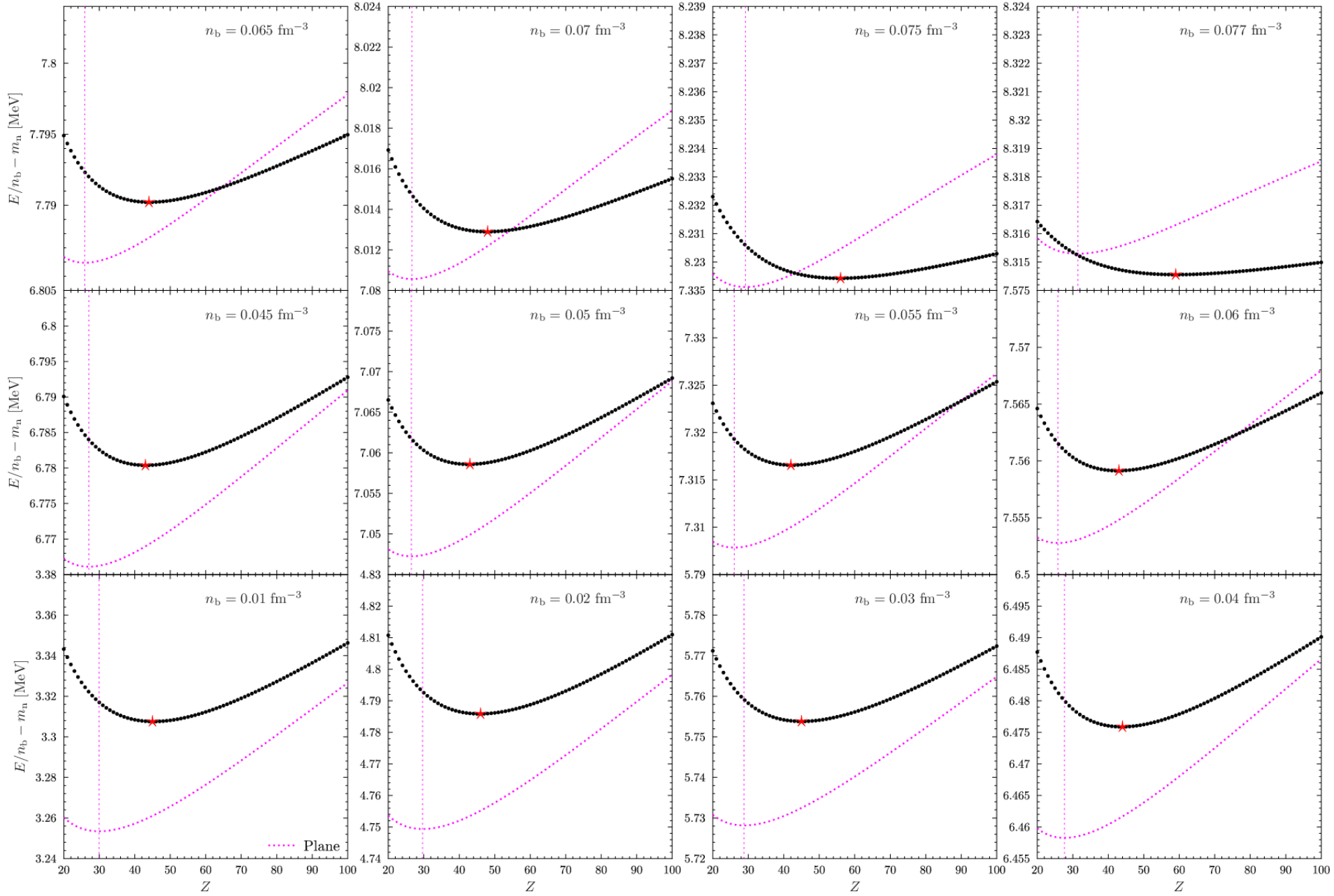
$$\varepsilon_{\mathrm{s}} \equiv \frac{E_S}{S} = \left(\sigma_{\mathrm{pl}}\left(\mu_{\mathrm{ns}}\right) + \frac{2\sigma_{\mathrm{cur}}\left(\mu_{\mathrm{ns}}\right)}{r_{\mathrm{p}}}\right)\frac{3u}{r_{\mathrm{p}}} + \left(\nu_{\mathrm{pl}}\left(\mu_{\mathrm{ns}}\right) + \frac{2\nu_{\mathrm{cur}}\left(\mu_{\mathrm{ns}}\right)}{r_{\mathrm{p}}}\right)\frac{3u\mu_{\mathrm{ns}}}{r_{\mathrm{p}}}$$

Surface parameters are given as integrals over plain-interface profiles:

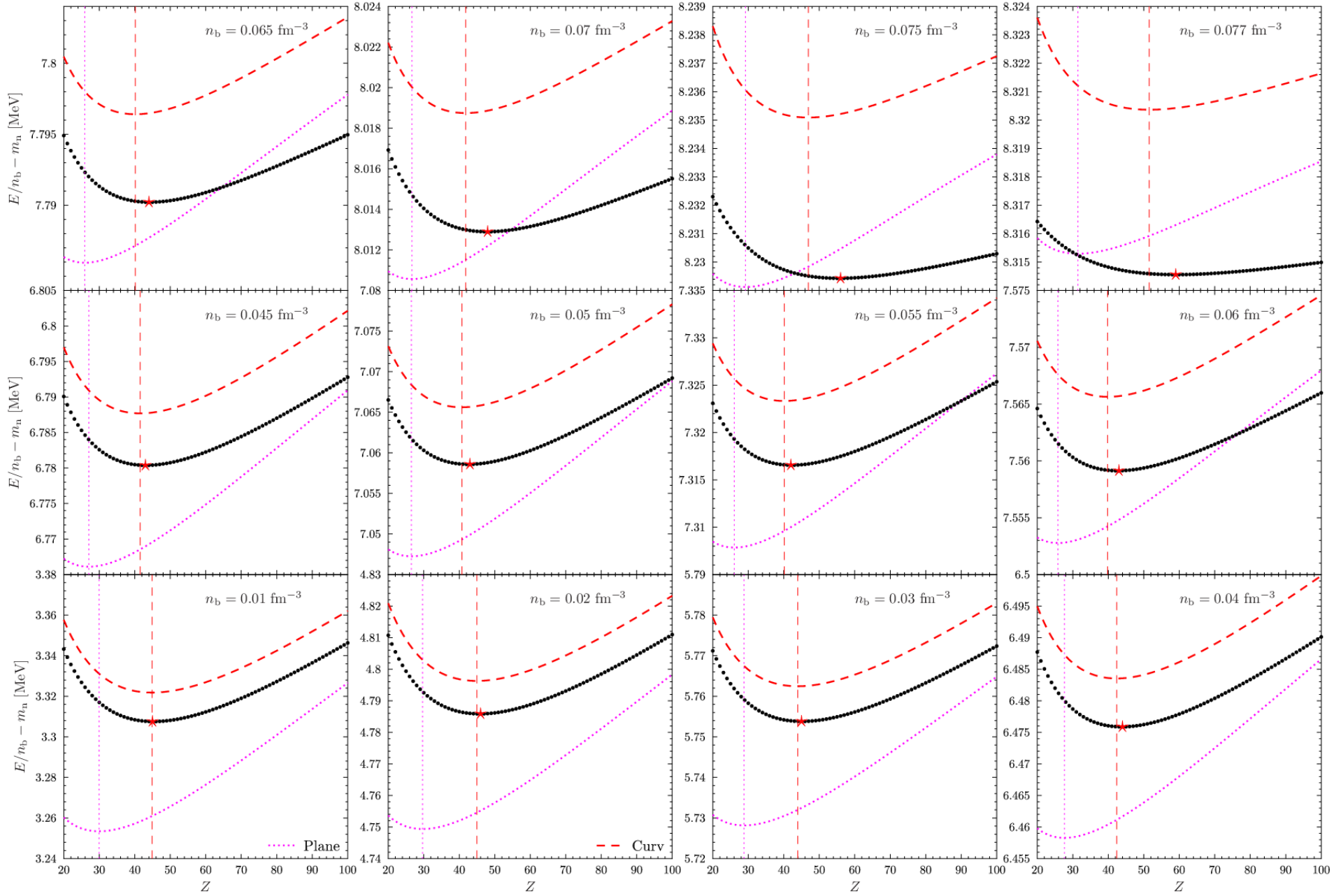
$$\sigma_{pl} = \int_{-\infty}^{\infty} \left\{\left[\epsilon(z) - \epsilon_{ref}(z)\right] - \mu_n\left[n_n(z) - n_{n,ref}(z)\right]\right\} \mathrm{d}z$$

$$\sigma_{cur} = \int_{-\infty}^{\infty} \left\{\left[\epsilon(z) - \epsilon_{ref}(z)\right] - \mu_n\left[n_n(z) - n_{n,ref}(z)\right]\right\} z\mathrm{d}z$$

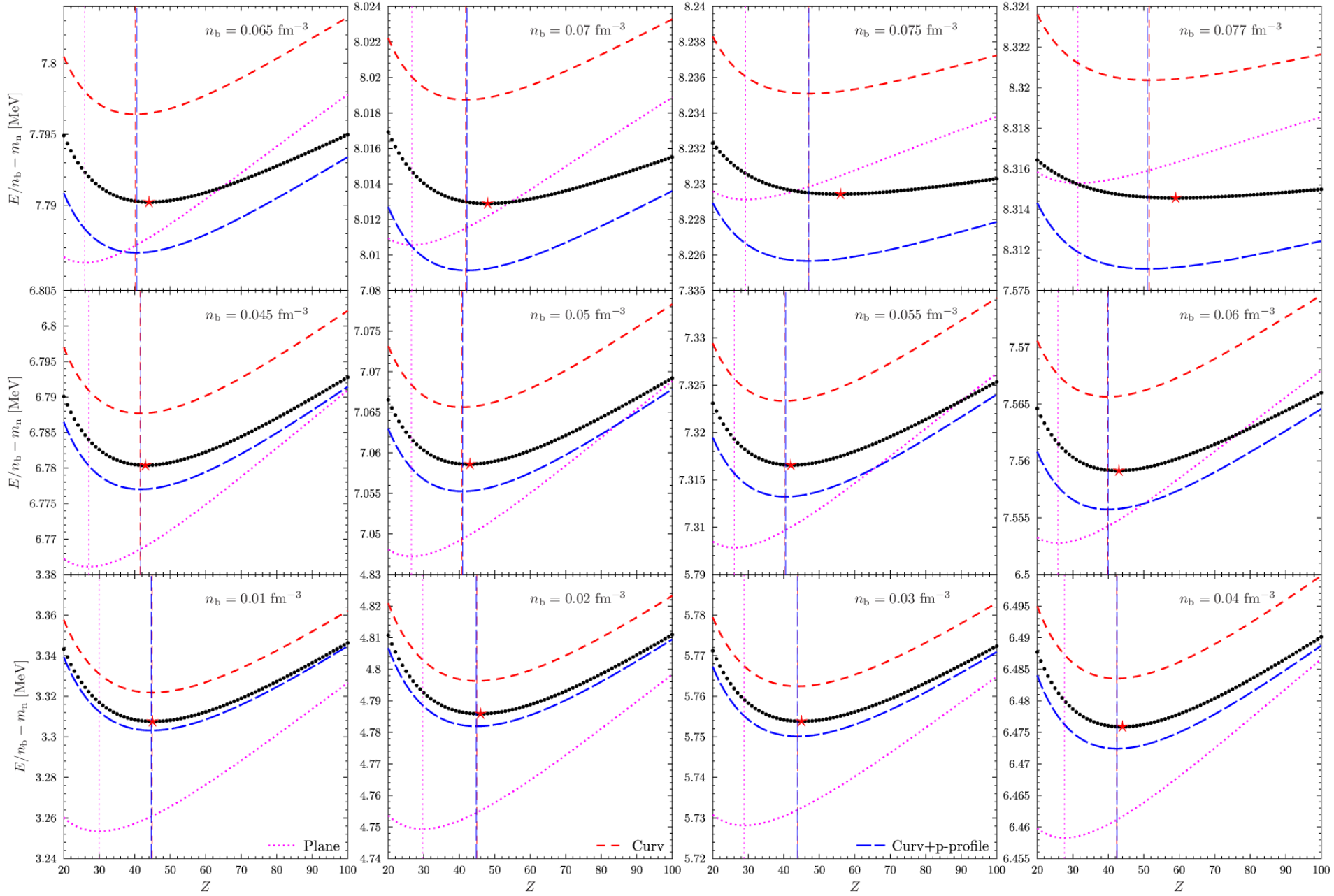
CLDM vs ETF: no curvature corrections (Gusakov&AIC 2020)



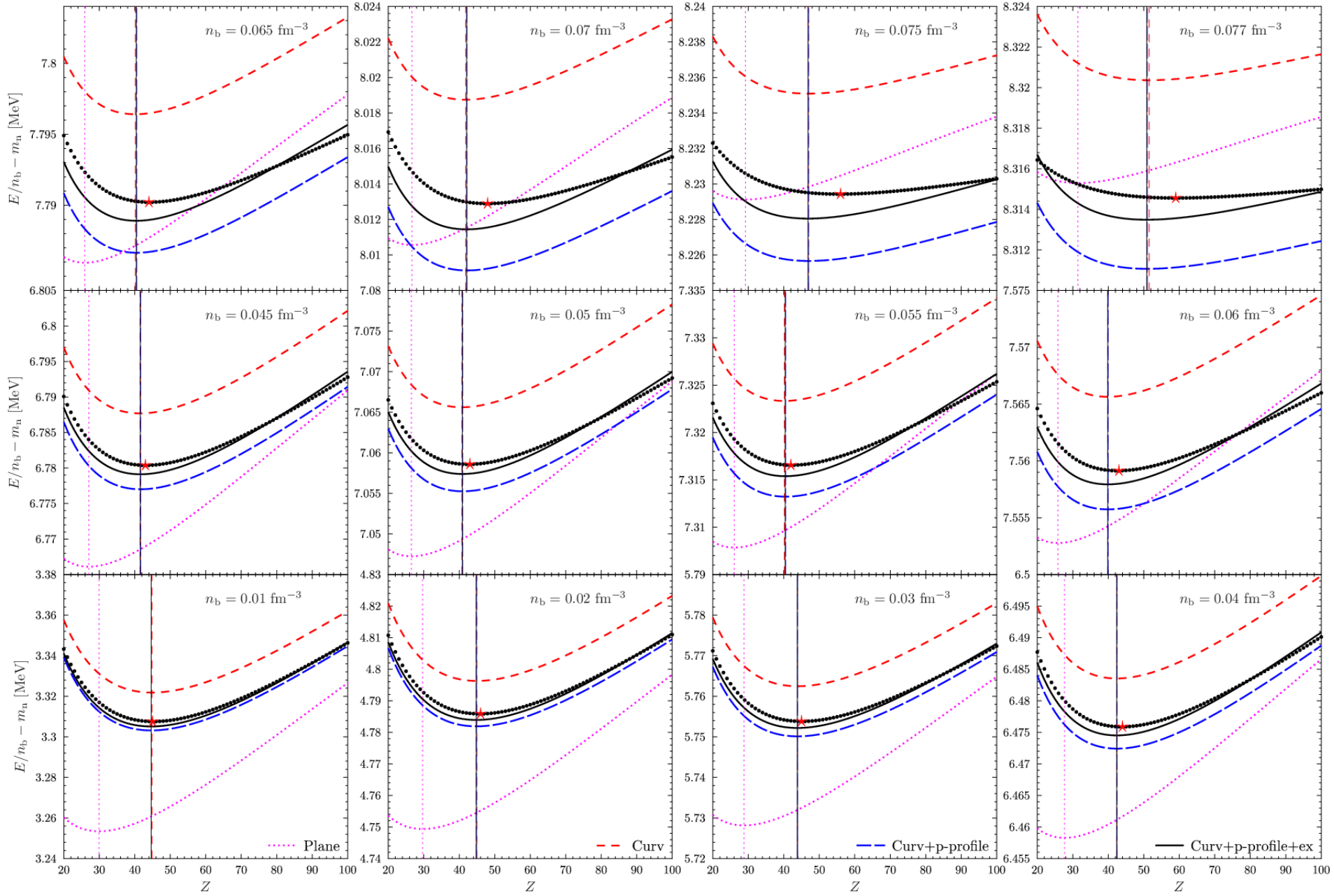
CLDM vs ETF: curvature corrections



CLDM vs ETF: curvature + p-profile (Coulomb energy)



CLDM vs ETF: curvature + p-profile+ exchange cor.



Thermodynamics of inner crust

Thermal corrections for electrons

$$F = F(T=0) + F_{\rm clust} + F_{\rm e} + F_{\rm n}$$

Previous slides
(CLDM, ETF, ETFSI)

Thermal excitations for neutrons

Thermodynamics for clusters:
System of charged spheres on neutralizing background

The same, as for outer crust!!!

$$F_{\rm i}(T, n_{\rm i}, Z, A) \neq F_{\rm id}(T, n_{\rm i})$$

Calculated in three assumptions

All nucleons inside cluster

All nucleons above background density

Band structure theory (Chamel 2008+)

$$C \propto (T/T_{\rm p})^3 \propto A^{3/2}$$

Crust of compact stars

Let’s inspect our spherical cows



Pressure within ion sphere model

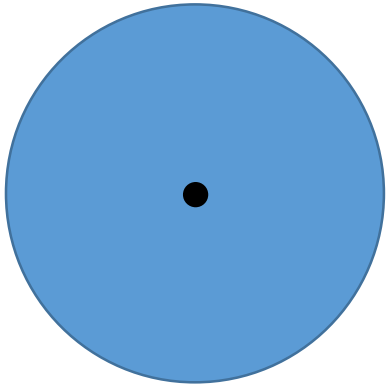
Thermodynamics

$$F = F_e + F_i$$

$$P = P_e + P_i$$

$$P_\mathrm{i} = -\frac{3}{10}\frac{Z^2 e^2}{a} n$$

Ion sphere model



Pressure is given by pressure at the boundary.
There are only electrons

$$P = P_e$$

Step back: small polarization

Screening cloud

$$\mu_e - e\phi = \mathrm{const}$$

$$\delta n_e = \frac{\partial n_e}{\partial \mu_e}\delta\mu_e = \frac{\partial n_e}{\partial \mu_e} e\phi$$

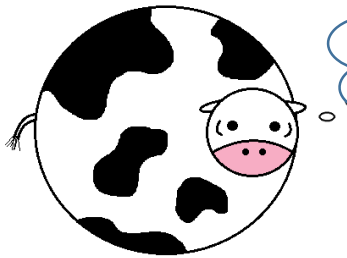
Potential

$$\phi = \frac{Ze^2}{a}\left[\frac{a}{r} - \left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{a^2}\right)\right]$$

$$\int n_e \mathrm{d}^3 r = Z \Rightarrow n_e(r=a) = n_e^{\mathrm{un}} - \frac{3}{10}\frac{\partial n_e}{\partial \mu_e}\frac{Ze^2}{a}$$

$$\frac{\partial P_e}{\partial \mu_e} = n_e = Zn$$

$$P = P_e(n_e^{\mathrm{un}}) - \frac{3}{10}\frac{Ze^2}{a}\frac{\partial P_e}{\partial n_e}\frac{\partial n_e}{\partial \mu_e}$$



Some problems can be solved analytically!

Note for students

- If you write down expression for thermodynamic function, all other thermodynamic functions should be calculated from this expression by thermodynamic relations

- If you have a brilliant idea how to calculate another thermodynamic quantity easily, check that it follows from thermodynamic relations and expression for thermodynamic function, mentioned in the first line.

A microphysical model, there thermodynamic functions are not in thermodynamically consistent be dangerous for applications:
it (potentially) can allow everything
(perpetuum mobile?)

How to build NS model?

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho,$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2Gm}{c^2\, r^2}\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1}.$$

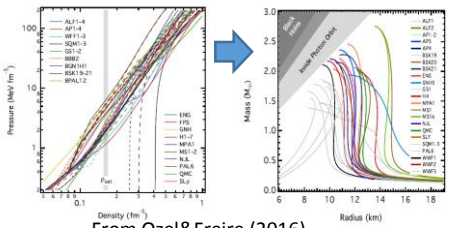
Common practice:

- Specify EOS everywhere (in the core and in the crust) $P(\rho)$
- Solve TOV equations with initial conditions $m(r=0)=0; \quad \rho(r=0)=\rho_c$
- Stop integration at $P=0$

NS radius $R = r|_{P=0}$

NS mass $M = m|_{P=0}$

$M(R)$ parametrized by ρ_c



From Ozel&Freire (2016)

How to build NS model?

Tolman-Oppenheimer-Volkoff equations

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Common practice:

- Specify EOS everywhere (in the core and in the crust) $P(\rho)$
- Solve TOV equations with initial conditions $m(r=0)=0; \quad \rho(r=0)=\rho_c$
- Stop integration at $P=0$

Problem: EOS in the crust is required!

Ideally, it should be calculated consistently with core EOS $\epsilon = \epsilon(n_p(\boldsymbol{r}), n_n(\boldsymbol{r}), n_e(\boldsymbol{r}))$

Can we build NS model without specifying crust?

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$

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Thermodynamics:

$$\mu_b^{\infty} = \mu_b \exp(\nu/2) = \mathrm{const}$$

$$\mathrm{d}P = n_b \mathrm{d}\mu_b;\ \ \rho c^2 + P = \Phi = n_b \mu_b$$

$$n_b \frac{\mathrm{d}\mu_b}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}r} = -2\left(\rho c^2 + P\right)\frac{\mathrm{d}\nu}{\mathrm{d}r} = 2 n_b \mu_b \frac{\mathrm{d}\nu}{\mathrm{d}r}$$

Can we build NS model without specifying crust?

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$
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Thermodynamics: $\mu_b^{\infty} = \mu_b \exp(\nu/2) = \mathrm{const}$

Crust-core interface: well predicted by instability of uniform matter ➡ μ_b^{cc}

Surface: ground state composition is ^{56}Fe $\quad \mu_b^{\mathrm{surf}} = m(^{56}\mathrm{Fe})/56$

Known (for given core EOS)

$$\frac{\mu_b^{\mathrm{cc}}}{\mu_b^{\mathrm{surf}}} = \exp\left(\frac{\nu^{\mathrm{surf}}}{\nu^{\mathrm{cc}}}\right)$$

Red-shift at the surface: $\nu^{\mathrm{surf}} = \log(1 - 2GM/c^2R)$

Can we build NS model without specifying crust?

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$
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Thermodynamics:

Known
(for given core EOS)

$$\frac{\mu_b^{\mathrm{cc}}}{\mu_b^{\mathrm{surf}}} = \exp\left(\frac{\nu^{\mathrm{surf}}}{\nu^{\mathrm{cc}}}\right)$$

Red-shift at the surface:

$$\nu^{\mathrm{surf}} = \log(1 - 2GM/c^2R)$$

What is also known:

$P^{cc},\ \rho^{cc}$ Pressure and density at the crust-core interface (from core EOS)

$r^{cc},\ m^{cc}$ Radius of crust-core interface and the core mass
(from TOV solution in the core, for given central density)

TOV equation in the crust

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$

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≈ 1

TOV equation in the crust: mass of the crust

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2} \left(1 - \frac{2Gm}{c^2 r}\right)^{-1},$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho,$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2Gm}{c^2\, r^2} \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}.$$

$$\frac{\mathrm{d}m}{\mathrm{d}P} = -4\pi r^2 \left(\frac{Gm}{r^2}\right)^{-1} \left(1 - \frac{2Gm}{c^2 r}\right)$$

$$\Delta m_{\mathrm{crust}} = M_{\mathrm{crust}} = 4\pi R^2 \left(\frac{GM}{P_{\mathrm{cc}} R^2}\right)^{-1} \left(1 - \frac{2GM}{c^2 R}\right)$$

<u>Secondary school description:</u>

The gravitational force to the crust is balanced by the pressure at the crust bottom $P = \rho g h$

TOV equation in the crust: red-shift at cc interface

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

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$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho,$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2Gm}{c^2\, r^2} \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}.$$

Crust is thin and light, almost whole mass of the crust is located at the bottom of the crust:

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} \approx \frac{2GM}{c^2\, r^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1}; \quad M = M^{cc} + M^{\mathrm{crust}}$$

The same equation as in vacuum!

$$\nu^{\mathrm{cc}} = \log(1 - 2GM/c^2 r^{\mathrm{cc}})$$

Building NS model without specifying crust

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1},$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho,$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2Gm}{c^2 r^2}\left(1+\frac{4\pi P r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1}.$$

Thermodynamics:

Known
(for given core EOS)

$$\frac{\mu_b^{\mathrm{cc}}}{\mu_b^{\mathrm{surf}}} = \exp\left(\frac{\nu^{\mathrm{surf}}}{\nu^{\mathrm{cc}}}\right) = \frac{1-2GM/c^2R}{1-2GM/c^2 r^{cc}}$$

Red-shift at the surface:

$$\nu^{\mathrm{surf}} = \log(1-2GM/c^2R)$$

Red-shift at the cc interface:

$$\nu^{\mathrm{cc}} = \log(1-2GM/c^2 r^{\mathrm{cc}})$$

NS mass

$$M = M^{cc} + M^{\mathrm{crust}}$$

The only unknown quantity is NS radius! => it can be determined

Building NS model without specifying crust

Zdunik et al., A&A 599 (2017), A119

Tolman-Oppenheimer-Volkoff equations

Thermodynamics:

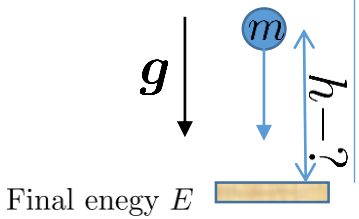
Known
(for given core EOS)

$$\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}} = \exp\left(\frac{\nu^{\rm surf}}{\nu^{\rm cc}}\right) = \frac{1-2GM/c^2R}{1-2GM/c^2r^{\rm cc}}$$

Analogies

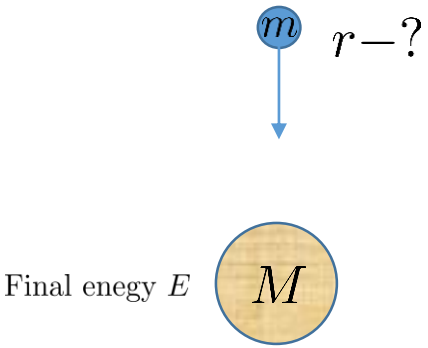
Junior high school

A body of mass m is dropped from some height above the floor.
When it reaches the floor, it has energy E. From what height was the body dropped?



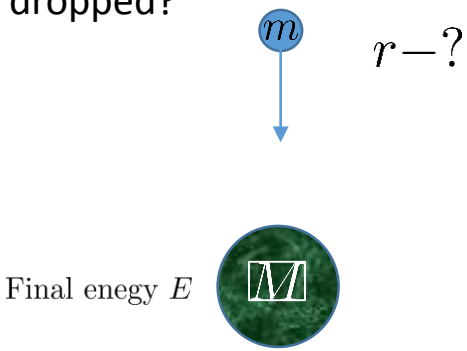
Senior high school

A body of mass m is dropped from some distance from the center of a planet with mass M.
When it reaches the planet, it has energy E. From what distance the body dropped?

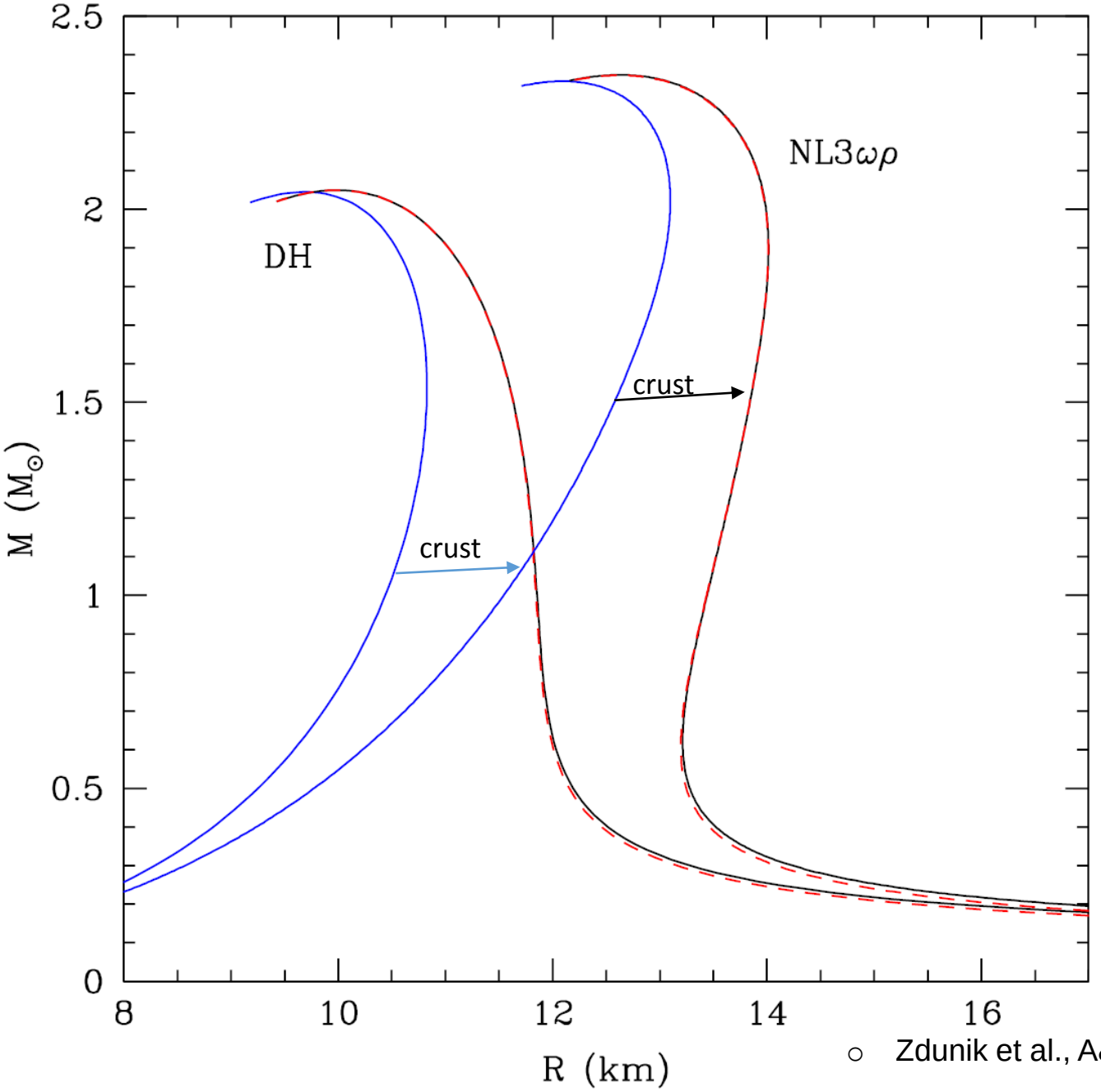


University (GR effects)

A body of mass m is dropped from some distance form the center of a NS with mass M.
When it reaches NS, it has locallt measured energy E (including the rest mass). From what distance the body dropped?



Building NS model without specifying crust



- Zdunik et al., A&A 599 (2017), A119

Can we do even better?

The main approximations:

- The mass of the crust was estimated neglecting variation of *r*
- Whole mass of the crust assumed to be located at the crust-core interface

<u>How to improve:</u>

Apply some physically-motivated analytical model to improve description of the innermost part of the crust (the most massive part):

<u>Polytrope EOS:</u>

$$P = P^{cc}\left(\frac{\rho}{\rho^{cc}}\right)^{\gamma}; \quad \gamma = 1 + \frac{1}{n}$$

Density distribution in the crust (neglecting self-gravity):

$$\rho(r) = \rho^{cc}\left[1 + \left(1 - \frac{1}{\gamma}\right)\frac{\rho^{cc}c^2}{P^{cc}}\ln\left(\frac{1 - r_G^{cc}/r^{cc}}{1 - r_G^{cc}/r}\right)\right]^n$$

$$r_G^{cc} = \frac{2GM^{cc}}{c^2},$$

- K.-E. Bekker, AIC, D.D. Ofengeim

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TOV equation:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho,$$
$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2Gm}{c^2\,r^2}\left(1 - \frac{r_G}{r}\right)^{-1}.$$

$$\frac{\mu_b^{\mathrm{cc}}}{\mu_b^{\mathrm{surf}}} = \exp\left(\frac{\nu^{\mathrm{surf}}}{\nu^{\mathrm{cc}}}\right)$$

Improved estimates for

$$M^{\mathrm{crust}}, \quad \nu^{\mathrm{cc}}$$

Improved estimated for *R*

- K.-E. Bekker, AIC, D.D. Ofengeim

M(R) within polytropic approximation for crust

Zdunik et al., A&A 599 (2017), A119:

$$M^0_{\rm crust} = 4\pi\,(r^{cc})^3\,\frac{P^{cc}}{c^2}\,\frac{r^{cc}}{r^{cc}_G}\left(1-\frac{R^{cc}_G}{r^{cc}}\right) \qquad R^{cc}_G = 2\frac{G\,M^{cc}}{c^2}$$

$$R^0 = R^0_G \Big/ \left\{1-\left(1-\frac{R^0_G}{r^{cc}}\right)\,\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}}\right\}, \qquad R^0_G = 2\frac{G\left(M^{cc}+M^0_{\rm crust}\right)}{c^2}$$

Polytropic approximation (K.-E. Bekker, AIC, D.D. Ofengeim, 2026):

$$M^{\rm poly}_{\rm crust} = M^0_{\rm crust}\left[1+\frac{n+1}{n+2}\frac{P^{cc}}{c^2\rho^{cc}}\,\frac{r^{cc}}{R^{cc}_G}\left(4-3\,\frac{R^{cc}_G}{r^{cc}}\right)\right]$$

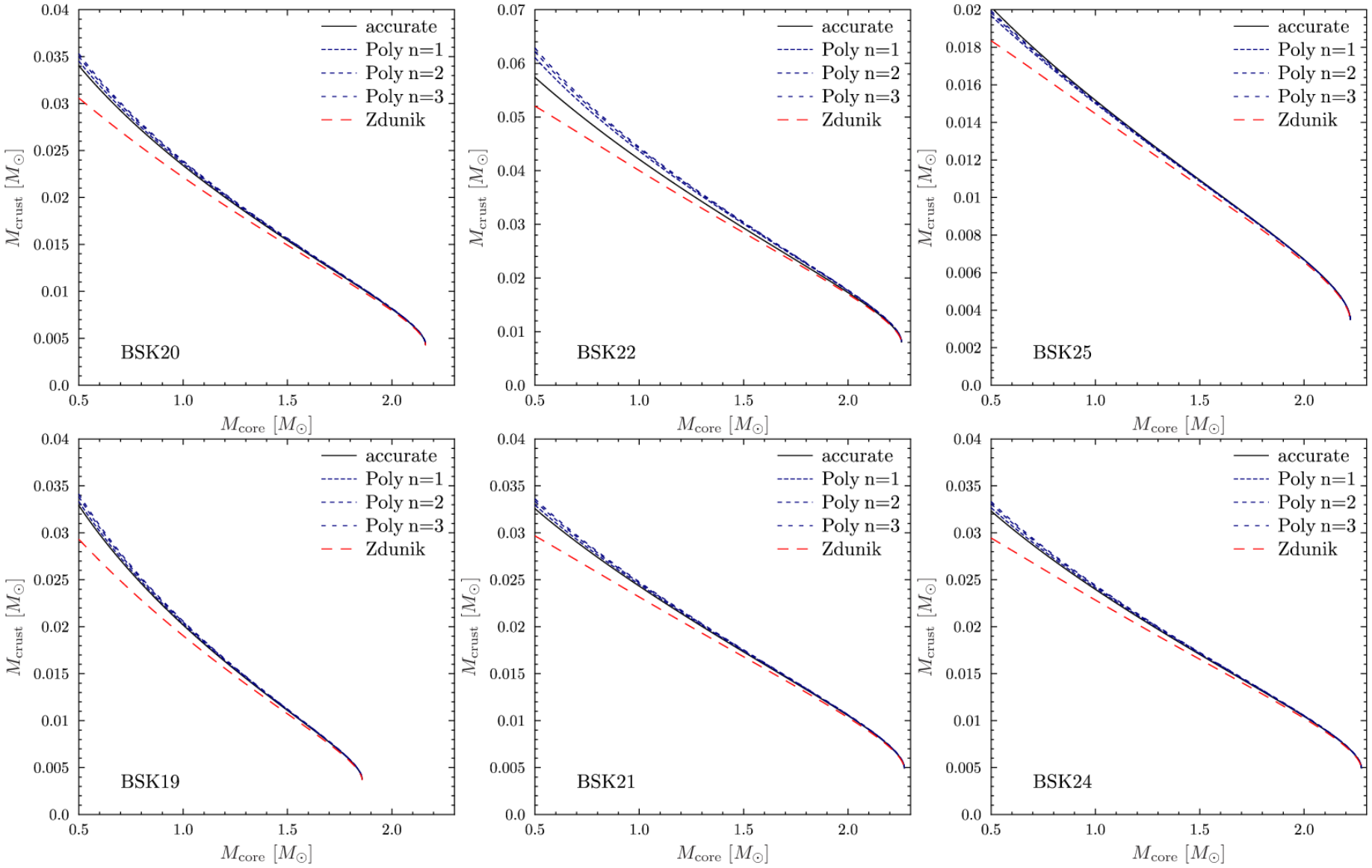
$$R_G = 2\frac{G\left(M^{cc}+M^{\rm poly}_{\rm crust}\right)}{c^2}$$

$$R^{\rm poly} = R_G \Big/ \left\{1-\left(1-\frac{R_G}{r^{cc}}\right)\,\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}}\right\} + 2\frac{n+1}{n+2}\,\frac{2G\,M^0_{\rm crust}}{c^2}\,\frac{P^{cc}}{c^2\rho_{cc}}\left(\frac{r^{cc}}{R_G}\right)^2\frac{\mu_b^{\rm surf}}{\mu_b^{\rm cc}}$$

Polytropic correction is not sensitive for n:

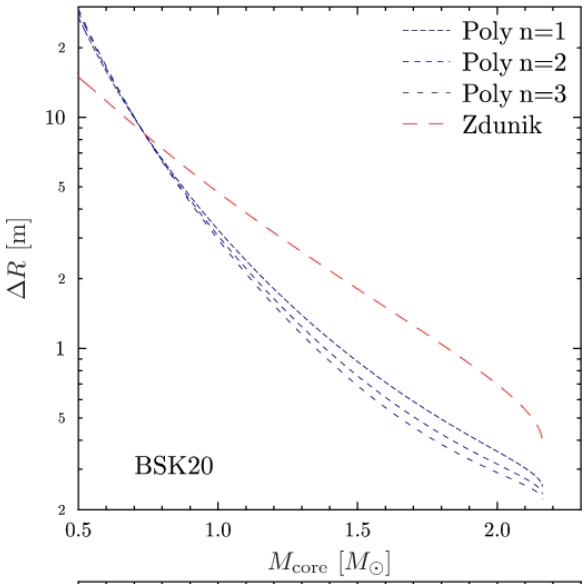
$$\propto \frac{n+1}{n+2} = (0.67 \div 0.8) \text{ for } n = 1 \div 3$$

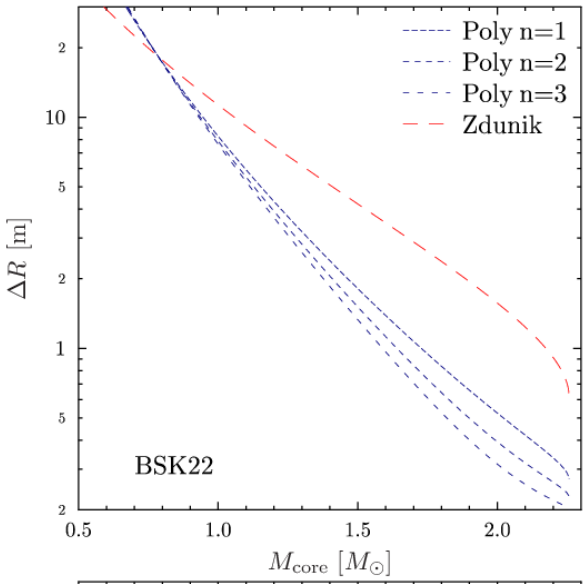
M(R) within polytropic approximation for crust

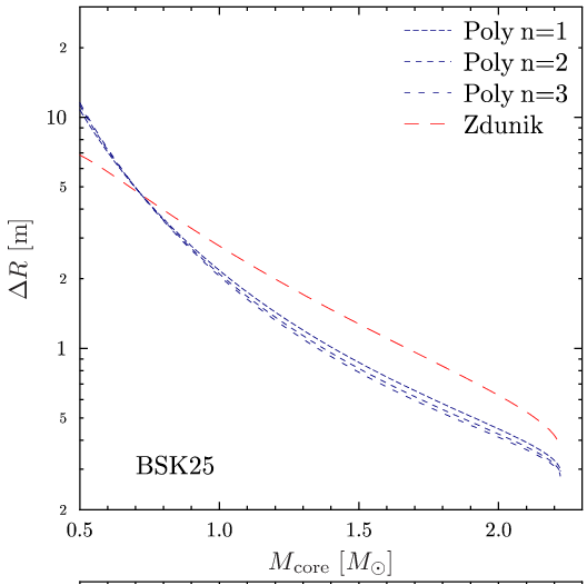


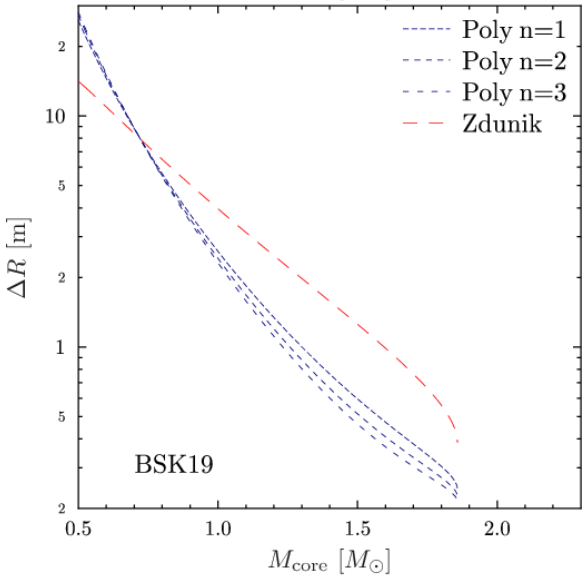
- Mass of the crust depend on EOS, but well predicted by polytrophic model for $n \gtrsim 1$
  - K.-E. Bekker, AIC, D.D. Ofengeim

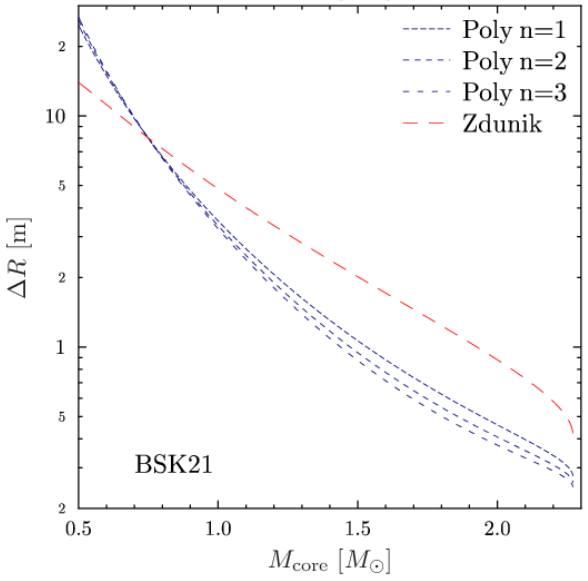
M(R) within polytropic approximation for crust

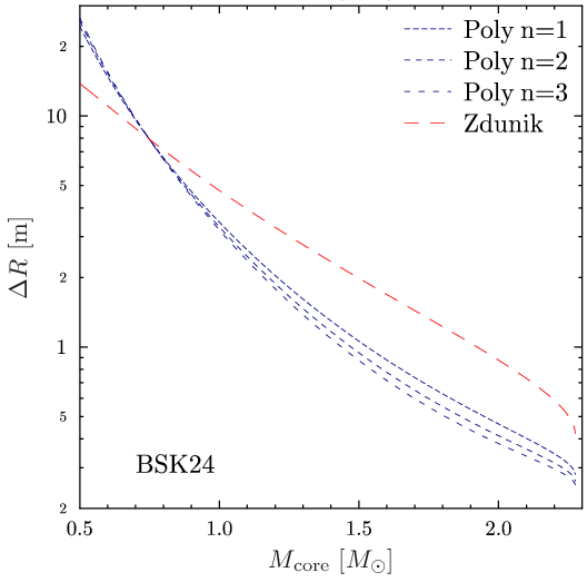






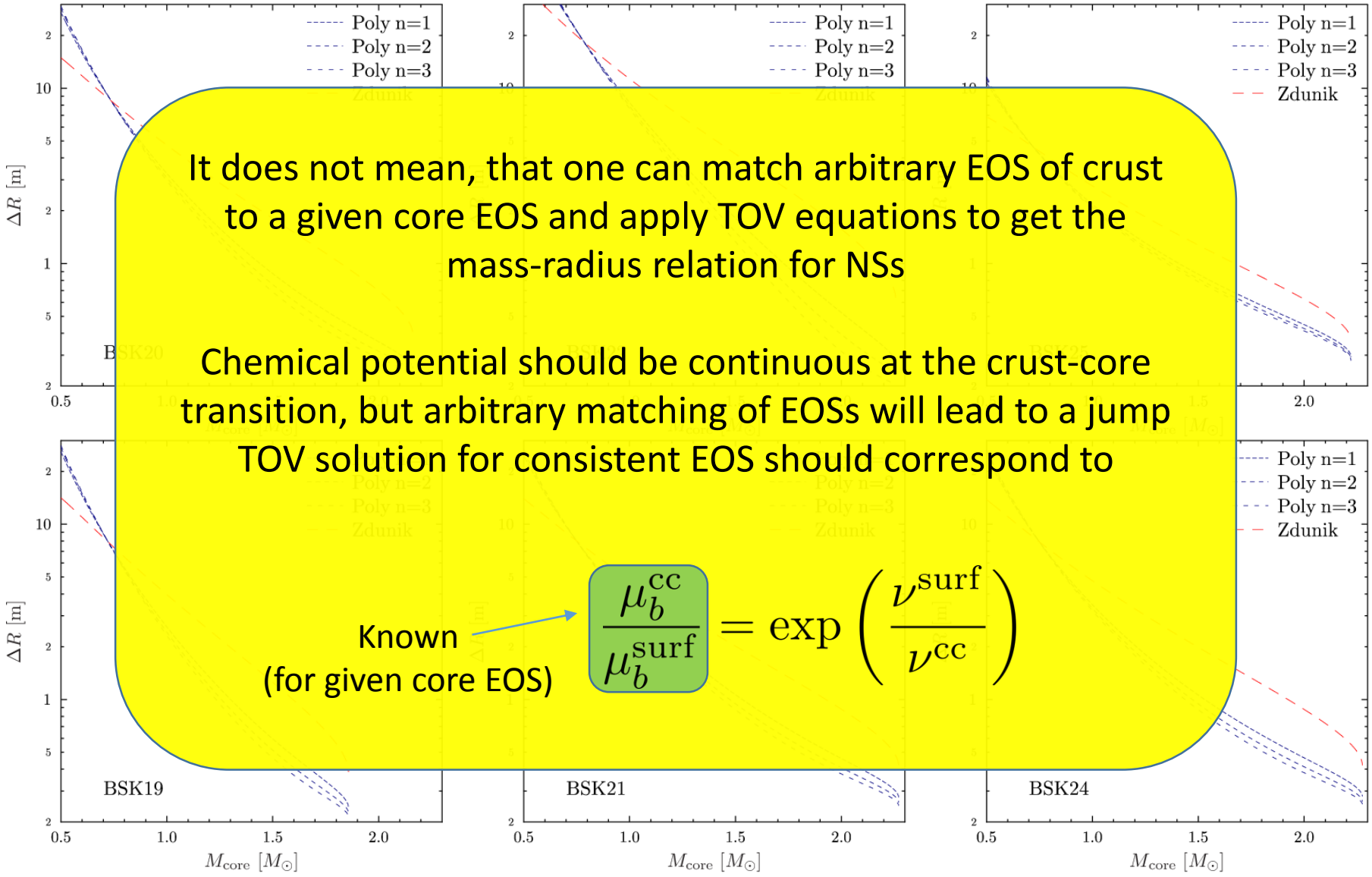






- Radius is well predicted by polytrophic model for $n \gtrsim 1$
  - K.-E. Bekker, AIC, D.D. Ofengeim

M(R) within polytropic approximation for crust



It does not mean, that one can match arbitrary EOS of crust to a given core EOS and apply TOV equations to get the mass-radius relation for NSs

Chemical potential should be continuous at the crust-core transition, but arbitrary matching of EOSs will lead to a jump

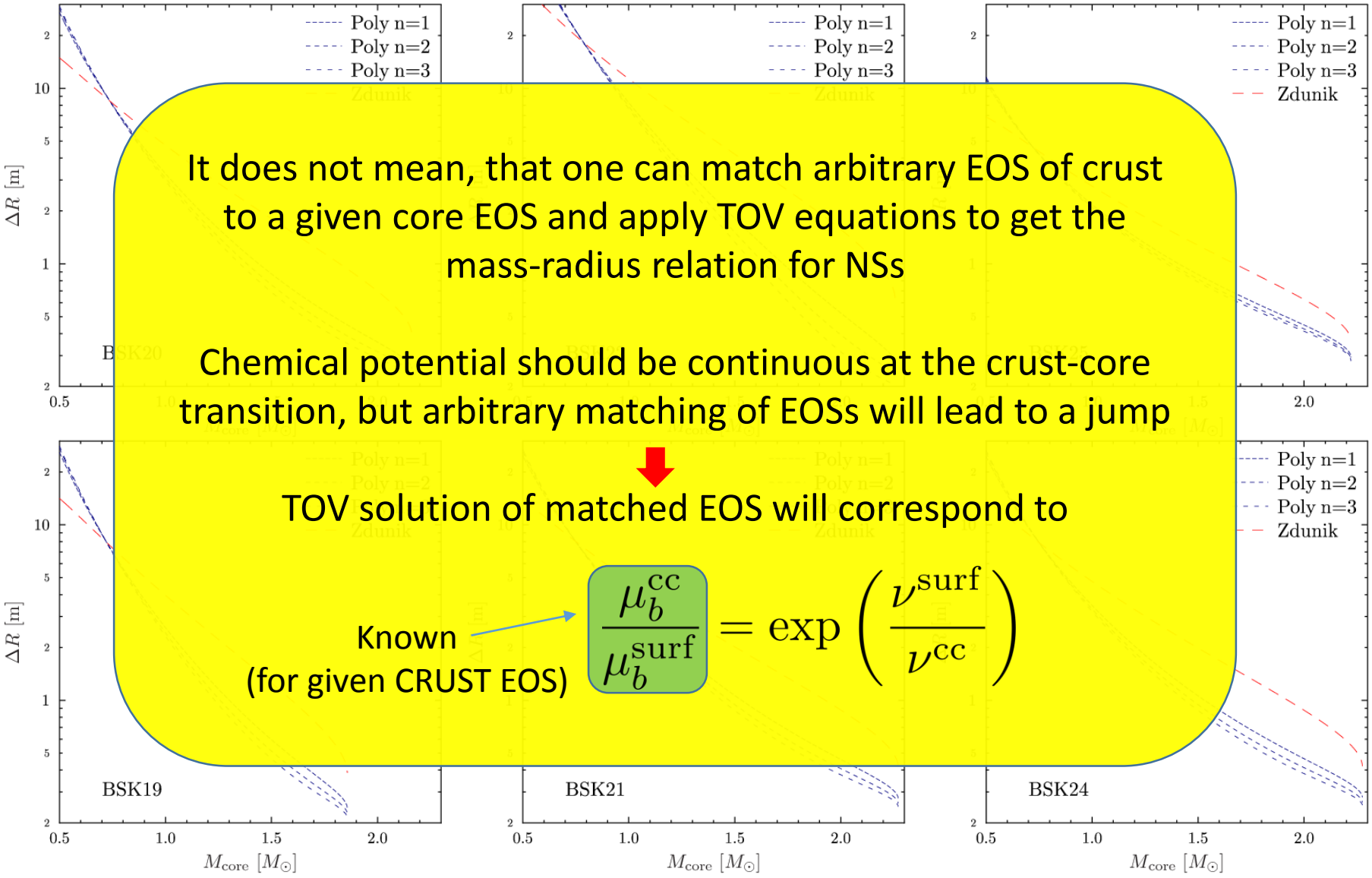
TOV solution for consistent EOS should correspond to

Known (for given core EOS)

$$\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}} = \exp\left(\frac{\nu^{\rm surf}}{\nu^{\rm cc}}\right)$$

- Radius is well predicted by polytrophic model for $M \gtrsim 1 M_\odot$ and $n \gtrsim 1$
  - K.-E. Bekker, AIC, D.D. Ofengeim

M(R) within polytropic approximation for crust



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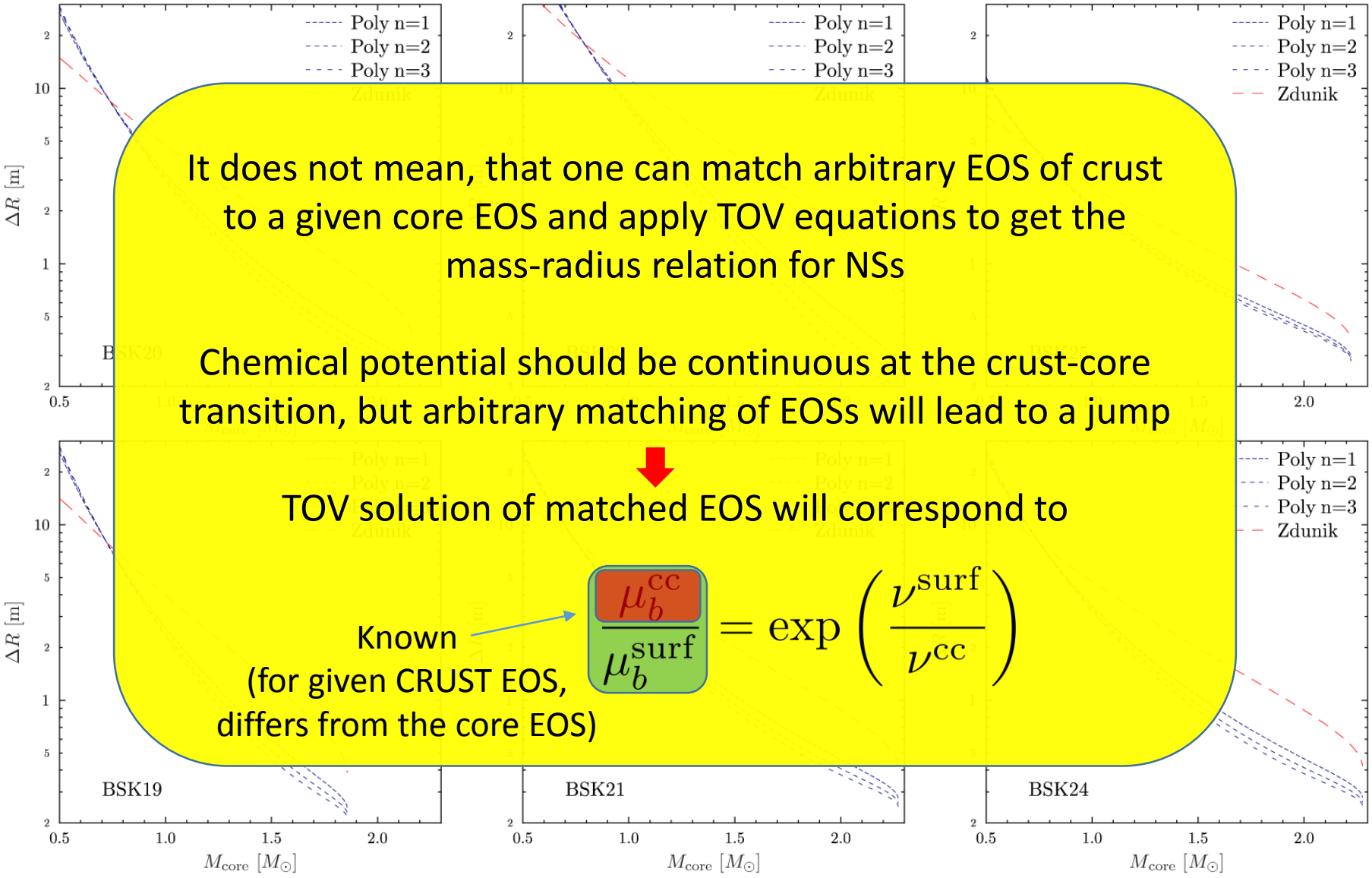
TOV solution of matched EOS will correspond to

Known (for given CRUST EOS)

$$\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}} = \exp\left(\frac{\nu^{\rm surf}}{\nu^{\rm cc}}\right)$$

- Radius is well predicted by polytrophic model for $M \gtrsim 1 M_\odot$ and $n \gtrsim 1$
  - K.-E. Bekker, AIC, D.D. Ofengeim

M(R) within polytropic approximation for crust



It does not mean, that one can match arbitrary EOS of crust to a given core EOS and apply TOV equations to get the mass-radius relation for NSs

Chemical potential should be continuous at the crust-core transition, but arbitrary matching of EOSs will lead to a jump

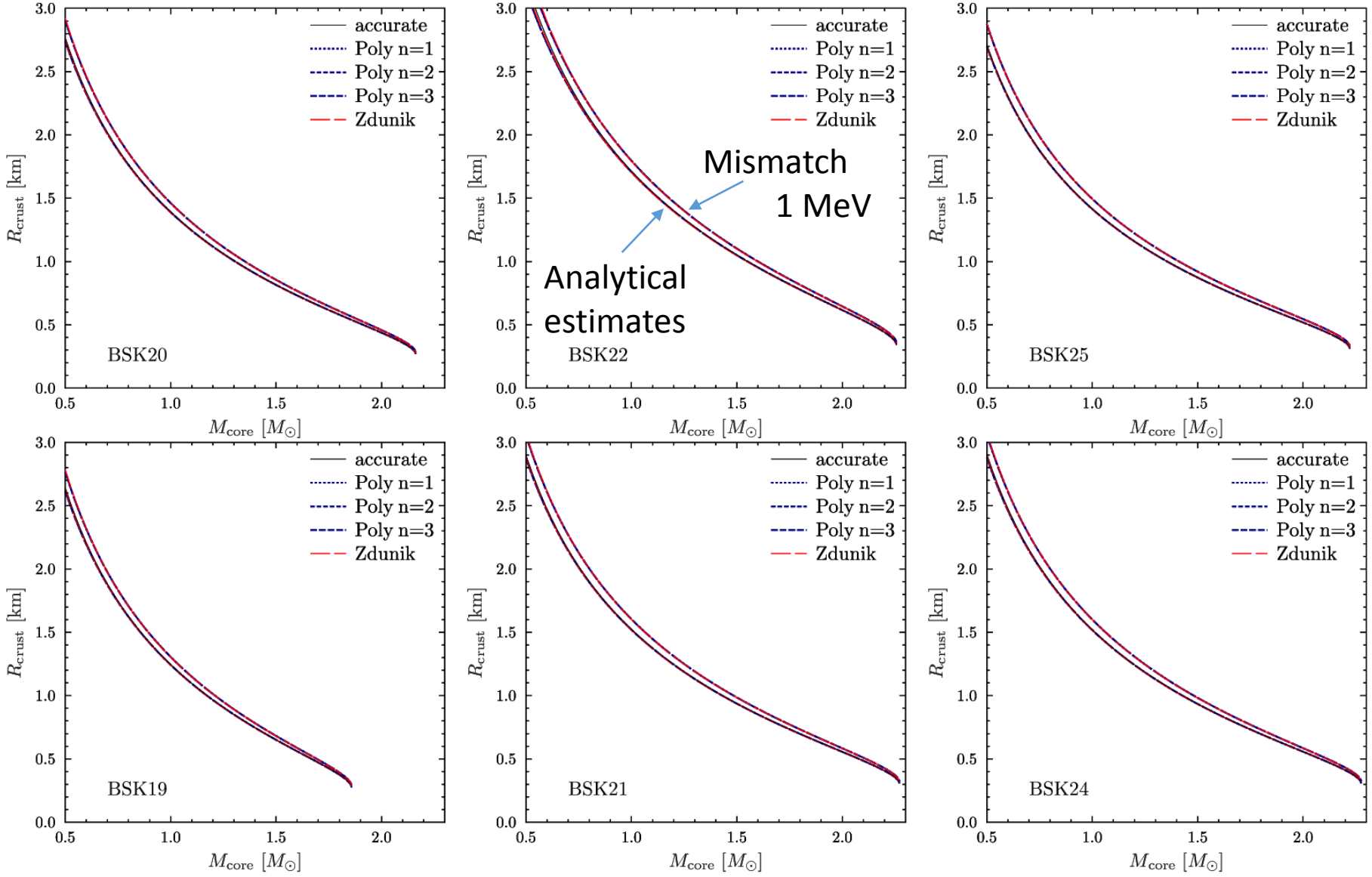
TOV solution of matched EOS will correspond to

Known (for given CRUST EOS, differs from the core EOS)

$$\frac{\mu_b^{\rm cc}}{\mu_b^{\rm surf}} = \exp\left(\frac{\nu^{\rm surf}}{\nu^{\rm cc}}\right)$$

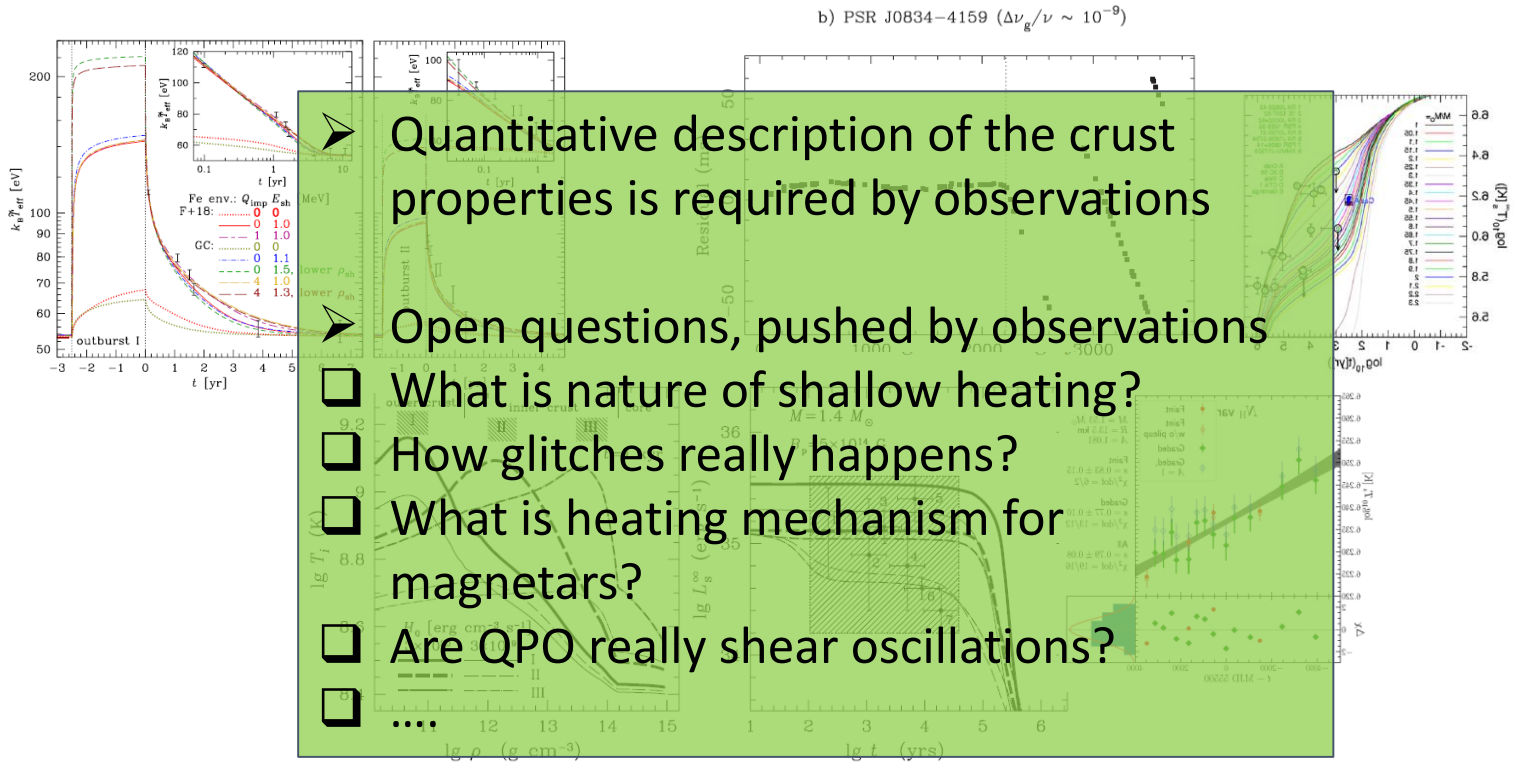
- Radius is well predicted by polytrophic model for $M \gtrsim 1 M_\odot$ and $n \gtrsim 1$
  - K.-E. Bekker, AIC, D.D. Ofengeim

M(R) within polytropic approximation for crust



1 MeV mismatch of chemical potential affects radius! It is better to apply analytics
(to beat 1m precision EOS should be known at 10^{-3} accuracy)

Future of crust physics



- Quantitative description of the crust properties is required by observations
- Open questions, pushed by observations
  - ❑ What is nature of shallow heating?
  - ❑ How glitches really happens?
  - ❑ What is heating mechanism for magnetars?
  - ❑ Are QPO really shear oscillations?
  - ❑ ….

Physics of a neutron star crust: summary

"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein



- Physics of NS crust deals with almost the same problems as terrestrial solid state physics. Similarly, the (unattainable) goal is to describe all properties on the base of ab initio simulations, and we done it for OCP (but don't forget the limitations). Generally, we need to appeal to simplified models.



- Some models (degenerate electrons, ion sphere model, Voigt shear modulus) can be solved analytically (even in 2020s). It's amazing!
- We need carefully inspect models: Even a small details can reveal inconsistencies/errors. If it is a case, one should make a back step



- The progress in observations puts higher requirements for the models: models which was 'possible' yesterday can become too simple tomorrow.



- Some observations of neutron stars reveal that our models are simpler, than it is possible. We need to work!



One can calculate M(R)
without dealing with crust EOS

