$$\mathcal{D}f_a \equiv \frac{\partial f_a}{\partial t} + \frac{\partial \varepsilon_a}{\partial \boldsymbol{p}} \nabla f_a - \left( \frac{\partial \varepsilon_a}{\partial \mathbf{r}} - q_a \left( \boldsymbol{E} + [\boldsymbol{v}_a \times \boldsymbol{B}] \right) \right) \nabla_{\boldsymbol{p}} f_a = I_a[\{f_b\}]$$

Yesterday: System of **integral-differential** kinetic equation was reduced to a system of linearized **integral** kinetic equations with different tensor dimensions decoupled

$$-\frac{1}{T} f'_F(x) \sum_{k|\ell_k=\ell} \sqrt{\frac{4\pi}{2\ell+1}} D^a_k(p) (\mathcal{X}_k)_{\ell m} = I^\ell_a[\{(\Phi_b)_{\ell m}\}] + i \frac{1}{T} f'_F(x) m \omega_{\mathrm{B}a}(p) (\Phi_a)_{\ell m}$$

# METHODS OF SOLUTION

# Relaxation time approximation

Consider for simplicity **single-component case**

The relaxation time approximation

$$I_a^\ell[\Phi_a] = -\frac{\overline{\delta f_a(\boldsymbol{p})}}{\tau_a^{(\ell)}(\varepsilon_a)} = \frac{1}{T} f_F'(x_a) \frac{(\Phi_a)_{\ell m}}{\tau_a^{(\ell)}(\varepsilon_a)}.$$

Energy-dependent relaxation times, different in general for different $\ell$
Can violate charge conservation, good for elastic collisions (quasiparticle energy unchanged during scattering)

$$-\frac{1}{T} f_F'(x) \sum_{k|\ell_k=\ell} \sqrt{\frac{4\pi}{2\ell+1}} D_k^a(p) (\mathcal{X}_k)_{\ell m} = \frac{1}{T} f_F'(x) \left( \frac{1}{\tau_a^{(\ell)}(\varepsilon_a)} + im\omega_{\mathrm{B}a}(p) \right) (\Phi_a)_{\ell m}$$

$$(\Phi_a)_{\ell m}(p) = \sum_k \left(\mathcal{G}_k^a(p)\right)_{\ell m} (\mathcal{X}_k)_{\ell m} \qquad (\mathcal{Y}_k)_{\ell m} = -\sum_{k'} L_{\ell m}^{kk'} (\mathcal{X}_{k'})_{\ell m}$$

$$L_{\ell m}^{kk'} = -\frac{g_{sa}}{2\pi^2(2\ell+1)} \int p^2 dp \, \frac{f_F'(x)}{T} D_k^a(p) D_{k'}^a(p) \frac{\tau_a^{(\ell)}(\varepsilon_a)}{1 + im\omega_{\mathrm{B}a}(p)\tau_a^{(\ell)}(\varepsilon_a)}$$

$g_{sa} = 2$

Symmetry relations

$$L_{\ell m}^{k'k} = L_{\ell m}^{kk'}$$

# Relaxation time approximation. Vector sector

As an example we consider thermal and electric conduction (i.e. driving force includes temperature gradient and the electric field). Tilde denotes m=1 component.

Linear flows

$$\widetilde{J}_q = -\widetilde{L}_{qq}\left(\widetilde{\frac{\nabla T}{T}} + \dot{\widetilde{V}}\right) + q_a\widetilde{L}_{qa}\widetilde{E}, \quad \widetilde{J} = -q_a\widetilde{L}_{qa}\left(\widetilde{\frac{\nabla T}{T}} + \dot{\widetilde{V}}\right) + q_a^2\widetilde{L}_{qq}\widetilde{E}$$

One usually defines thermal conductivity as a coefficient in absence of current, then constitutive relations are

$$\widetilde{J}_q = -\widetilde{\kappa}\left(\widetilde{\nabla T} + T\dot{\widetilde{V}}\right) - \widetilde{Q}_T T\widetilde{J}, \quad \widetilde{E} = \frac{\widetilde{J}}{\widetilde{\sigma}} - \widetilde{Q}_T\left(\widetilde{\nabla T} + T\dot{\widetilde{V}}\right)$$

$$T\widetilde{\kappa} = \widetilde{L}_{qq} - \frac{\widetilde{L}_{qa}^2}{\widetilde{L}_{aa}}, \quad \widetilde{\sigma} = q_a^2\widetilde{L}_{aa}, \quad T\widetilde{Q}_T = -\frac{\widetilde{L}_{aq}}{q_a\widetilde{L}_{qq}}$$

$$\left\{\begin{array}{c}\widetilde{L}_{qq}(B)\\ \widetilde{L}_{qa}(B)\\ \widetilde{L}_{aa}(B)\end{array}\right\} = -\frac{1}{3\pi^2}\int p^2 v_a^2 dp \frac{f_F'(x)}{T}\left\{\begin{array}{c}(\varepsilon_a - h_a)^2\\ (\varepsilon_a - h_a)\\ 1\end{array}\right\}\frac{\tau_a^{(1)}(\varepsilon_a)}{1 + i\omega_{\mathrm{B}a}\tau_a^{(1)}(\varepsilon_a)}.$$

$$\widetilde{\kappa} = \frac{\pi^2 n_a T}{3m_a^*}\frac{\tau_a^{(1)}(\mu_a)}{1 + i\omega_{\mathrm{BF}a}\tau_a^{(1)}(\mu_a)}$$

$$\widetilde{\sigma} = \frac{q_a^2 n_a}{m_a^*}\frac{\tau_a^{(1)}(\mu_a)}{1 + i\omega_{\mathrm{BF}a}\tau_a^{(1)}(\mu_a)},$$

In degenerate matter $h \approx \mu$

$$f_F'(x) \approx -\delta(x) - \frac{\pi^2}{6}\delta''(x)$$

Thermal diffusion suppressed

$$\widetilde{L}_{aq},\ \widetilde{Q}_T \approx 0$$

Wiedemann-Franz rule

$$\widetilde{\kappa}/\widetilde{\sigma} = \frac{q_a^2\pi^2 T}{3}$$

# Relaxation time approximation. Tensor sector

$$\ell = 2$$

Viscous stress tensor $\Pi_{2m} = -2\widetilde{\eta}(mB)V_{2m}$

Five shear viscosity coefficients are given by a single function of B

$$\widetilde{\eta} = -\frac{1}{15\pi^2}\int p^4 v_a^2 dp \frac{f_F'(x)}{T}\frac{\tau_a^{(2)}(\varepsilon_a)}{1+i\omega_{\mathrm{B}a}\tau_a^{(2)}(\varepsilon_a)},$$

In strongly degenerate matter $\widetilde{\eta} = \dfrac{n_a p_{\mathrm{F}a}^2}{5m_a^*}\dfrac{\tau_a^{(2)}(\mu_a)}{1+i\omega_{\mathrm{BF}a}\tau_a^{(2)}(\mu_a)}$

# Relaxation time approximation.

$$\widetilde{\kappa} = \frac{\pi^2 n_a T}{3m_a^*} \frac{\tau_a^{(1)}(\mu_a)}{1 + i\omega_{\mathrm{BF}a}\tau_a^{(1)}(\mu_a)}$$

$$\widetilde{\eta} = \frac{n_a p_{\mathrm{F}a}^2}{5m_a^*} \frac{\tau_a^{(2)}(\mu_a)}{1 + i\omega_{\mathrm{BF}a}\tau_a^{(2)}(\mu_a)}$$

Strong magnetization limit $x_{\mathrm{Hall}} \gg 1$

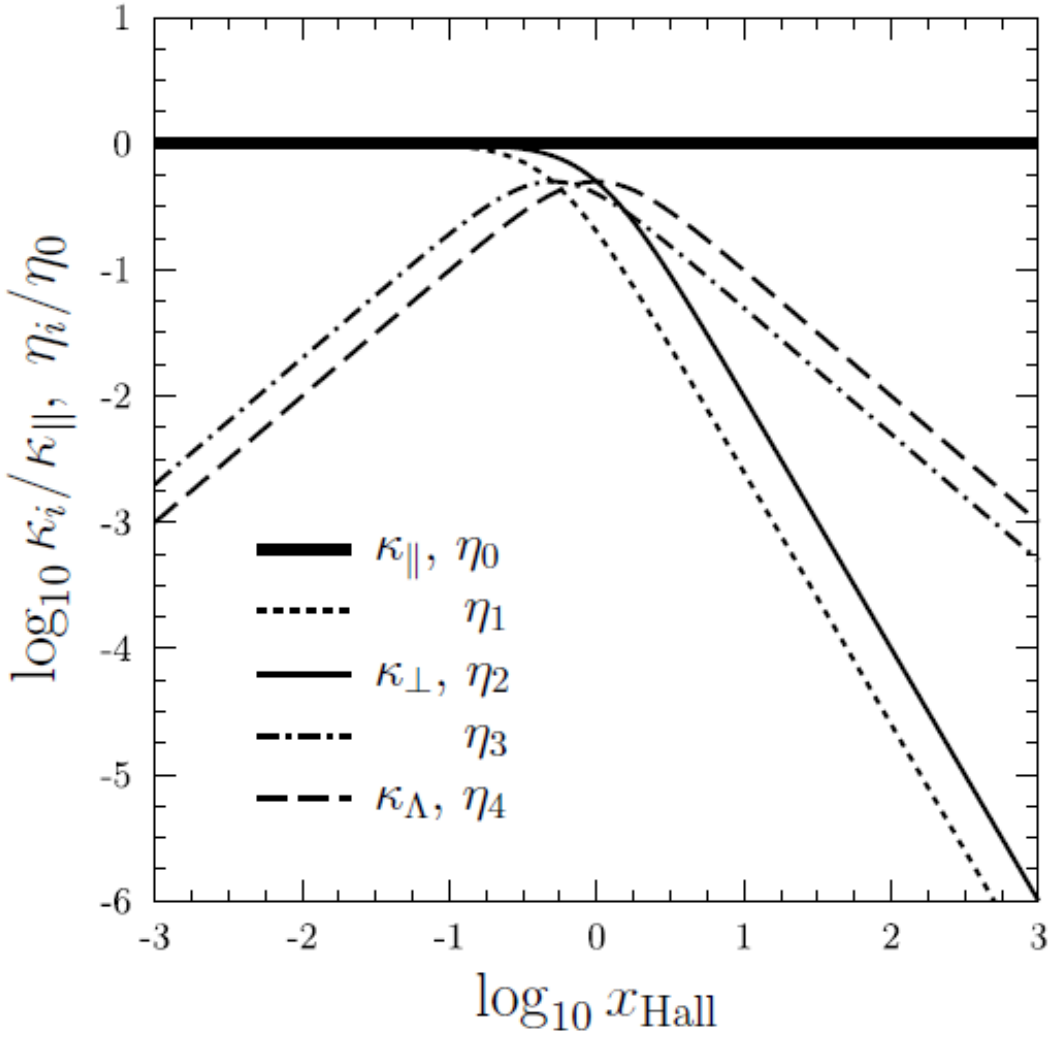
$$\kappa_\perp = \frac{\pi^2 n_a T}{3m_a^*} \frac{1}{\omega_{\mathrm{BF}a}^2 \tau_a^{(1)}(\mu_a)} \propto B^{-2}$$

$$\kappa_\Lambda = \frac{\pi^2 n_a T}{3m_a^*} \frac{1}{\omega_{\mathrm{BF}a}} \propto B^{-1}$$

$$\eta_1 = \frac{\eta_2}{4} = \frac{n_a p_{\mathrm{F}a}^2}{5m_a^*} \frac{1}{4\omega_{\mathrm{BF}a}^2 \tau_a^{(2)}(\mu_a)}$$

$$\eta_3 = \frac{\eta_4}{2} = \frac{n_a p_{\mathrm{F}a}^2}{5m_a^*} \frac{1}{2\omega_{\mathrm{BF}a}}$$

Hall parameter $x_{\mathrm{Hall}} = \omega_{\mathrm{BF}a}\tau_a(\mu_a)$



# Variational principle in kinetic theory

Let us write kinetic equation in a form $X = (\hat{L} + \hat{M})\Phi$ and omit M for a moment $X = \hat{L}\Phi$

Scalar product $\langle \Psi | \Phi \rangle = \int_{\boldsymbol{p}} \Psi(\boldsymbol{p})^* \Phi(\boldsymbol{p})$ Remind, that L is self-adjoint $\langle \Psi | \hat{L} | \Phi \rangle = \langle \Phi | \hat{L} | \Psi \rangle$

Entropy production functional $-T\varsigma[\Phi] = \langle \Phi | \hat{L} | \Phi \rangle$

Consider class of functions that satisfy $\langle \widetilde{\Phi} | X \rangle = \langle \widetilde{\Phi} | \hat{L} | \widetilde{\Phi} \rangle = -T\varsigma[\widetilde{\Phi}], \quad \widetilde{\Phi} \in \mathrm{Ker}^{\perp} \hat{L}$

Variation principle states that entropy production is maximal for the solution of the transport equation, i.e. $\varsigma[\widetilde{\Phi}] \leqslant \varsigma[\Phi]$

Proof $0 \leqslant -\langle (\Phi - \widetilde{\Phi}) | \hat{L} | (\Phi - \widetilde{\Phi}) \rangle = T\varsigma[\Phi] + T\varsigma[\widetilde{\Phi}] + 2\langle \widetilde{\Phi} | \hat{L} | \Phi \rangle = T\varsigma[\Phi] + T\varsigma[\widetilde{\Phi}] + 2\langle \widetilde{\Phi} | X \rangle = T\varsigma[\Phi] - T\varsigma[\widetilde{\Phi}]$

$$\Phi = \sum_k \mathcal{G}_k \cdot \mathcal{X}_k \qquad L_{kk'} = -\langle \mathcal{G}_k | \hat{L} | \mathcal{G}_{k'} \rangle$$

Gives upper bound on diagonal transport coefficients $0 \leqslant \widetilde{L}_{kk} \leqslant L_{kk}$ $\kappa \gtrsim 0, \ \eta \gtrsim 0, \ \zeta \gtrsim 0$

Single transport process $\widetilde{\Phi} = \mathcal{G}_k \cdot \mathcal{X}_k \qquad \widetilde{\mathcal{Y}}_k = \langle \widetilde{\Phi_k} | X \rangle \qquad \widetilde{L}_{kk} = \dfrac{T\varsigma[\widetilde{\Phi}]}{\mathcal{X}_k \cdot \mathcal{X}_k} = \dfrac{\mathcal{Y}_k \cdot \mathcal{Y}_k}{T\varsigma[\widetilde{\Phi}]}$

# Variational principle in kinetic theory

$$\langle \widetilde{\Phi} | X \rangle = \langle \widetilde{\Phi} | \hat{L} | \widetilde{\Phi} \rangle$$

In practice one expands the solution over the appropriate set of functions

$$\widetilde{\Phi} = \sum_{n=1}^{N} c_n \Phi_n$$

Variation over $c_n$

$$\langle \Phi_n | X \rangle = \sum_m \langle \Phi_n | \hat{L} | \Phi_m \rangle c_m \quad (*)$$

This form works for symmetric operators and variational principle guarantees convergence

In magnetized case, the magnetic operator does not contribute to the entropy production and the formulation of the variational principle is ambiguous. One still uses the approximation (*). This is known as Bubnov-Galerkin method. One uses expansion in appropriate functions (these turn out to be associated Laguerre (Sonine) polynomials in classical gases with Maxwell equilibrium distribution)

$$\langle \Phi_n | X \rangle = \sum_m \langle \Phi_n | \hat{L} + \hat{M} | \Phi_m \rangle c_m$$

For standard Fermi-liquids the appropriate basis functions can be found (at least in the non-magnetic case) and the expressions for the transport coefficients can be given in form converging series as we see below. Moreover, the simplest variational approximation based on a single basis function is usually enough.

# Strongly degenerate matter

# Simplifications for degenerate matter

$$-\frac{1}{T}f'_F(x)\sum_{k|\ell_k=\ell}\sqrt{\frac{4\pi}{2\ell+1}}D^a_k(p)(\mathcal{X}_k)_{\ell m}=I^\ell_a[\{(\Phi_b)_{\ell m}\}]+i\frac{1}{T}f'_F(x)m\omega_{\mathrm{B}a}(p)(\Phi_a)_{\ell m}$$

$$I^\ell_{ab} = -\frac{1}{T(1+\delta_{ab})}\int_{\boldsymbol{p}_{1'}}\int_{\boldsymbol{p}_2}\int_{\boldsymbol{p}_{2'}} w_{ab}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_{1'},\boldsymbol{p}_{2'})\mathcal{F}_{ab}\left[(\Phi_a(p_1))_{\ell m}+(\Phi_b(p_2))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)-(\Phi_a(p_{1'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{1'})-(\Phi_b(p_{2'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{2'})\right]$$

$$\mathcal{F}_{ab}=f_1^{\mathrm{eq}}f_2^{\mathrm{eq}}(1-f_{1'}^{\mathrm{eq}})(1-f_{2'}^{\mathrm{eq}}) \qquad w_{ab}=(2\pi)^4\delta(\varepsilon_1+\varepsilon_2-\varepsilon_{1'}-\varepsilon_{2'})\delta(\boldsymbol{p}_{1'}+\boldsymbol{p}_{2'}-\boldsymbol{p}_1-\boldsymbol{p}_2)|T_{ab}(12\to 1'2')|^2$$

The factor $\mathcal{F}_{ab}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_{1'}-\varepsilon_{2'})$ ensures that all quasiparticles scatter in the vicinity of the Fermi surface

$$\mu_a+\mu_b\gtrsim\varepsilon_1+\varepsilon_2=\varepsilon_{1'}+\varepsilon_{2'}\gtrsim\mu_a+\mu_b$$

Angular-energy decomposition

$$\frac{1}{T}f'_F(x_a)\approx-\delta(\varepsilon_a-\mu_a) \qquad h\approx\mu \qquad \mathrm{d}\boldsymbol{p}_1\mathrm{d}\boldsymbol{p}_2\mathrm{d}\boldsymbol{p}_{2'}\delta(\Delta\varepsilon)\propto T^2\Rightarrow\tau\propto T^{-2}$$

$$\mathrm{d}\boldsymbol{p}=m^*p_FT\mathrm{d}x\mathrm{d}\Omega_{\boldsymbol{p}}$$

$$x=\frac{\varepsilon-\mu}{T}\to-\infty\ldots\infty$$

Collision integral conserves parity in x hence thermal diffusion vanishes in a first approximation

| Force $(k)$ | $\ell$ | $D^a_k$ | $\mathcal{X}_k$ |
|---|---|---|---|
| $\kappa$ | 1 | $v_a(\varepsilon_a-h_a)$ | $\frac{\nabla T}{T}+\dot{\boldsymbol{V}}$ |
| $Da$ | 1 | $v_a$ | $\boldsymbol{d}_{(a)}+\frac{h_a}{h\eta}[\boldsymbol{J}\times\boldsymbol{B}]$ |
| $\eta$ | 2 | $\sqrt{2/3}\,(pv_a)$ | $\hat{V}$ |
| $\zeta$ | 0 | $1/3(pv_a)+nT\frac{\mathrm{d}}{\mathrm{d}n}(\varepsilon_a-\mu_a)/T$ | $\mathrm{div}\boldsymbol{V}$ |

# Variational solution for thermal conductivity and shear viscosity

$$-\frac{1}{T}f'_F(x)\sum_{k|\ell_k=\ell}\sqrt{\frac{4\pi}{2\ell+1}}D_k^a(p)(\mathcal{X}_k)_{\ell m}=I_a^\ell[\{(\Phi_b)_{\ell m}\}]+i\frac{1}{T}f'_F(x)m\omega_{\mathrm{B}a}(p)(\Phi_a)_{\ell m}$$

| Force $(k)$ | $\ell$ | $D_k^a$ | $\mathcal{X}_k$ |
|---|---|---|---|
| $\kappa$ | 1 | $v_aTx_a$ | $\frac{\nabla T}{T}+\dot{\boldsymbol{V}}$ |
| $Da$ | 1 | $v_a$ | $\boldsymbol{d}_{(a)}+\frac{h_a}{h\eta}[\boldsymbol{J}\times\boldsymbol{B}]$ |
| $\eta$ | 2 | $\sqrt{2/3}\,(pv_a)$ | $\hat{V}$ |
| $\zeta$ | 0 | $1/3(pv_a)+nT\frac{\mathrm{d}}{\mathrm{d}n}(\varepsilon_a-\mu_a)/T$ | $\mathrm{div}\boldsymbol{V}$ |

**Simplest anzats**

$$\left(\Phi_a^k(p_1)\right)_{\ell 1}=-\sqrt{\frac{4\pi}{2\ell+1}}D_k(p)\widetilde{\lambda}_a^k\,(\mathcal{X}_k)_{\ell 1}$$

**Variational parameter (complex mean free path)** $\lambda_a^k=\tau_a^k v_{\mathrm{F}a}\propto T^{-2}$

$$\langle\Phi_n|X\rangle=\sum_{m=1}^{N}\langle\Phi_n|\hat{L}|\Phi_m\rangle c_m\rightarrow\quad 1=\sum_b\left(\Lambda_{ab}^k\widetilde{\lambda}_a^k+\Lambda'^k_{ab}\widetilde{\lambda}_b^k\right)+i\frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}}\widetilde{\lambda}_a^k$$

**Transport matrix**

$$1=\left(\Lambda_a^k+i\frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}}\right)\widetilde{\lambda}_a^k+\sum_{b\neq a}\Lambda'^k_{ab}\widetilde{\lambda}_b^k,\qquad \Lambda_a^k=\sum_b\Lambda_{ab}^k+\Lambda'^k_{aa}.$$

**Elements of transport matrix are some averages of transition probability. May be called generalized transport cross-section**

**Thermal conductivity**

$$\widetilde{\kappa}_a=\frac{\pi^2 n_a T}{3p_{\mathrm{F}a}}\widetilde{\lambda}_a^\kappa\propto T^{-1}$$

**Shear viscosity**

$$\widetilde{\eta}_a=\frac{n_a p_{\mathrm{F}a}}{5}\widetilde{\lambda}_a^\eta\propto T^{-2}$$

Now the system of transport equations has reduced to a system of ordinary linear equations

# Diffusion

$$-\frac{1}{T}f'_F(x_a)\boldsymbol{v}_a\left(\boldsymbol{d}_{(a)}+\frac{h_a}{hn}[\boldsymbol{J}\times\boldsymbol{B}]\right)=\sum_b I_{ab\,\mathrm{lin}}[\Phi]+\frac{q_a}{T}f'_F(x_a)\,[\boldsymbol{v}_a\times\boldsymbol{B}]\,\nabla_{\boldsymbol{p}_1}\Phi_a(\boldsymbol{p}_1)$$

Simplest anzats

$$\Phi_a(\boldsymbol{p}_1)=\boldsymbol{p}_1\boldsymbol{w}_{(a)}$$

Diffusion velocity

$$\Delta\boldsymbol{j}_{(a)}=n_a\boldsymbol{w}_{(a)}$$

Multiplying by $\boldsymbol{p}_1$ and integrating one obtains the so-called generalized Ohm law (Stephan-Maxwell form of linear relations)

$$n_a\left(\boldsymbol{d}_{(a)}+\frac{h_a}{hn}\left[\boldsymbol{J}\times\boldsymbol{B}\right]\right)=\int_{\boldsymbol{p}_1}\boldsymbol{p}_1 I_a[\{\Phi_b\}]+q_a\left[\Delta\boldsymbol{j}_{(a)}\times\boldsymbol{B}\right]=\sum_b J_{ab}(\boldsymbol{w}_b-\boldsymbol{w}_a)+n_a q_a[\boldsymbol{w}_a\times\boldsymbol{B}]$$

$$\boldsymbol{d}_{(a)}=T\nabla\frac{\mu_a}{T}-T\frac{h_a}{h}\sum_b y_b\nabla\frac{\mu_b}{T}-q_a\boldsymbol{E}\qquad\sum_a\boldsymbol{d}_{(a)}=0$$

Momentum transfer rates (friction coefficients)

$$J_{ab}\propto T^2$$

Entropy production in the diffusion

$$T\varsigma=\frac{1}{2}\sum_{ab}J_{ab}(\boldsymbol{w}_b-\boldsymbol{w}_a)^2\geqslant 0$$

# Diffusion

$$n_a\left(\boldsymbol{d}_{(a)}+\frac{h_a}{hn}\left[\boldsymbol{J}\times\boldsymbol{B}\right]\right)=\sum_b J_{ab}(\boldsymbol{w}_b-\boldsymbol{w}_a)+n_a q_a[\boldsymbol{w}_a\times\boldsymbol{B}] \qquad J_{ab}=J_{ba}$$

$$\boldsymbol{d}_{(a)}=T\nabla\frac{\mu_a}{T}-T\frac{h_a}{h}\sum_b y_b\nabla\frac{\mu_b}{T}-q_a\boldsymbol{E} \qquad \sum_a \boldsymbol{d}_{(a)}=0$$

The system is not readily invertible because of the linear dependence of left-hand side

Let us fix the frame by some non-zero constants $\sum_a \mathfrak{Z}_a\boldsymbol{w}_a=0,\quad \sum_a \mathfrak{Z}_a=1$

In other frame

First – non-magnetized case $n_a\boldsymbol{d}_a=\sum_b J_{ab}(\boldsymbol{w}_b-\boldsymbol{w}_a)$

$$\overline{\mathfrak{D}}_{ab}=\mathfrak{D}_{ab}-\sum_c \overline{\mathfrak{Z}}_c\mathfrak{D}_{cb}-\sum_c \overline{\mathfrak{Z}}_c\mathfrak{D}_{ac}+\sum_{cd}\overline{\mathfrak{Z}}_c\overline{\mathfrak{Z}}_d\mathfrak{D}_{cd}$$

Assume that we constructed one of the inversions $\boldsymbol{w}_a=-\sum_b \mathfrak{D}_{ab}n_b\boldsymbol{d}_b$

Let us demand from aesthetic point of view that the diffusion tensor should be symmetric. Then it also follows the condition

$$\sum_a \mathfrak{Z}_a\mathfrak{D}_{ab}=\sum_b \mathfrak{D}_{ab}\mathfrak{Z}_b=0$$

Curtiss et al have shown that one can obtain symmetric diffusion tensor by solving the following well-defined system

$$\sum_c\left(J_{ac}-\frac{\mathfrak{Z}_c}{\mathfrak{Z}_a}J_{aa}\right)\mathfrak{D}_{cb}=-\delta_{ab}+\mathfrak{Z}_a$$

Here $\sum_a J_{ab}=0$

# Diffusion/electric conductivity. Magnetized case

Center of mass frame

$$\boldsymbol{u}_g = \sum_a \mathfrak{x}_a \boldsymbol{w}_a, \quad \mathfrak{x}_a = \frac{n_a h_a}{nh}$$

Going to circular components of vector, the generalized Ohm law reads

$$n_a \widetilde{d}_a - i\mathfrak{x}_a \widetilde{J} B = \sum_b J_{ab}(\widetilde{w}_b - \widetilde{w}_a) - iq_a n_a \widetilde{w}_a B$$

$$\widetilde{J} = \sum_a n_a q_a \widetilde{w}_a$$

Formal inversion based on the non-magnetic case

$$\widetilde{w}_a = -\sum \mathfrak{D}_{ab}\left(n_b \widetilde{d}_b + iq_b n_b \widetilde{w}_b B - i\mathfrak{x}_b B \widetilde{J}\right)$$

At this step it is possible to expand current in diffusion velocities and perform another inversion. The simplest way to do this in the c.m. frame

Assuming now that $\boldsymbol{d}_{(a)} = -q_a \boldsymbol{E}$ $\quad \widetilde{J} = \widetilde{\sigma}\widetilde{E}$

Electrical conductivity in the c.m. frame reads

$$\widetilde{\sigma} = \sum_{abc} q_a n_a \mathfrak{R}_{ac} \mathfrak{D}_{cb} n_b q_b$$

$$\mathfrak{R}_{ab} = (\delta_{ab} - i\mathfrak{D}_{ab} q_b B)^{-1}$$

# Electrical conductivity

Example solution for 3-component *npe* matter

$$\widetilde{R} = \widetilde{\sigma}^{-1} = \frac{J_{ep} - ien_e(1-2\mathfrak{X}_e^2)B}{n_e e^2} + \frac{\mathfrak{X}_n^2}{J_{pn}} B^2$$

One observes an increase of resistivity in magnetized multicomponent plasma with neutral species. Sometimes this effect is referred to as ambipolar diffusion

Example solution for 4-component *npe*μ matter

$$\begin{aligned}
\widetilde{R} = \widetilde{\sigma}^{-1} &= \frac{d_0 + iBd_1 - B^2 d_2}{a_0 + iBa_1} + \frac{\mathfrak{X}_n^2}{J_{pn}} B^2, \\
a_0 &= e^2(J_{ep}n_\mu^2 + J_{\mu p}n_e^2 + J_{e\mu}n_p^2), \\
a_1 &= -e^3 n_e n_\mu n_p, \\
d_0 &= J_{ep}J_{\mu p} + J_{e\mu}J_{\mu p} + J_{e\mu}J_{ep}, \\
d_1 &= -en_e(1-2\mathfrak{X}_e)J_{\mu p} - en_\mu(1-2\mathfrak{X}_\mu)J_{ep} \\
&\quad -en_p(1-2\mathfrak{X}_e-2\mathfrak{X}_\mu)J_{e\mu}, \\
d_2 &= e^2\left[n_e n_\mu(1-\mathfrak{X}_e-\mathfrak{X}_\mu)^2 - n_p(\mathfrak{X}_e^2 n_\mu + \mathfrak{X}_\mu^2 n_e)\right]
\end{aligned}$$

# Kinematics

Elements of the transport matrices, momentum transfer rates are certain weighted averages of the transition probabilities

$$\Lambda_{ab}^{k} \propto m_a^{*2} m_b^{*2} \langle D^k(\Omega, w) |T_{ab}|^2 \rangle$$

# Kinematics of collisions

Only two angular variables define the relative orientation of the system of four vectors of fixed length

$$1 + 2 \to 1' + 2'$$

Transferred momentum $\quad \boldsymbol{q} = \boldsymbol{p}_1' - \boldsymbol{p}_1 \qquad \boldsymbol{q}' = \boldsymbol{p}_{2'} - \boldsymbol{p}_1$

Total momentum $\quad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2'$

Relative momenta $\quad \boldsymbol{p} = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \qquad \boldsymbol{p}' = \frac{1}{2}(\boldsymbol{p}_1' - \boldsymbol{p}_2')$

Traditionally one employs Abrikosov-Khalatnikov angles

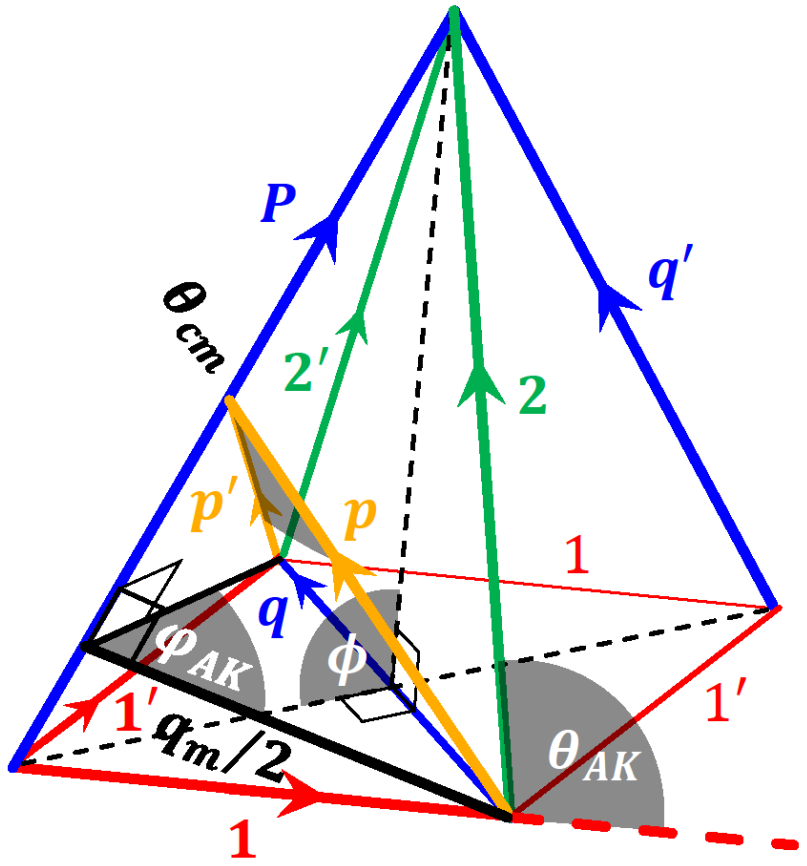
Not always convenient and over variables are of use, e.g. $q$ and $\phi$

$$q \in [q_{\min}, q_{\max}] \qquad \phi \in [0, 2\pi]$$

$$q_{\max} = \min(p_1 + p_{1'}, p_2 + p_{2'})$$

$$q_{\min} = \max(|p_1 - p_{1'}|, |p_2 - p_{2'}|)$$

$$|p_1 - p_{1'}|(p_1 + p_{1'}) = |p_1^2 - p_{1'}^2| \approx 2\omega\varepsilon_{F1} \Rightarrow |p_1 - p_{1'}| \approx \omega/v_{F1}$$



# Kinematics of collisions

Only two angular variables define the relative orientation of the system of four vectors of fixed length

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Transferred momentum $\boldsymbol{q} = \boldsymbol{p}_1' - \boldsymbol{p}_1 \qquad \boldsymbol{q}' = \boldsymbol{p}_{2'} - \boldsymbol{p}_1$

Example: $$\int \mathrm{d}\hat{\boldsymbol{p}}_1 \mathrm{d}\hat{\boldsymbol{p}}_{1'} \mathrm{d}\hat{\boldsymbol{p}}_2 \mathrm{d}\hat{\boldsymbol{p}}_{2'}\ A(\{\boldsymbol{p}_i\}) \delta(\boldsymbol{p}_{1'} + \boldsymbol{p}_{2'} - \boldsymbol{p}_1 - \boldsymbol{p}_2)$$

$$\delta(\boldsymbol{p}_{1'} + \boldsymbol{p}_{2'} - \boldsymbol{p}_1 - \boldsymbol{p}_2) = \int \mathrm{d}\boldsymbol{q} \delta(\boldsymbol{p}_{1'} - \boldsymbol{p}_1 - \boldsymbol{q}) \delta(\boldsymbol{p}_{2'} - \boldsymbol{p}_2 + \boldsymbol{q})$$

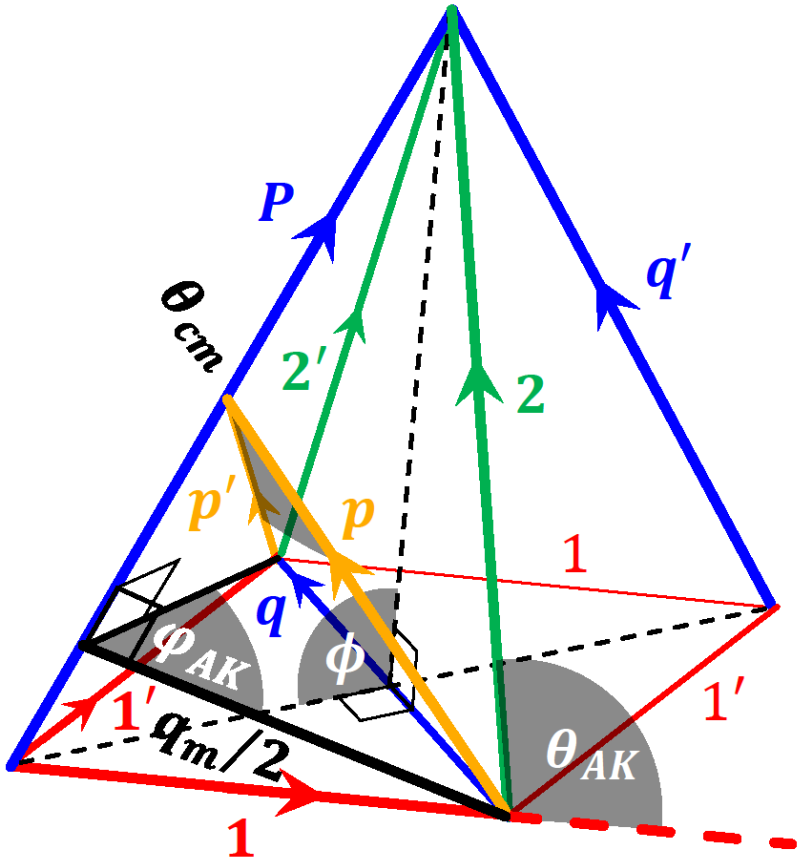
$(p_1 p_{1'})$ plane is rotated by angle $\phi$ about $(p_2 p_{2'})$ plane

$$\delta(\boldsymbol{p}_{1'} - \boldsymbol{p}_1 - \boldsymbol{q}) = \frac{1}{p_1^2} \delta(p_1 - \sqrt{p_{1'}^2 + q^2 - 2q p_{1'} c_{1'}}) \delta\left(c_1 - \frac{p_1' c_1' - q}{p_1}\right) \delta(\phi_{1'} - \phi_1)$$

$$\int_{-1}^{+1} \mathrm{d}c_{1'}\ \delta\left(p_1 - \sqrt{p_{1'}^2 + q^2 - 2 p_{1'} q c_{1'}}\right) = \frac{p_1}{q p_{1'}} \Theta\left(q - |p_1 - p_{1'}|\right) \Theta\left(p_1 + p_{1'} - q\right)$$

$$\int \mathrm{d}\hat{\boldsymbol{p}}_1 \mathrm{d}\hat{\boldsymbol{p}}_{1'} \mathrm{d}\hat{\boldsymbol{p}}_2 \mathrm{d}\hat{\boldsymbol{p}}_{2'}\ A(\{\boldsymbol{p}_i\}) \delta(\boldsymbol{p}_{1'} + \boldsymbol{p}_{2'} - \boldsymbol{p}_1 - \boldsymbol{p}_2) = \frac{16\pi^2}{p_1 p_2 p_{1'} p_{2'}} \int_{q_{\min}}^{q_{\max}} \mathrm{d}q \int_0^{\pi} \mathrm{d}\phi\, A(q, \phi)$$

Average $$\langle A(\{\boldsymbol{p}_i\}) \rangle = \frac{p_1 p_2 p_{1'} p_{2'}}{16\pi^2} \int \mathrm{d}\hat{\boldsymbol{p}}_1 \mathrm{d}\hat{\boldsymbol{p}}_{1'} \mathrm{d}\hat{\boldsymbol{p}}_2 \mathrm{d}\hat{\boldsymbol{p}}_{2'}\ A(\{\boldsymbol{p}_i\}) \delta(\boldsymbol{p}_{1'} + \boldsymbol{p}_{2'} - \boldsymbol{p}_1 - \boldsymbol{p}_2)$$



# Kinematics of collisions

Only two angular variables define the relative orientation of the system of four vectors of fixed length

$$1 + 2 \to 1' + 2'$$

Transferred momentum $\quad \boldsymbol{q} = \boldsymbol{p}'_1 - \boldsymbol{p}_1 \qquad \boldsymbol{q}' = \boldsymbol{p}_{2'} - \boldsymbol{p}_1$

Total momentum $\quad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2$

Relative momenta $\quad \boldsymbol{p} = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) \qquad \boldsymbol{p}' = \frac{1}{2}(\boldsymbol{p}'_1 - \boldsymbol{p}'_2)$

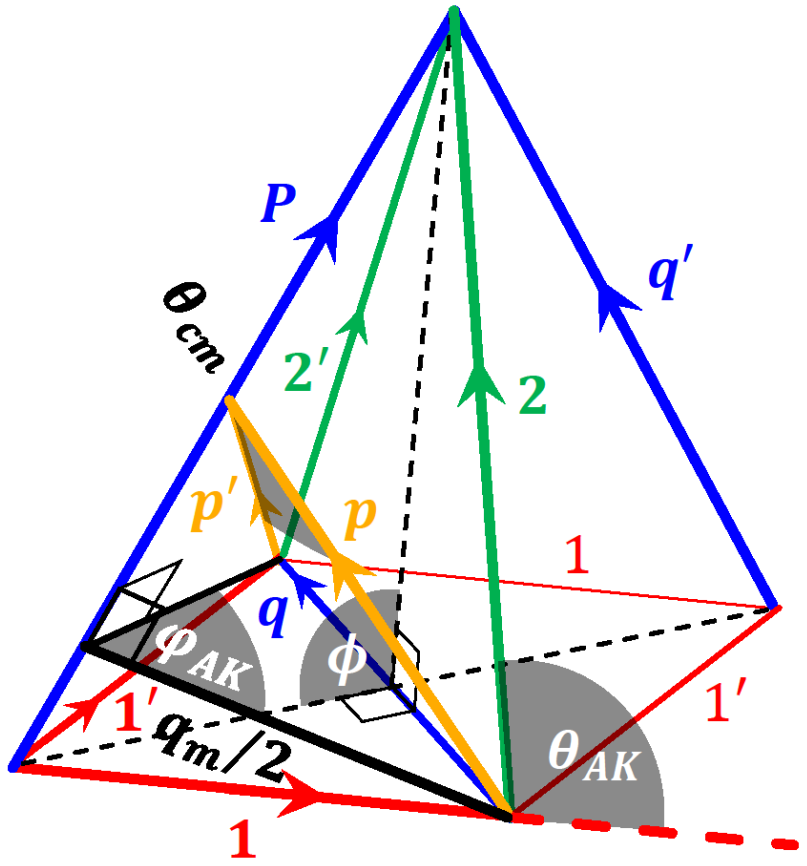
Traditionally one employs Abrikosov-Khalatnikov angles

Not always convenient and other variables are of use

$$\langle A(\{\boldsymbol{p}_i\})\rangle = \int_{q_{\min}}^{q_{\max}} \mathrm{d}q \int_0^{\pi} \mathrm{d}\phi\, A(q,\phi) \qquad P^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta_{AK}$$

$$\langle A(\{\boldsymbol{p}_i\})\rangle = \int_{P_{\min}}^{P_{\max}} \mathrm{d}P \int_0^{\pi} \mathrm{d}\phi_{AK}\, A(P,\phi_{AK}) = \int \frac{p_1 p_2 \sin\theta_{AK}\mathrm{d}\theta_{AK}}{P} \int_0^{\pi} \mathrm{d}\phi_{AK}\, A(\theta_{AK},\phi_{AK})$$

$$q = q_m \sin\frac{\phi_{AK}}{2} \qquad \langle A(\{\boldsymbol{p}_i\})\rangle = 2\int_{P_{\min}}^{P_{\max}} \mathrm{d}P \int_0^{q_m} \frac{\mathrm{d}q}{\sqrt{q_m^2 - q^2}} A(P,q)$$



# Exact solutions

# Exact solutions

Single-component case. B=0.
Thermal conductivity

$$-\frac{1}{T}f_F'(x)\sum_{k|\ell_k=\ell}\sqrt{\frac{4\pi}{2\ell+1}}D_k^a(p)(\mathcal{X}_k)_{\ell m}=I_a^\ell[\{(\Phi_b)_{\ell m}\}] \qquad \ell=1 \quad D_\kappa^a(p)=v_{Fa}Tx$$

$$I_{ab}^\ell = -\frac{1}{T(1+\delta_{ab})}\int_{\boldsymbol{p}_{1'}}\int_{\boldsymbol{p}_2}\int_{\boldsymbol{p}_{2'}} w_{ab}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_{1'},\boldsymbol{p}_{2'})\mathcal{F}_{ab}\left[(\Phi_a(p_1))_{\ell m}+(\Phi_b(p_2))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)-(\Phi_a(p_{1'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{1'})-(\Phi_b(p_{2'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{2'})\right]$$

$$\mathrm{d}\boldsymbol{p}=m^*p_FT\mathrm{d}x\mathrm{d}\Omega_{\boldsymbol{p}} \qquad x=\frac{\varepsilon-\mu}{T}\to-\infty\ldots\infty \qquad \mathcal{P}_1(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)=\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2$$

$$\mathcal{F}_{ab}=f_1^{\mathrm{eq}}f_2^{\mathrm{eq}}(1-f_{1'}^{\mathrm{eq}})(1-f_{2'}^{\mathrm{eq}}) \qquad w_{ab}=(2\pi)^4\delta(\varepsilon_1+\varepsilon_2-\varepsilon_{1'}-\varepsilon_{2'})\delta(\boldsymbol{p}_{1'}+\boldsymbol{p}_{2'}-\boldsymbol{p}_1-\boldsymbol{p}_2)|T_{ab}(12\to1'2')|^2$$

Dimensionless form

$$\Phi=\sqrt{\frac{4\pi}{3}}\lambda_{\mathrm{eff}}(\nabla T+T\dot{V})\Psi(x) \qquad \lambda_{\mathrm{eff}}=\frac{4\pi^4p_F^2}{m^{*4}T^2\langle\mathcal{Q}\rangle} \qquad \mathcal{Q}_{ab}=\frac{1}{4(1+\delta_{ab})}\sum_{\mathrm{spins}}|T_{ab}(12\to1'2')|^2$$

$$x_{1'}\to-x_{1'},\ x_{2'}\to-x_{2'}$$

$$f_F'(x_1)x_1=\int\mathrm{d}x_2\mathrm{d}x_{1'}\mathrm{d}x_{2'}f_F(x_1)f_F(x_2)f_F(x_{1'})f_F(x_{2'})\delta(x_1+x_2+x_{1'}+x_{2'})(\Psi(x_1)+\zeta_\kappa\Psi(x_2)) \qquad \zeta^\kappa=\frac{\langle\mathcal{Q}(1+2\cos\theta_{AK})\rangle}{\langle\mathcal{Q}\rangle}$$

$$f_F(x_1)(1-f_F(x_1))x_1=\frac{\pi^2+x_1^2}{2}f_F(x_1)(1-f_F(x_1))\Psi(x_1)+\zeta_\kappa\int_{-\infty}^{\infty}\mathrm{d}x_2f_F(x_1)f_F(x_2)\frac{(x_1+x_2)}{1-\mathrm{e}^{-(x_1+x_2)}}\Psi(x_1)$$

Can be brought to the soluble form

$$\Psi(x)=2\cosh\left(\frac{x}{2}\right)\psi(x)$$

$$\frac{x_1}{\cosh\frac{x_1}{2}}=(\pi^2+x_1^2)\psi(x_1)-\zeta_\kappa\int_{-\infty}^{\infty}\mathrm{d}x_2\frac{(x_1-x_2)}{\sinh\left(\frac{x_1-x_2}{2}\right)}\psi(x_2)$$

# Exact solutions. Thermal conductivity

**Integral equation with convolution kernel**

$$\frac{x_1}{\cosh\frac{x_1}{2}} = (\pi^2 + x_1^2)\psi(x_1) - \zeta_\kappa \int_{-\infty}^{\infty} \mathrm{d}x_2 \frac{(x_1 - x_2)}{\sinh\left(\frac{x_1 - x_2}{2}\right)}\psi(x_2)$$

**Eigenvalue problem**

$$\frac{1}{\alpha_\nu}(\pi^2 + x_1^2)\psi_\nu(x_1) = \int_{-\infty}^{\infty} \mathrm{d}x_2 \frac{(x_1 - x_2)}{\sinh\left(\frac{x_1 - x_2}{2}\right)}\psi_\nu(x_2) \qquad \int_{-\infty}^{\infty} \mathrm{d}x(\pi^2 + x^2)\psi_\nu(x)\psi_\mu^*(x) = \delta_{\mu\nu}$$

$$\psi(x) = \sum c_\nu \psi_\nu(x)$$

**Solution for the inhomogeneous equation**

$$\frac{x_1}{\cosh\frac{x_1}{2}} = (\pi^2 + x_1^2)\psi(x_1) \sum c_\nu \left(1 - \frac{\zeta_\kappa}{\alpha_\nu}\right)\psi_\nu(x_1) \Rightarrow c_\nu = \left(1 - \frac{\zeta_\kappa}{\alpha_\nu}\right)^{-1} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\cosh\frac{x}{2}}\,\psi_\nu^*(x)$$

$$\kappa = \frac{p_F^2 T \lambda_{\text{eff}}}{6\pi^2} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\cosh\frac{x}{2}}\,\psi(x) = \frac{p_F^2 T \lambda_{\text{eff}}}{6\pi^2} \sum_\nu \left(1 - \frac{\zeta_\kappa}{\alpha_\nu}\right)^{-1} \left| \int_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\cosh\frac{x}{2}}\,\psi_\nu(x) \right|^2$$

**It remains to solve the eigenvalue problem and calculate the integral**

$$K(k) = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\sinh\frac{x}{2}} \mathrm{e}^{-ikx} = \frac{2\pi^2}{\cosh^2 \pi k} \qquad \int_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\cosh\frac{x}{2}}\,\psi_\nu(x) = \pi i \int_{-\infty}^{+\infty} \mathrm{d}k \widetilde{\psi}(k) \frac{\tanh \pi k}{\cosh \pi k}$$

$$\frac{1}{\alpha_\nu}\left(\pi^2 - \frac{\partial^2}{\partial k^2}\right)\widetilde{\psi}_\nu(k) = K(k)\widetilde{\psi}_\nu(k)$$

$$z = \tanh \pi k$$

$$u_\nu(z) \equiv \widetilde{\psi}(k)$$

$$(1 - z^2)\frac{d^2 u_\nu}{dz^2} - 2z\frac{du_\nu}{dz} + \left(2\alpha_\nu - \frac{1}{1 - z^2}\right)u_\nu = 0$$

# Exact solutions. Thermal conductivity

This is associated Legendre equation with solution

$$(1-z^2)\frac{d^2u_\nu}{dz^2} - 2z\frac{du_\nu}{dz} + \left(2\alpha_\nu - \frac{1}{1-z^2}\right)u_\nu = 0$$

$$\widetilde{\psi}(k) = \frac{\sqrt{2\nu+1}}{\nu(\nu+1)}P_\nu^1(\tanh \pi k) \qquad \alpha_\nu = \nu(\nu+1), \quad \nu = 1,2,\dots$$

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x \frac{x}{\cosh\frac{x}{2}}\,\psi_\nu(x) = i\pi \int\limits_{-\infty}^{+\infty} \mathrm{d}k \widetilde{\psi}(k)\frac{\tanh \pi k}{\cosh \pi k} = i\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}\int\limits_{-1}^{+1}\mathrm{d}z\frac{z}{\sqrt{1-z^2}}P_\nu^1(z) = -i\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}\int\limits_{-1}^{+1}\mathrm{d}z z\frac{\mathrm{d}}{\mathrm{d}z}P_\nu(z) = -2i\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}, \quad \nu \text{ even}$$

Final answer

$$\kappa = \frac{p_F^2 T\lambda_{\mathrm{eff}}}{6\pi^2}\sum_\nu\left(1-\frac{\zeta_\kappa}{\alpha_\nu}\right)^{-1}\left|\int\limits_{-\infty}^{+\infty}\mathrm{d}x\frac{x}{\cosh\frac{x}{2}}\,\psi_\nu(x)\right|^2 = \frac{\pi^2}{3}\frac{n}{p_F}T\lambda_{\mathrm{eff}}\left[\frac{6}{\pi^2}\sum_{\nu=2,4,\dots}\frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\zeta_\kappa]}\right]$$

Only one parameter

$$\zeta^\kappa = \frac{\langle \mathcal{Q}(1+2\cos\theta_{AK})\rangle}{\langle \mathcal{Q}\rangle} \qquad -1 \leqslant \zeta_\kappa \leqslant 3$$

Variational solution corresponds to the first term

$$C_{var} = \frac{5}{2\pi^2}\frac{1}{3-\zeta_\kappa}$$

Corrections are not so large (from astrophysicist point of view

$$C_{ex}/C_{var} = 1.35\dots 1$$

# Exact solutions. Shear viscosity

$$-\frac{1}{T}f_F'(x)\sum_{k|\ell_k=\ell}\sqrt{\frac{4\pi}{2\ell+1}}D_k^a(p)(\mathcal{X}_k)_{\ell m}=I_a^\ell[\{(\Phi_b)_{\ell m}\}] \qquad \ell=2 \qquad D_\eta^a(p)=\sqrt{2/3}\,(p_{Fa}v_{Fa})$$

$$I_{ab}^\ell = -\frac{1}{T(1+\delta_{ab})}\int_{\boldsymbol{p}_{1'}}\int_{\boldsymbol{p}_2}\int_{\boldsymbol{p}_{2'}} w_{ab}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_{1'},\boldsymbol{p}_{2'})\mathcal{F}_{ab}\left[(\Phi_a(p_1))_{\ell m}+(\Phi_b(p_2))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)-(\Phi_a(p_{1'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{1'})-(\Phi_b(p_{2'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{2'})\right]$$

$$\Phi_{2m}=\sqrt{\frac{4\pi}{5}}\sqrt{\frac{2}{3}}p_F v_F\lambda_{\rm eff}\hat{V}_{2m}\Psi(x)$$

$$\mathcal{P}_2(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{1'})+\mathcal{P}_2(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{2'})-\mathcal{P}_2(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)=1-3\sin^4\frac{\theta_{AK}}{2}\sin^2\phi_{AK} \qquad \zeta^\eta=\frac{\langle\mathcal{Q}(1-3\sin^4\frac{\theta_{AK}}{2}\sin^2\phi_{AK})\rangle}{\langle\mathcal{Q}\rangle} \qquad -2\leqslant\zeta_\eta\leqslant 1$$

Resulting equation differs in lhs only

$$\frac{1}{\cosh\frac{x_1}{2}}=(\pi^2+x_1^2)\psi(x_1)-\zeta_\eta\int_{-\infty}^{\infty}\mathrm{d}x_2\frac{(x_1-x_2)}{\sinh\left(\frac{x_1-x_2}{2}\right)}\psi(x_2)$$

The answer is

$$\eta=\frac{1}{5}np_F\lambda_{\rm eff}\sum_\nu\frac{1}{2}\left(1-\frac{\zeta_\eta}{\alpha_\nu}\right)^{-1}\left|\int_{-\infty}^{+\infty}\mathrm{d}x\frac{\psi_\nu(x)}{\cosh\frac{x}{2}}\right|^2=\frac{1}{5}np_F\lambda_{\rm eff}\left[\frac{2}{\pi^2}\sum_{\nu=1,3,\ldots}\frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\zeta_\eta]}\right]$$

$$\int_{-\infty}^{+\infty}\mathrm{d}x\frac{\psi_\nu(x)}{\cosh\frac{x}{2}}=\int_{-\infty}^{+\infty}\mathrm{d}k\frac{\tilde{\psi}(k)}{\cosh\pi k}=\frac{1}{\pi}\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}\int_{-1}^{+1}\mathrm{d}z\frac{P_\nu^1(z)}{\sqrt{1-z^2}}=-\frac{1}{\pi}\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}\int_{-1}^{+1}\mathrm{d}z\frac{\mathrm{d}}{\mathrm{d}z}P_\nu(z)=-\frac{2}{\pi}\frac{\sqrt{2\nu+1}}{\nu(\nu+1)},\quad \nu\ \text{odd}$$

Comparison with variational soultion

$$C_{var}=\frac{3}{2\pi^2}\frac{1}{1-\zeta_\eta} \qquad C_{ex}/C_{var}=1.24\ldots 1$$

# Exact solutions. Bulk viscosity

$$-\frac{1}{T}f_F'(x)\sum_{k|\ell_k=\ell}\sqrt{\frac{4\pi}{2\ell+1}}D_k^a(p)(\mathcal{X}_k)_{\ell m}=I_a^\ell[\{(\Phi_b)_{\ell m}\}] \qquad \zeta_\zeta=1$$

$$I_{ab}^\ell = -\frac{1}{T(1+\delta_{ab})}\int_{\boldsymbol{p}_{1'}}\int_{\boldsymbol{p}_2}\int_{\boldsymbol{p}_{2'}} w_{ab}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_{1'},\boldsymbol{p}_{2'})\mathcal{F}_{ab}\left[(\Phi_a(p_1))_{\ell m}+(\Phi_b(p_2))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_2)-(\Phi_a(p_{1'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{1'})-(\Phi_b(p_{2'}))_{\ell m}\mathcal{P}_\ell(\hat{\boldsymbol{p}}_1\hat{\boldsymbol{p}}_{2'})\right]$$

Driving term

$$\frac{1}{T}f_F'(x)\left[\frac{1}{3}pv-nT\frac{\mathrm{d}}{\mathrm{d}n}\frac{\varepsilon-\mu}{T}\right]\mathrm{div}\boldsymbol{V}$$

Should be orthogonal to any summation invariant

$$\alpha+\beta\varepsilon=\alpha'+\beta(\varepsilon-\mu)$$

Formal expansion

$$\frac{1}{3}pv-nT\frac{\mathrm{d}}{\mathrm{d}n}\frac{\varepsilon-\mu}{T}=a+b(\varepsilon-\mu)+s_2(\varepsilon-\mu)^2+s_3(\varepsilon-\mu)^3+\ldots$$

Sommerfeld expansion

$$\frac{1}{T}f(1-f)\approx-\delta(\varepsilon-\mu)-\frac{\pi^2}{6}T^2\delta''(\varepsilon-\mu)\Rightarrow a=a'T^2,\quad b=b'T^2 \qquad a'+\frac{\pi^2}{3}s_2=O(T^2)$$

In the leading order in T

$$D_0=\frac{1}{3}pv-nT\frac{\mathrm{d}}{\mathrm{d}n}\frac{\varepsilon-\mu}{T}=s_2T^2\left(x^2-\frac{\pi^2}{3}\right)+O(T^3) \qquad s_2=\frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\left[\frac{1}{3}pv+n\int\frac{\mathrm{d}\Omega_{p'}}{4\pi}f^s_{\boldsymbol{p}\boldsymbol{p}'}\right]_{p=p'=pF}$$

$$\Phi_0=\sqrt{4\pi}s_2T^2\lambda_{\mathrm{eff}}\mathrm{div}\boldsymbol{V}\Psi(x) \qquad \frac{x^2-\pi^2/3}{\cosh\frac{x}{2}}=(\pi^2+x_1^2)\psi(x_1)-\int_{-\infty}^{\infty}\mathrm{d}x_2\frac{(x_1-x_2)}{\sinh\left(\frac{x_1-x_2}{2}\right)}\psi(x_2)$$

Final result ($\nu$ sum can be taken exactly)

$$\zeta=\frac{2}{27}p_F^3m^*T^4\left(\frac{\partial^2}{\partial\varepsilon^2}\left[\frac{1}{3}pv+n\int\frac{\mathrm{d}\Omega_{p'}}{4\pi}f^s_{\boldsymbol{p}\boldsymbol{p}'}\right]_{p=p'=pF}\right)^2$$

Bulk viscosity can be neglected! $\zeta/\eta\propto T^2$

Is zero in non-interacting system with homogeneous dispersion relation

# Remarks on the exact solutions

The construction of exact solution can be generalized to the case of multicomponent system within the same principle scheme

Can be generalized to magnetized problems as well (much more cumbersome equations)

In NS cores this solution fails due to peculiarities in electromagnetic interaction

Difference between the simplest variational solution and exact one is expected to be small, We further will rely on the variational solution only

# Variational solutions. Explicit expressions for transport matrices

$$1=\sum_b\left(\Lambda^k_{ab}\widetilde{\lambda}^k_a+\Lambda'^k_{ab}\widetilde{\lambda}^k_b\right)+i\frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}}\widetilde{\lambda}^k_a \qquad k=\kappa,\,\eta \qquad \int_w=\int_0^\infty \mathrm{d}w\frac{(w/2)^2}{\sinh^2(w/2)}$$

$$1+2\to 1'+2'$$

Transport matrices

$$\mathcal{Q}_{ab}=\mathcal{Q}_{ab}(w)$$

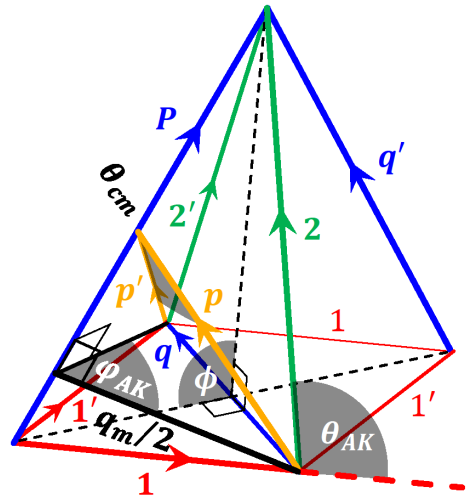
$$\begin{aligned}
\Lambda^\kappa_{ab} &= \frac{3T^2m^{*2}_am^{*2}_b}{4\pi^4p^2_{\mathrm{F}a}}\int_w\left\langle\left(\frac{w^2}{\pi^2}+\left[\frac{1}{3}-\frac{w^2}{6\pi^2}\right]\frac{q^2}{p^2_{\mathrm{F}a}}\right)\mathcal{Q}_{ab}\right\rangle\\
\Lambda'^\kappa_{ab} &= -\frac{3T^2m^{*2}_am^{*2}_b}{4\pi^4p^2_{\mathrm{F}a}}\int_w\frac{w^2}{\pi^2}\left\langle\left(\frac{\boldsymbol{p}_1(\boldsymbol{p}_2+\boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}}\right)\mathcal{Q}_{ab}\right\rangle,\\
\Lambda^\eta_{ab} &= \frac{3T^2m^{*2}_am^{*2}_b}{4\pi^4p^2_{\mathrm{F}a}}\int_w\left\langle\frac{q^2}{p^2_{\mathrm{F}a}}\left(1-\frac{q^2}{4p^2_{\mathrm{F}a}}\right)\mathcal{Q}_{ab}\right\rangle\\
\Lambda'^\eta_{ab} &= -\frac{3T^2m^{*2}_am^{*2}_b}{4\pi^4p^2_{\mathrm{F}a}}\int_w\left\langle\frac{q^2}{p^2_{\mathrm{F}a}}\left(\frac{\boldsymbol{p}_1(\boldsymbol{p}_2+\boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}}\right)\mathcal{Q}_{ab}\right\rangle
\end{aligned}$$

$$J_{ab}=\frac{T^2m^{*2}_am^{*2}_b}{12\pi^6}\int_w\langle q^2\mathcal{Q}_{ab}\rangle$$

$$\mathcal{Q}_{ab}\neq\mathcal{Q}_{ab}(w)$$

$$\begin{aligned}
\Lambda^\kappa_{ab} &= \frac{T^2m^{*2}_am^{*2}_b}{5\pi^2p^2_{\mathrm{F}a}}\left\langle\left(1+\frac{q^2}{4p^2_{\mathrm{F}a}}\right)\mathcal{Q}_{ab}\right\rangle\\
\Lambda'^\kappa_{ab} &= -\frac{T^2m^{*2}_am^{*2}_b}{5\pi^2p^2_{\mathrm{F}a}}\left\langle\left(\frac{\boldsymbol{p}_1(\boldsymbol{p}_2+\boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}}\right)\mathcal{Q}_{ab}\right\rangle\\
\Lambda^\eta_{ab} &= \frac{T^2m^{*2}_am^{*2}_b}{4\pi^2p^2_{\mathrm{F}a}}\left\langle\frac{q^2}{p^2_{\mathrm{F}a}}\left(1-\frac{q^2}{4p^2_{\mathrm{F}a}}\right)\mathcal{Q}_{ab}\right\rangle\\
\Lambda'^\eta_{ab} &= -\frac{T^2m^{*2}_am^{*2}_b}{4\pi^2p^2_{\mathrm{F}a}}\left\langle\frac{q^2}{p^2_{\mathrm{F}a}}\left(\frac{\boldsymbol{p}_1(\boldsymbol{p}_2+\boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}}\right)\mathcal{Q}_{ab}\right\rangle
\end{aligned}$$

$$J_{ab}=\frac{T^2m^{*2}_am^{*2}_b}{36\pi^4}\langle q^2\mathcal{Q}_{ab}\rangle$$




$$w=\frac{\varepsilon_{1'}-\varepsilon_1}{T}$$

$$\boldsymbol{q}=\boldsymbol{p}_{1'}-\boldsymbol{p}_1$$

General structure

$$\Lambda^k_{ab}\propto m^{*2}_am^{*2}_b\langle D^k(\Omega,w)\mathcal{Q}_{ab}\rangle$$

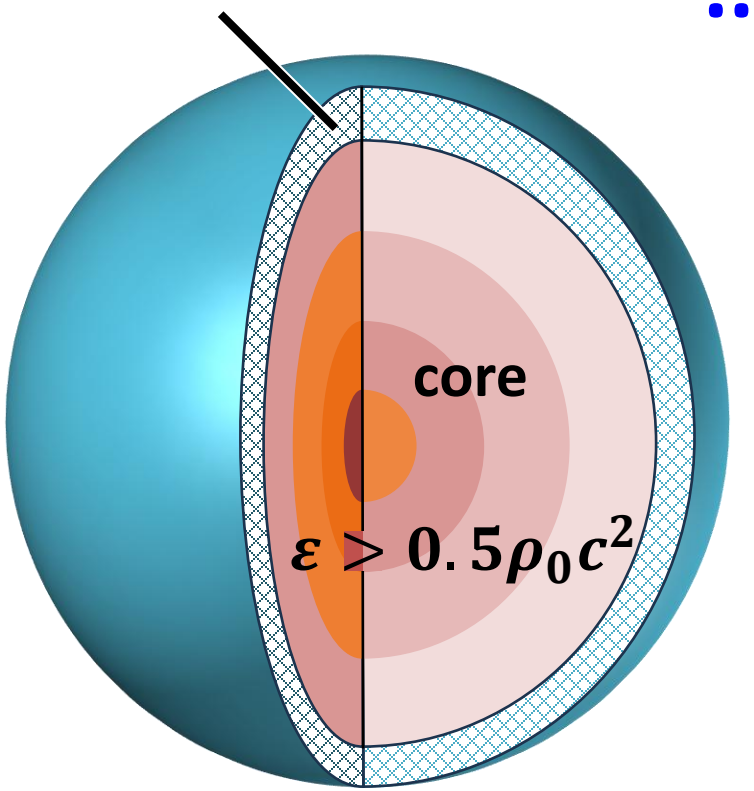
$$\mathcal{Q}_{ab}=\frac{1}{4(1+\delta_{ab})}\sum_{\text{spins}}|T_{ab}(12\to 1'2')|^2$$

# Application to neutron star cores

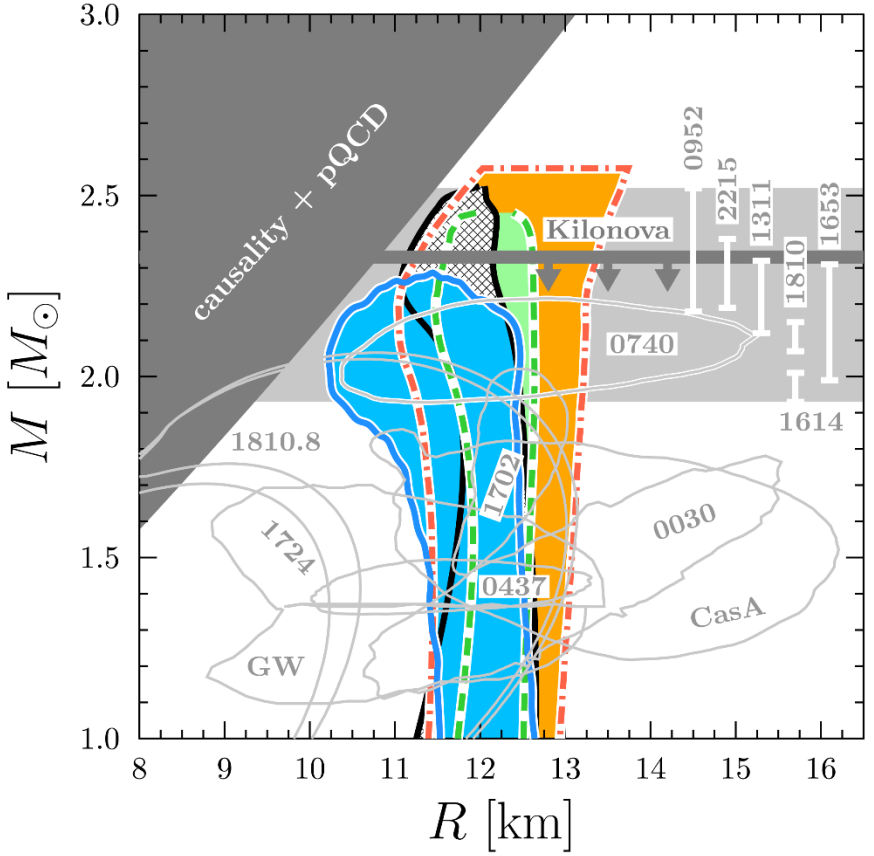
# Towards neutron star conditions

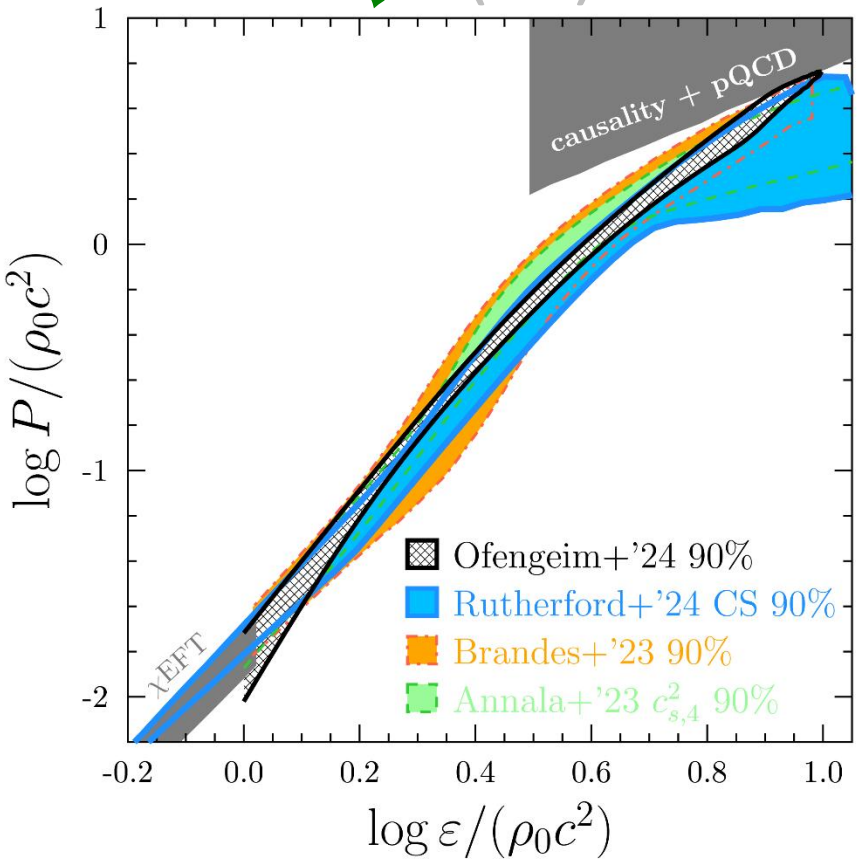
**EoS = Equation of State**

*thermal properties* ..... *composition* ? ***mechanical properties*** ✓ (?..)




$\rho_0 c^2 \approx 150\ \mathrm{MeV\ fm^{-3}}$

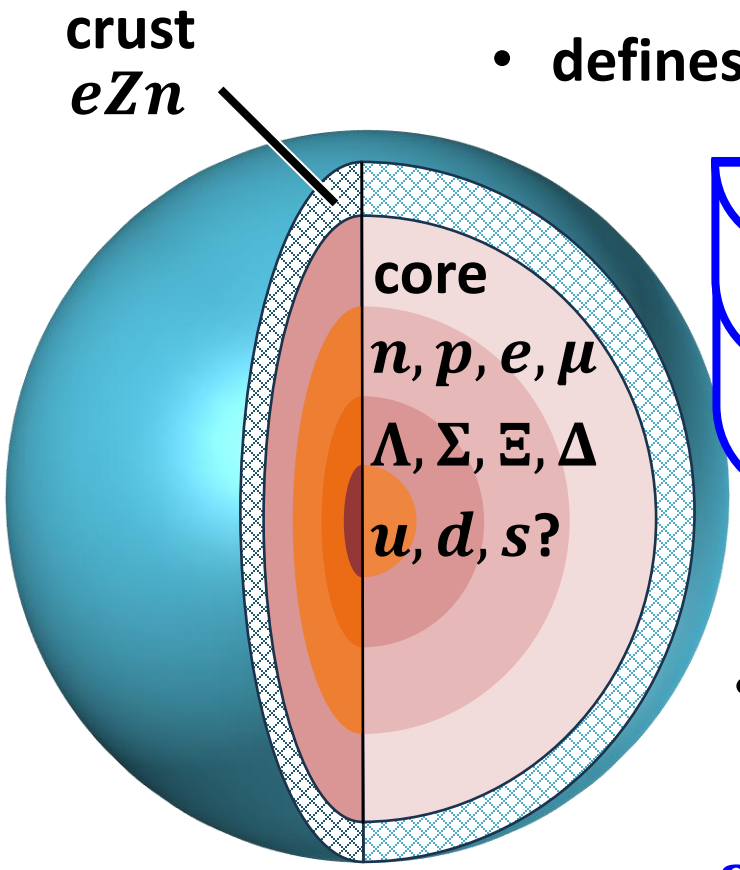


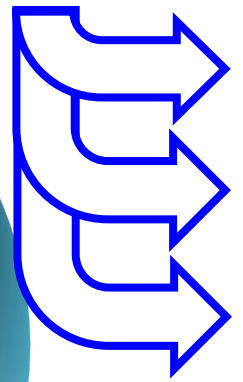




Ofengeim, Kolomeitsev, Shternin, Neutron star equation of state. In Encyclopedia of Astrophysics; 2026; Vol. 3, pp. 160–204

# Neutron star core composition

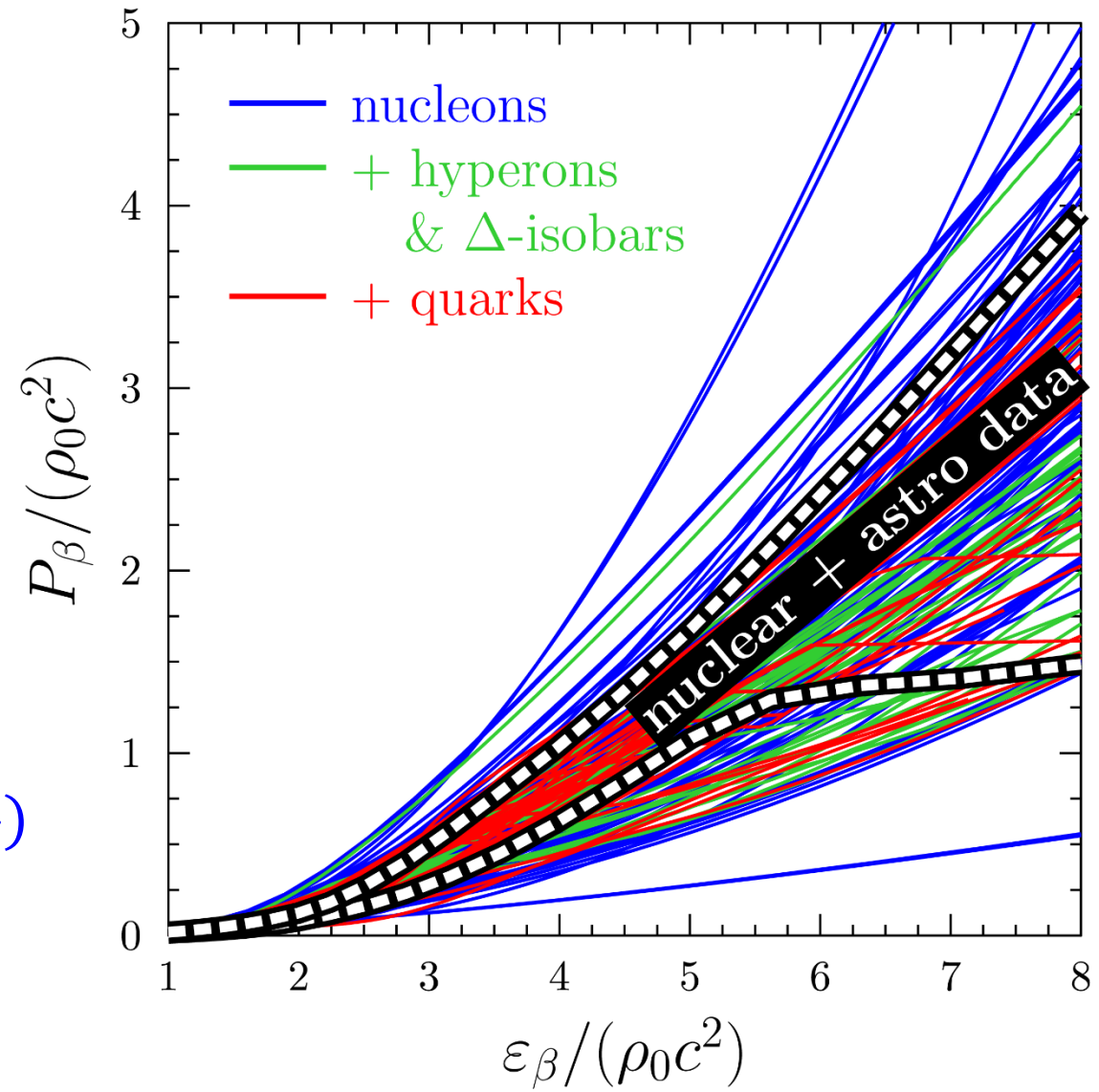
- $\beta$-equilibrium
- strongly model-dependent
- defines transport properties




- determined by $\varepsilon(\{n_i\})$

$\varepsilon(\{n_i\})$ ✓ / ✗ $P_\beta(\varepsilon_\beta)$

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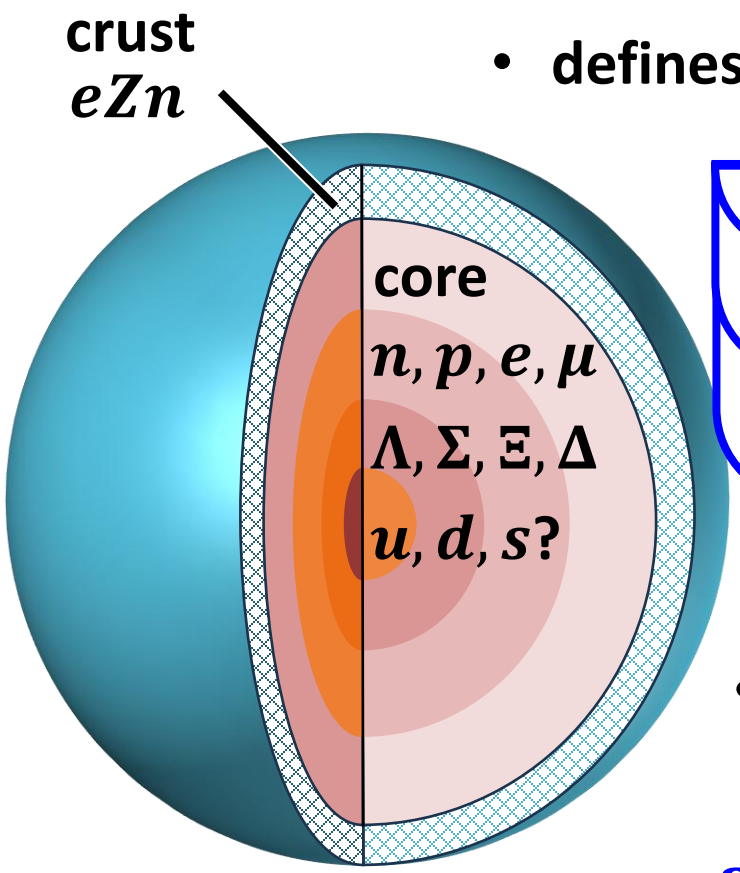
- strongly model-dependent
- defines transport properties



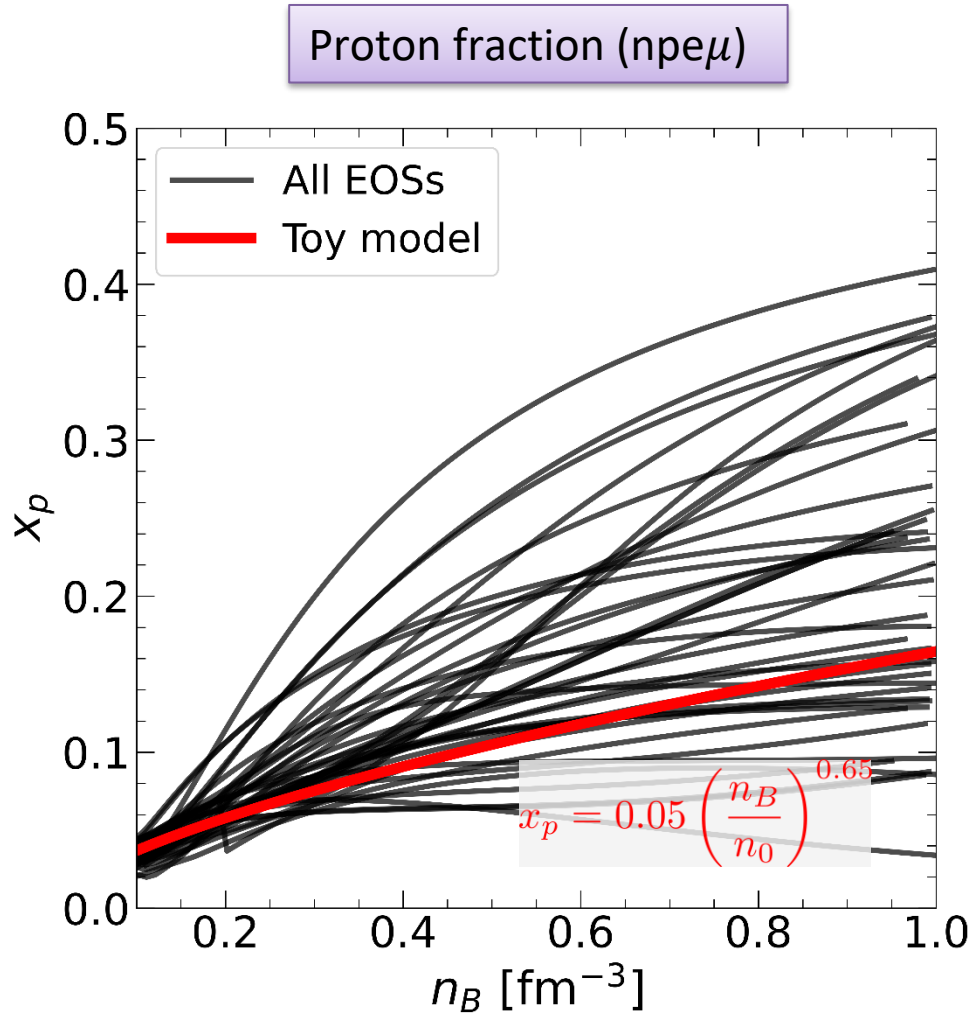

*cooling*

$\vec{B}$ *evolution*

*pulsations damping*

- determined by $\varepsilon(\{n_i\})$

$\varepsilon(\{n_i\})$ ✓ ✗ $P_\beta(\varepsilon_\beta)$

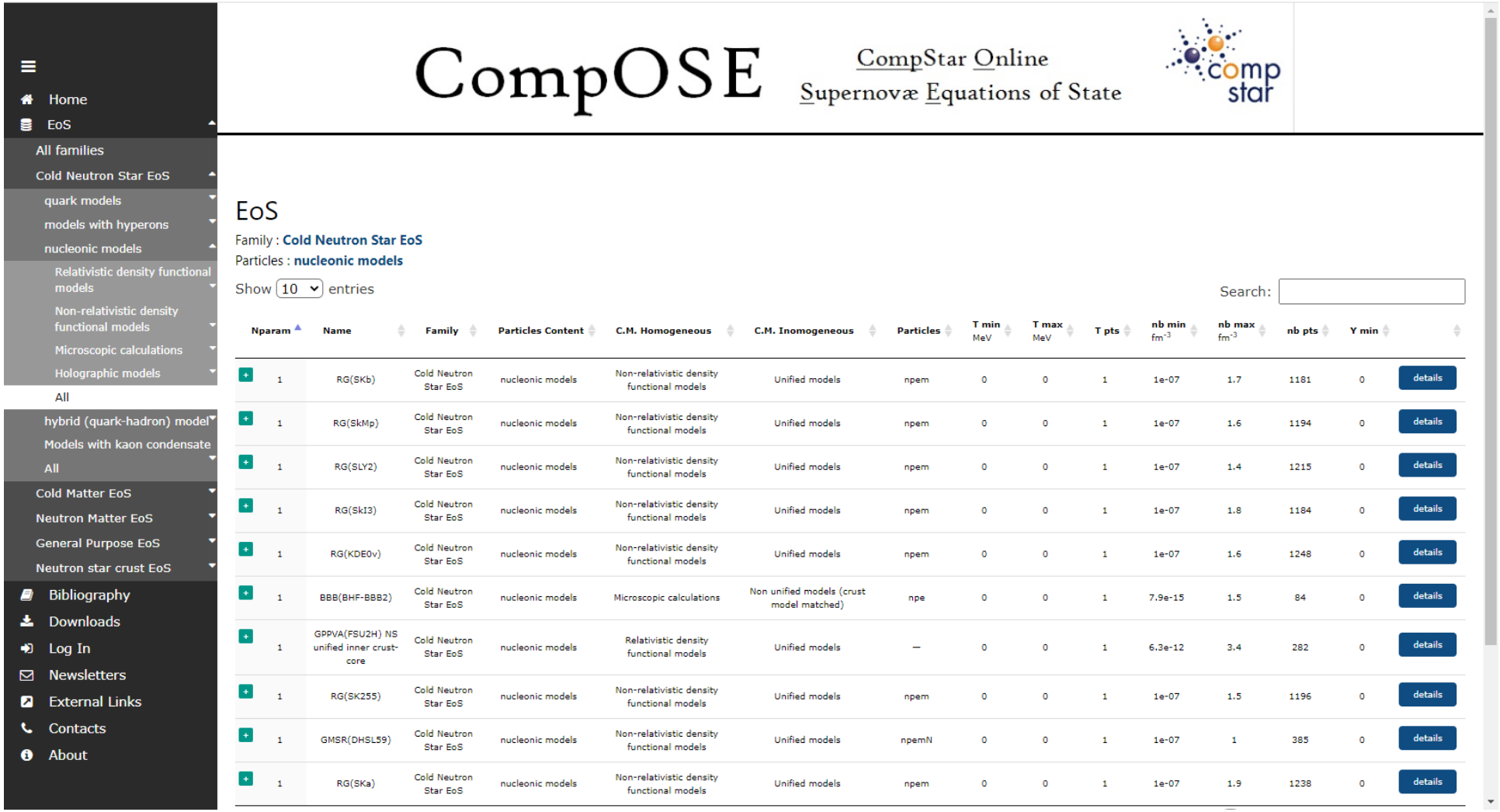



Ofengeim, Kolomeitsev, Shternin, Neutron star equation of state. In Encyclopedia of Astrophysics; 2026; Vol. 3, pp. 160–204

# Diversity of equations of state

CompOSE database http://compose.obspm.fr

We selected purely nucleonic models




| Nparam | Name | Family | Particles Content | C.M. Homogeneous | C.M. Inomogeneous | Particles | T min MeV | T max MeV | T pts | nb min fm$^{-3}$ | nb max fm$^{-3}$ | nb pts | Y min | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | RG(SKb) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.7 | 1181 | 0 | details |
| 1 | RG(SkMp) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.6 | 1194 | 0 | details |
| 1 | RG(SLY2) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.4 | 1215 | 0 | details |
| 1 | RG(SkI3) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.8 | 1184 | 0 | details |
| 1 | RG(KDE0v) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.6 | 1248 | 0 | details |
| 1 | BBB(BHF-BBB2) | Cold Neutron Star EoS | nucleonic models | Microscopic calculations | Non unified models (crust model matched) | npe | 0 | 0 | 1 | 7.9e-15 | 1.5 | 84 | 0 | details |
| 1 | GPPVA(FSU2H) NS unified inner crust-core | Cold Neutron Star EoS | nucleonic models | Relativistic density functional models | Unified models | – | 0 | 0 | 1 | 6.3e-12 | 3.4 | 282 | 0 | details |
| 1 | RG(SK255) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.5 | 1196 | 0 | details |
| 1 | GMSR(DHSL59) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npemN | 0 | 0 | 1 | 1e-07 | 1 | 385 | 0 | details |
| 1 | RG(SKa) | Cold Neutron Star EoS | nucleonic models | Non-relativistic density functional models | Unified models | npem | 0 | 0 | 1 | 1e-07 | 1.9 | 1238 | 0 | details |

# Diversity of equations of state

20 CompOSE models

+

6 BSk models (20—22; 24—26)

APR model and 4 variations (HHJ)

4 PAL models

Another 4 models
DDME2, NL3wr, SLy4, BHF (av14+uix)

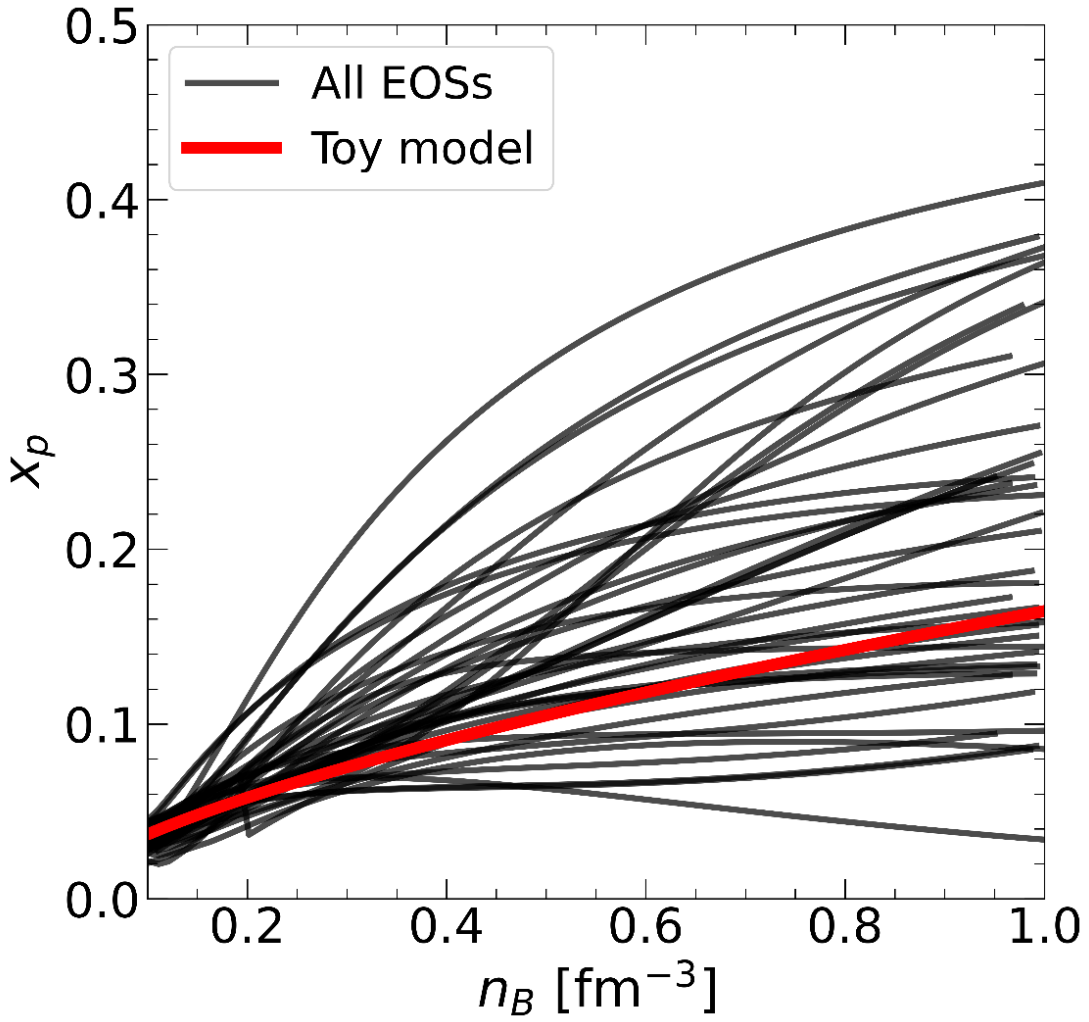
=

39 models in total

Toy model

$$x_p = 0.05 \left( \frac{n_B}{n_0} \right)^{0.65}$$

Proton fraction



# Neutron star conditions: General picture for $npe\mu$ matter

"Kinetic equations"

$$1 = \sum_b \left( \Lambda^k_{ab} \widetilde{\lambda}^k_a + \Lambda'^k_{ab} \widetilde{\lambda}^k_b \right) + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \widetilde{\lambda}^k_a, \quad k = \kappa, \eta$$

$$\widetilde{\lambda}^k_a = \widetilde{\tau}^k_a v_{\mathrm{F}a}$$

Microscopic data: collision probabilities, effective masses

$$\widetilde{\kappa}_a = \frac{\pi^2 n_a T}{3 p_{\mathrm{F}a}} \widetilde{\lambda}^\kappa_a(B)$$

$$\widetilde{\eta}_a = \frac{n_a p_{\mathrm{F}a}}{5} \widetilde{\lambda}^\eta_a(B)$$

Structure of the transport matrix

$$\Lambda^k_{ab} \propto m^{*2}_a m^{*2}_b \langle D^k(\Omega, w) |T_{ab}|^2 \rangle$$

Nuclear sector
Strong interactions (np)
Strong dependence on the model (EOS)

$$|T_{ab}|^2 \neq |T_{ab}(w)|^2 \Rightarrow \Lambda^k_{ab} \propto T^2$$

$$\kappa_B \propto T^{-1}$$
$$\eta_B \propto T^{-2}$$

Lepton sector
Electromagnetic interactions (peμ)
Relatively controlled dependence on EOS

$$|T_{ab}|^2 = |T_{ab}|^2(w)$$

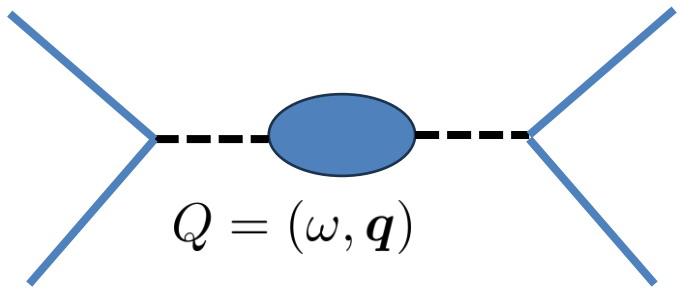
$$\kappa_\ell = \mathrm{const}(T)$$
$$\eta_\ell \propto T^{-5/3}$$

In absence of B-field

$$\kappa = \kappa_{e\mu}[ee, e\mu, ep] + \kappa_n[nn, np] \qquad \eta = \eta_{e\mu}[ee, e\mu, ep] + \eta_n[nn, np]$$

$$\kappa_{e\mu} \ll \kappa_n \qquad \eta_{e\mu} \gg \eta_n$$

# Electromagnetic interactions




Photon propagator in isotropic media

$$D_{\mu\nu}(\omega, \boldsymbol{q}) = \frac{(\mathcal{P}_L)_{\mu\nu}}{Q^2 - \Pi_L(\omega, \boldsymbol{q})} + \frac{(\mathcal{P}_T)_{\mu\nu}}{Q^2 - \Pi_T(\omega, \boldsymbol{q})} + \text{cal.}$$

Contribution from one fermion species

$$\Pi_L^{HDL}(x) = -\frac{Q^2}{|\boldsymbol{q}|^2} q_{TF}^2 \left[1 - x \ln \frac{x+1}{x-1}\right]$$

$$\Pi_T^{HDL}(q_0, q) = (q_{TF} v_F)^2 \left[x^2 + \frac{x(1-x^2)}{4} \ln \frac{x+1}{x-1}\right]$$

$$x = \frac{\omega}{q v_F}$$

$$q_{TF}^2 = \frac{4\alpha_f}{\pi} \frac{p_F^2}{v_F}$$

Projectors

$$\mathcal{P}_T^{00} = \mathcal{P}_T^{0i} = \mathcal{P}_T^{i0} = 0$$

$$\mathcal{P}_T^{ij} = \delta^{ij} - \frac{q^i q^j}{\boldsymbol{q}^2}$$

$$\mathcal{P}_L^{\mu\nu} = \frac{Q^\mu Q^\nu}{Q^2} - g^{\mu\nu} - \mathcal{P}_T^{\mu\nu}$$

In multicomponent system need to sum over the charged species

The screening momentum is generally small, so small q (small-angle, large impact parameters) collisions dominate. This is a consequence of the long-range nature of the electromagnetic force

# Electromagnetic interactions

$$D_{\mu\nu}(\omega,\boldsymbol{q}) = \frac{(\mathcal{P}_L)_{\mu\nu}}{K^2 - \Pi_L(\omega,\boldsymbol{q})} + \frac{(\mathcal{P}_T)_{\mu\nu}}{K^2 - \Pi_T(\omega,\boldsymbol{q})}$$

$$\omega \ll q v_{\mathrm{F}j} \qquad K^2 \approx -q^2$$

Static longitudinal screening (Thomas-Fermi, Debye mass)

$$\Pi_L(\omega,q) \approx \Pi_L(0,q) = q_{TF}^2 \equiv \frac{4\alpha_f}{\pi} \sum_{j=e,p,\mu} \frac{p_{\mathrm{F}j}^2}{v_{Fj}}$$

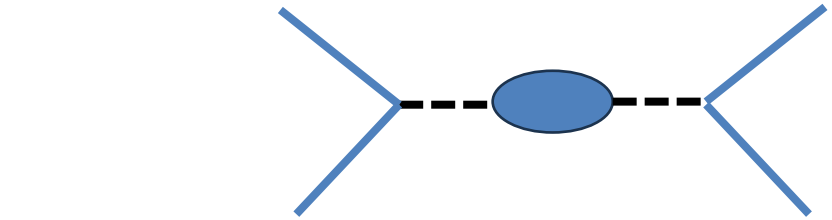
Transverse screening is dynamical (Landau damping)

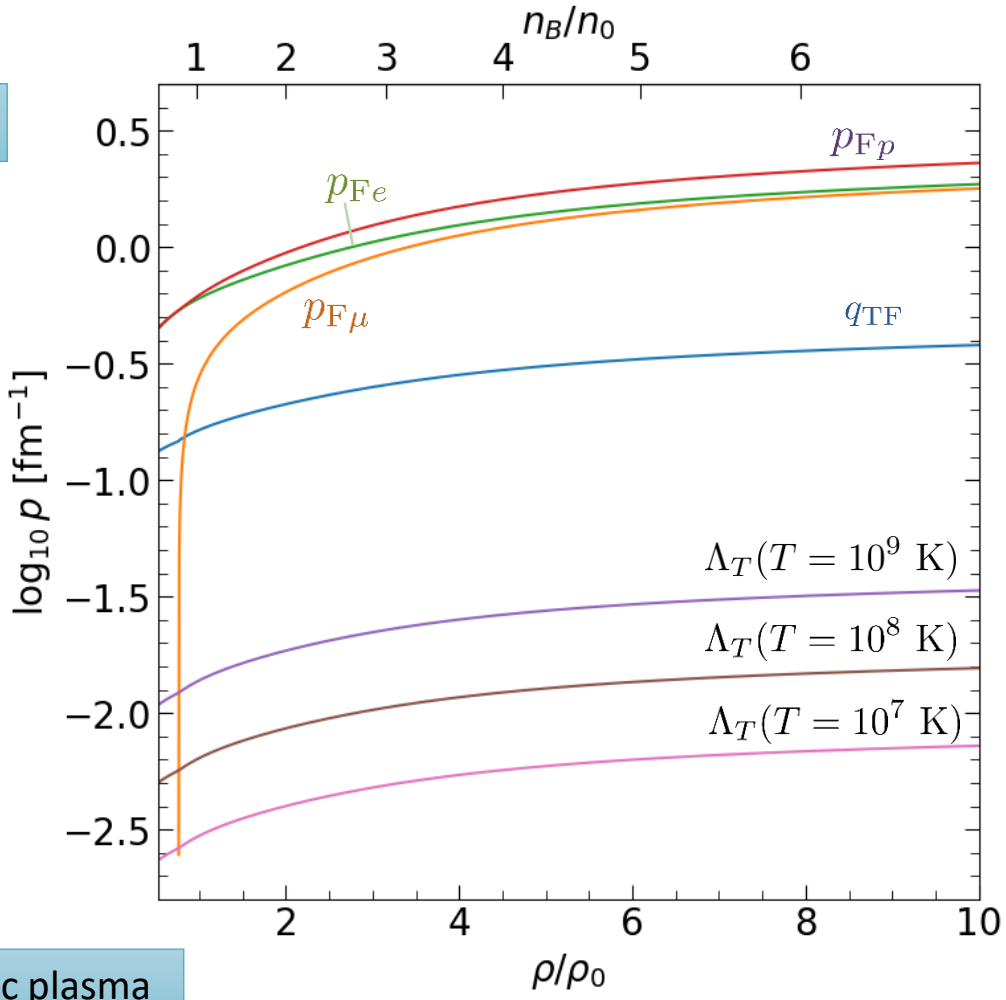
$$\Pi_T(\omega,q) = i\frac{\omega}{q}\alpha_f \sum_{j=e,p,\mu} p_{\mathrm{F}j}^2 \equiv i\Lambda^3/q = iw\Lambda_T^3/q$$

Characteristic transverse screening wavenumber

$$q_t^2 = \sum_a \frac{4\alpha_f}{\pi} p_{Fa}^2 \qquad \Lambda_T = \left(T\alpha_f \sum_j p_{\mathrm{F}j}^2\right)^{1/3} \ll q_{\mathrm{TF}}$$

Transverse plasmon exchange dominates the collisions in relativistic plasma





# Electromagnetic interactions

Matrix element of the electromagnetic interaction

$$T_{ab} = 4\pi\alpha_f \mathcal{J}_a^\mu D_{\mu\nu} \mathcal{J}_b^\nu$$

Electromagnetic current (we immediately disregard the formfactors)

$$\mathcal{J}_a^\mu = \overline{u}_a(p_{1'}) \left( F_{a1}(-q^2)\gamma^\mu + i\frac{F_{a2}(-q^2)}{2m_a}\sigma^{\mu\nu}q_\nu \right) u_a(p_1)$$

$$\mathcal{Q}_{ab} = 16\pi^2\alpha_f^2 \left( \frac{L_l}{\left|q^2 + \Pi_l(\omega,q)\right|^2} - 2\Re\frac{v_a v_b L_{tl}}{(q^2+\Pi_l(\omega,q))(q^2+iw\Lambda_T^3/k)^*} + \frac{v_a^2 v_b^2 L_t}{\left|q^2+iw\Lambda_T^3/q\right|^2} \right)$$

Spin-averaged collision rate contains contributions from transverse, longitudinal and mixed screening channels

Kinematic factors

$$L_l = \left(1-\frac{q^2}{4m_a^{*2}}\right)\left(1-\frac{q^2}{4m_b^{*2}}\right) \approx 1 \qquad L_{tl} = \sqrt{\left(1-\frac{q^2}{4p_a^2}\right)\left(1-\frac{q^2}{4p_b^2}\right)}\cos\phi \approx \cos\phi$$

$$L_t = \left(1-\frac{q^2}{4p_a^2}\right)\left(1-\frac{q^2}{4p_b^2}\right)\cos^2\phi + \frac{q^2}{4p_a^2} + \frac{q^2}{4p_b^2} \approx \cos^2\phi$$

The transverse channel dominates, which makes the transition amplitudes frequency-dependent leading to the non-Fermi liquid behavior of transport coefficients (in a sense of powers of temperature)

$$\Lambda_T \ll q_{\mathrm{TF}}$$

# Decomposition of the transport matrices

$$1 = \sum_b \left( \Lambda^k_{ab} \widetilde{\lambda}^k_a + \Lambda'^k_{ab} \widetilde{\lambda}^k_b \right) + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \widetilde{\lambda}^k_a \qquad k = \kappa,\ \eta$$

$$\mathcal{Q}_{ab} = 16\pi^2 \alpha_f^2 \left( \frac{1}{|q^2 + q_{TF}^2|^2} - 2\Re \frac{v_a v_b \cos\phi}{(q^2 + q_{TF})(q^2 + iw\Lambda_T^3/k)^*} + \frac{v_a^2 v_b^2 \cos^2\phi}{|q^2 + iw\Lambda_T^3/q|^2} \right)$$

Transport matrices contain contributions from different screening channels

$$\Lambda_{ab} = \Lambda^l_{ab} + \Lambda^t_{ab} \qquad \Lambda^l_{ab} \propto T^2 \qquad \Lambda^t_{ab} \propto T^{2-\xi}$$

$$I_l^{(n)} = \int_0^{q_m} \frac{q^n \mathrm{d}q}{|q^2 + q_l^2|^2} \qquad I_{tl}^{(n)} = \int_0^{q_m} \frac{q^{n+4} \mathrm{d}q}{(q^2 + q_l^2)(q^6 + w^2\Lambda_T^6)} \qquad I_t^{(n)} = \int_0^{q_m} \frac{q^{n+2} \mathrm{d}q}{q^6 + w^2\Lambda_T^6}$$

Leading order $\kappa: \ n = 0 \quad \eta: \ n = 2$

Asymptotic expressions for small screening momenta (very good approximation)

$$I_t^{(0)} = \frac{\pi}{6|w|\Lambda_T^3} + \mathcal{O}(\Lambda_T^{-1}), \quad I_t^{(2)} = \frac{\pi}{3|w|^{1/3}\Lambda_T} + \mathcal{O}(\Lambda_T),$$

$$I_{tl}^{(0)} = \frac{\pi^2}{3q_l^2|w|^{1/3}\Lambda_T} + \mathcal{O}(\Lambda^0) \qquad I_{tl}^{(2)} \approx I_l^{(2)}$$

$$I_l^{(n)} \approx \frac{\pi}{4q_{TF}^{3-n}}$$

$$\Lambda^\kappa_{ab} = \frac{3T^2 m_a^{*2} m_b^{*2}}{4\pi^4 p_{\mathrm{F}a}^2} \int_w \left\langle \left( \frac{w^2}{\pi^2} + \left[ \frac{1}{3} - \frac{w^2}{6\pi^2} \right] \frac{q^2}{p_{\mathrm{F}a}^2} \right) \mathcal{Q}_{ab} \right\rangle$$

$$\Lambda'^\kappa_{ab} = -\frac{3T^2 m_a^{*2} m_b^{*2}}{4\pi^4 p_{\mathrm{F}a}^2} \int_w \frac{w^2}{\pi^2} \left\langle \left( \frac{\boldsymbol{p}_1(\boldsymbol{p}_2 + \boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}} \right) \mathcal{Q}_{ab} \right\rangle,$$

$$\Lambda^\eta_{ab} = \frac{3T^2 m_a^{*2} m_b^{*2}}{4\pi^4 p_{\mathrm{F}a}^2} \int_w \left\langle \frac{q^2}{p_{\mathrm{F}a}^2} \left( 1 - \frac{q^2}{4p_{\mathrm{F}a}^2} \right) \mathcal{Q}_{ab} \right\rangle$$

$$\Lambda'^\eta_{ab} = -\frac{3T^2 m_a^{*2} m_b^{*2}}{4\pi^4 p_{\mathrm{F}a}^2} \int_w \left\langle \frac{q^2}{p_{\mathrm{F}a}^2} \left( \frac{\boldsymbol{p}_1(\boldsymbol{p}_2 + \boldsymbol{p}_{2'})}{2p_{\mathrm{F}a}p_{\mathrm{F}b}} \right) \mathcal{Q}_{ab} \right\rangle$$

$$J_{ab} = \frac{T^2 m_a^{*2} m_b^{*2}}{36\pi^4} \langle q^2 \mathcal{Q}_{ab} \rangle$$

$$\int_w = \int_0^\infty \mathrm{d}w \frac{(w/2)^2}{\sinh^2(w/2)}$$

$$\int_w w^k = \Gamma(3+k)\zeta(2+k)$$

# Electromagnetic interactions

$$1=\sum_b\left(\Lambda^k_{ab}\widetilde{\lambda}^k_a+\Lambda'^k_{ab}\widetilde{\lambda}^k_b\right)+i\frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}}\widetilde{\lambda}^k_a \qquad \Lambda_{ab}=\Lambda^l_{ab}+\Lambda^t_{ab} \qquad \Lambda^l_{ab}\propto T^2 \quad \Lambda^t_{ab}\propto T^{2-\xi} \qquad q_t^2=\sum_a\frac{4\alpha_f}{\pi}p_{Fa}^2$$

The leading-order expressions read

$$\Lambda^t\gg\Lambda^l,\,\Lambda^{lt} \qquad \Lambda^\kappa\gg\Lambda^\eta$$

Thermal conductivity

$$\Lambda^{\kappa,t}_{ab}=\frac{24\zeta(3)T}{\pi^3 q_t^2}\alpha_f^2 p_{\mathrm{F}b}^2$$

Shear viscosity

$$\Lambda^{\eta,t}_{ab}=2\left(\frac{4}{\pi}\right)^{1/3}\Gamma(8/3)\zeta(5/3)\alpha_f^2\frac{p_{\mathrm{F}b}^2T^{5/3}}{p_{\mathrm{F}a}^2q_t^{2/3}}$$

Momentum transfer rates

$$J^t_{ab}=\frac{n_a p_{Fa}}{3}\Lambda^{\eta,t}_{ab}$$

Thermal conductivity in leading order does not depend on T!

$B=0$

$$\kappa^t_a=\frac{\pi^2}{54\zeta(3)}\frac{k_B p_{Fa}^2}{\alpha_f}$$

$$\eta^t_a=\frac{1.1}{\alpha_f}\frac{n_a^2}{q_t^{1/3}}(k_BT)^{-5/3}$$

$$\Lambda'^\kappa_{ab}=\left(\frac{4}{\pi}\right)^{1/3}\Gamma\left(\frac{14}{3}\right)\zeta\left(\frac{11}{3}\right)\alpha_f^2\frac{m_a^*m_b^*p_{\mathrm{F}b}T^{5/3}}{p_{\mathrm{F}a}q_l^2q_t^{2/3}}$$

$$\Lambda^{\kappa,l}_{ab}=\frac{4\pi^2\alpha_f^2T^2m_a^{*2}m_b^{*2}}{5p_{\mathrm{F}a}^2q_l^3} \qquad \Lambda^{\eta,l}_{ab}=\frac{\pi^2\alpha_f^2T^2m_a^{*2}m_b^{*2}}{p_{\mathrm{F}a}^4q_l}$$

$$J^l_{ab}=\frac{\alpha_f^2T^2m_a^{*2}m_b^{*2}}{9q_l} \qquad \Lambda'^\eta_{ab}=\frac{2\pi^2\alpha_f^2T^2m_a^*m_b^*p_b}{p_{\mathrm{F}a}^3q_l}$$

$$\Lambda^\kappa_{\mathrm{em}}=\begin{pmatrix} e & \mu & p \\ 2.35 & 0.009 & 0.04 \\ 0.013 & 2.35 & 0.05 \\ 0.033 & 0.028 & 2.56 \end{pmatrix}\times10^6\ \mathrm{cm}^{-1}.$$

$n_B=0.35\ \mathrm{fm}^{-3}$

$x_p=0.15$

$T=10^8\ \mathrm{K}$

$$\Lambda^\eta_{\mathrm{em}}=\begin{pmatrix} e & \mu & p \\ 2.5 & 0.05 & 0.3 \\ 0.11 & 3.7 & 0.47 \\ 0.15 & 0.12 & 5.45 \end{pmatrix}\times10^3\ \mathrm{cm}^{-1}.$$

# Electromagnetic mean free path in NS cores

$k = \kappa,\ \eta$

$$1 = \sum_b \left( \Lambda^k_{ab} \widetilde{\lambda}^k_a + \Lambda'^k_{ab} \widetilde{\lambda}^k_b \right) + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \widetilde{\lambda}^k_a$$

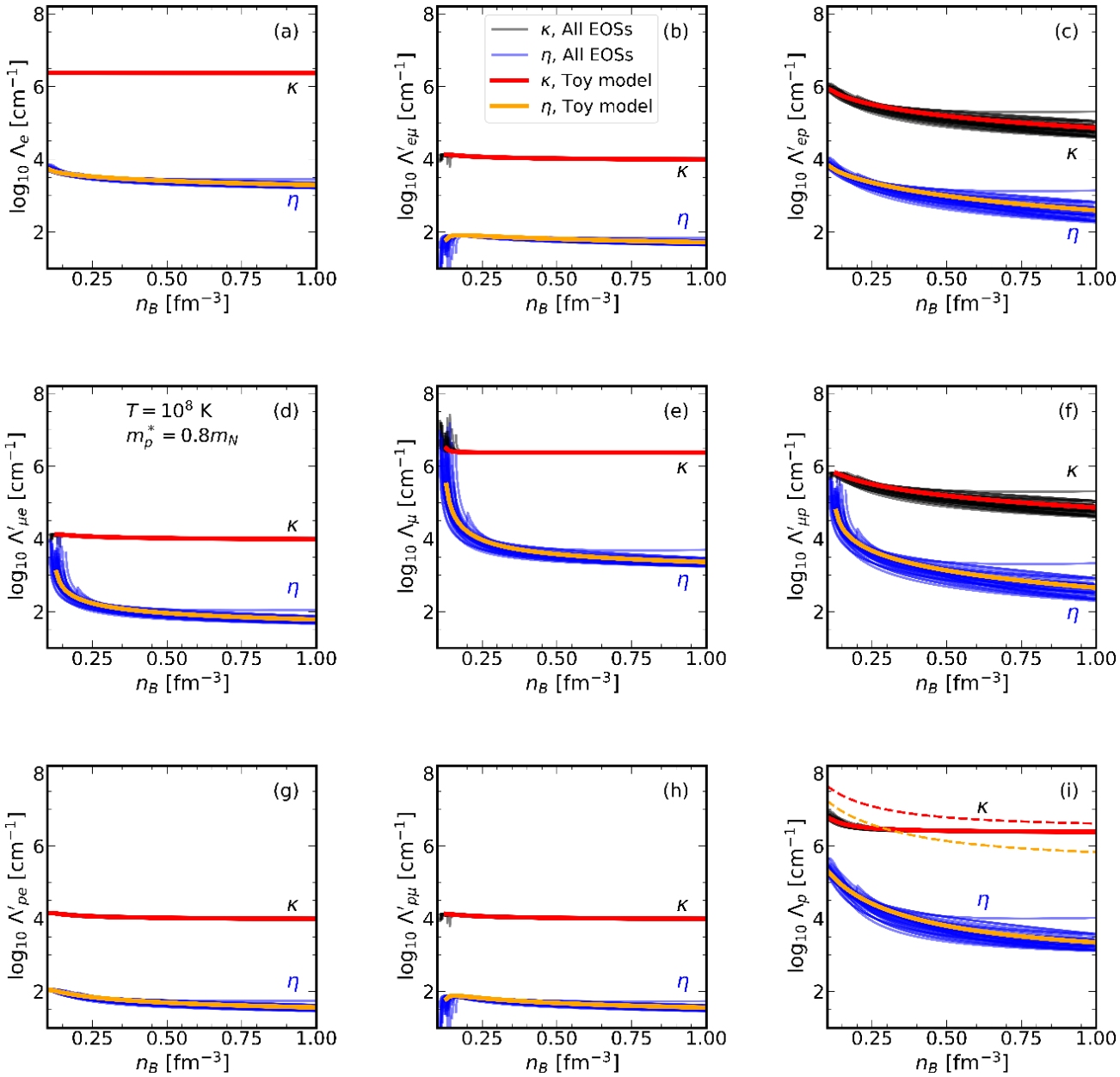
$$1 = \left( \Lambda^k_a + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \right) \widetilde{\lambda}^k_a + \sum_{b \neq a} \Lambda'^k_{ab} \widetilde{\lambda}^k_b,$$

$$\Lambda^k_a = \sum_b \Lambda^k_{ab} + \Lambda'^k_{aa}.$$

One can neglect non-diagonal matrix elements so partial equations are decoupled

Resembles single-particle relaxation-time approach

$$\widetilde{\lambda}^k_\ell = \frac{1}{\Lambda^k_\ell + i\omega_{\mathrm{BF}\ell}/v_{\mathrm{F}\ell}}, \quad \ell = e,\ \mu$$



# Electromagnetic interactions in relativistic plasmas

$$D_{\mu\nu}(\omega,\boldsymbol{q}) = \frac{(\mathcal{P}_L)_{\mu\nu}}{K^2 - \Pi_L(\omega,\boldsymbol{q})} + \frac{(\mathcal{P}_T)_{\mu\nu}}{K^2 - \Pi_T(\omega,\boldsymbol{q})}$$

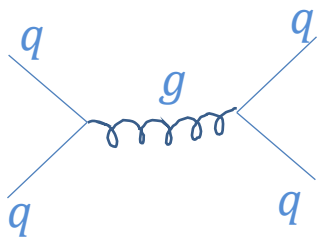
From kinematics, transferred momentum varies as $q \in [\omega/v_{\rm F}, 2p_{\rm F}]$ with $\omega \sim T$

Temperature scale: the plasma temperature $T_{pe} = \dfrac{\hbar\omega_{pe}}{k_B} = \sqrt{\dfrac{4\pi\hbar^2 e^2 n_e}{k_B^2 m_e^*}}$

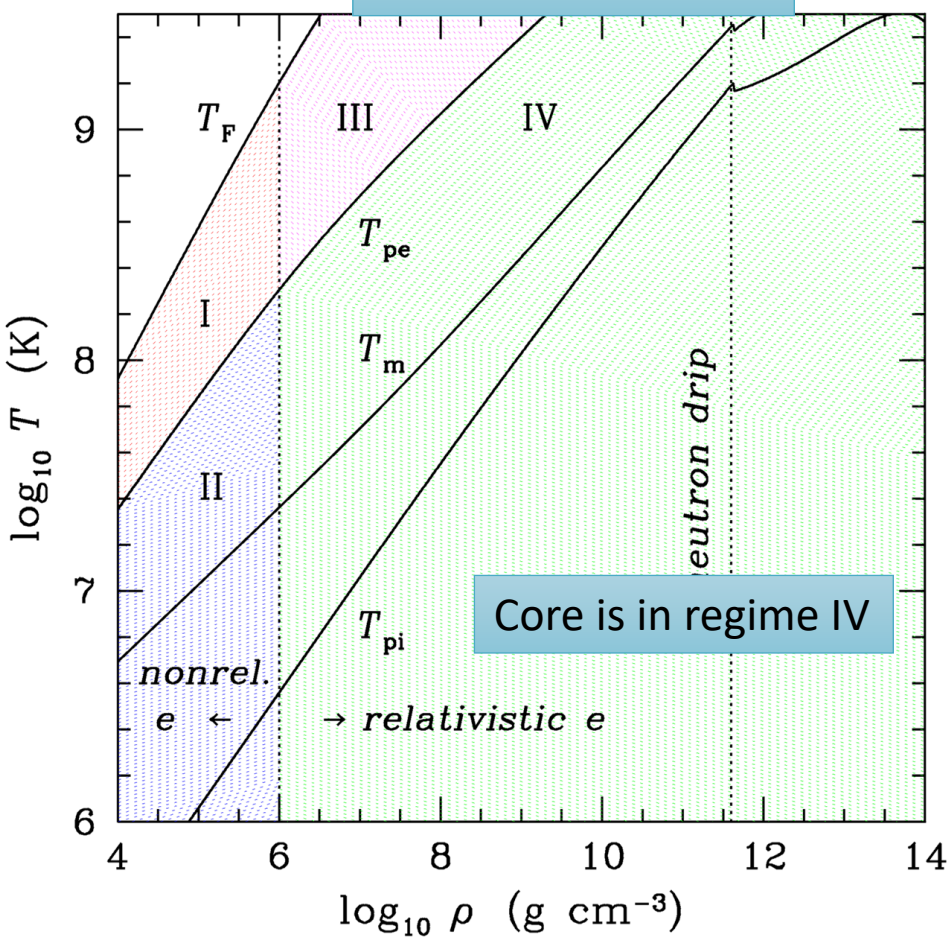
Screening depends on the relation between $T$ and $T_{pe}$

| Regime | Electron velocity | Temperature | Main contribution | $T$-dependence of $\kappa_{ee}$ |
|---|---|---|---|---|
| I | $v_e \ll c$ | $T \gtrsim T_{pe}$ | $\Lambda^l$ | $T^2/\ln(T/T_{pe})$ |
| II | $v_e \ll c$ | $T \ll T_{pe}$ | $\Lambda^l$ | $1/T$ |
| III | $v_e \approx c$ | $T \gtrsim T_{pe}$ | $\Lambda^l + \Lambda^t + \Lambda^{lt}$ | $T^2/\ln(T/T_{pe})$ |
| IV | $v_e \approx c$ | $T \ll T_{pe}$ | $\Lambda^t$ | const |

General form in relativistic plasmas (quark matter in weak-coupling regime), anomalous metals




Electrons in NS crust



# Nucleon sector

# Strategies in the nucleon sector

- IDEAL (HARD): calculate EOS and transport based on the same microscopic models

Rarely available

- CHEATING: calculate transport from some favorite microscopic model for in-medium scattering but using particle fractions (and probably m* if available) from all EOS

BHF calculations, 5 NN potentials, Interpolation of $\Lambda(n_B, x_p)$

# Strategies in the nucleon sector

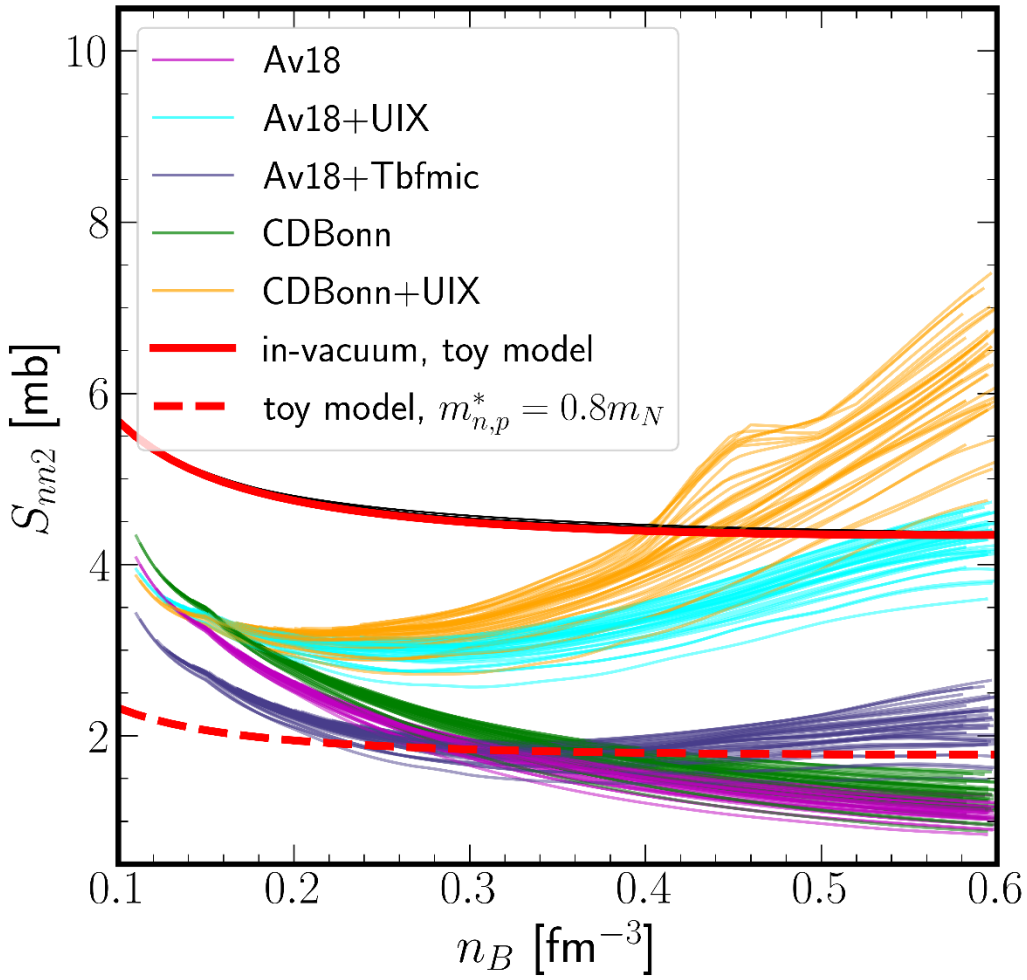
Vacuum (black and red) and in-medium (colored) transport cross-sections for 39 EOS

Only one many-body method: BHF calculations
5 NN potentials
Interpolation of $\Lambda(n_B, x_p)$

Shternin & Baldo *PRD* 102, 063010 (2020)

Unfortunate situation

$$\Lambda^{\kappa}_{nn} + \Lambda'^{\kappa}_{nn} = \frac{64T^2 m_n^{*4}}{5m_N^2 p_{\mathrm{F}n}} S_{nn2}(p_{\mathrm{F}n})$$



# Strategies in the nucleon sector

- IDEAL (HARD): calculate EOS and transport based on the same microscopic models

Rarely available

- CHEATING: calculate transport from some favorite microscopic model for in-medium scattering but using particle fractions (and probably m* if available) from all EOS

BHF calculations, 5 NN potentials, Interpolation of $\Lambda(n_B, x_p)$

- QUALITATIVE: use in-vacuum NN cross-sections

$$\frac{\mathrm{d}\sigma_{ab}}{\mathrm{d}\Omega}(E_{\mathrm{lab}}, \theta_{\mathrm{cm}}) = \frac{m_N^2}{16\pi^2}(1+\delta_{ab})\mathcal{Q}_{ab}$$

- POOR MAN: use in-vacuum NN cross-sections and toy model EOS

Allows to construct single-parameter fitting expressions $(n_B)$

# Effective transport cross-sections

$$\Lambda_{ab} \propto m_a^{*2} m_b^{*2} T^2 S_{ab}$$

DO NOT LOOK AT THESE EXPRESSIONS

## Transport matrix

$$\begin{aligned}
\Lambda^{\kappa}_{nn} + \Lambda'^{\kappa}_{nn} &= \frac{64T^2 m_n^{*4}}{5m_N^2 p_{\mathrm{F}n}} S_{nn2}(p_{\mathrm{F}n}) \\
\Lambda^{\kappa}_{pp} + \Lambda'^{\kappa}_{pp} &= \frac{64T^2 m_p^{*4}}{5m_N^2 p_{\mathrm{F}p}} S_{nn2}(p_{\mathrm{F}p}) \\
\Lambda^{\kappa}_{np} &= \frac{64T^2 m_n^{*2} m_p^{*2}}{5m_N^2 p_{\mathrm{F}n}} (S_{np1} + S_{np2}) \\
\Lambda^{\kappa}_{pn} &= \frac{64T^2 m_n^{*2} m_p^{*2} p_{\mathrm{F}n}}{5m_N^2 {p_{\mathrm{F}p}}^2} \left( S_{np1} + \frac{p_{\mathrm{F}n}^2}{p_{\mathrm{F}p}^2} S_{np2} \right) \\
\Lambda'^{\kappa}_{np} &= \frac{p_{\mathrm{F}p}^2}{p_{\mathrm{F}n}^2} \Lambda'^{\kappa}_{pn} = \frac{64T^2 m_n^{*2} m_p^{*2}}{5m_N^2 p_{\mathrm{F}p}} S'_{np2} \\
\Lambda^{\eta}_{aa} + \Lambda'^{\eta}_{aa} &= \frac{64T^2 m_a^{*4}}{m_N^2 p_{\mathrm{F}a}} S_{nn4}(p_{\mathrm{F}a}) \\
\Lambda^{\eta}_{np} &= \frac{64T^2 m_n^{*2} m_p^{*2}}{m_N^2 p_{\mathrm{F}n}} (S_{p2} - S_{p4}) \\
\Lambda^{\eta}_{pn} &= \frac{64T^2 m_n^{*2} m_p^{*2}}{m_N^2 p_{\mathrm{F}p}} \left( \frac{p_{\mathrm{F}n}^3}{p_{\mathrm{F}p}^3} S_{np2} - \frac{p_{\mathrm{F}n}^5}{p_{\mathrm{F}p}^5} S_{np4} \right) \\
\Lambda'^{\eta}_{np} &= \frac{p_{\mathrm{F}p}^4}{p_{\mathrm{F}n}^4} \Lambda'^{\eta}_{pn} = \frac{64T^2 m_n^{*2} m_p^{*2}}{m_N^2 p_{\mathrm{F}p}} S'_{np4} \\
J_{np} &= J_{pn} = \frac{64}{9\pi^2} T^2 \frac{m_n^{*2} m_p^{*2}}{m_N^2} p_{\mathrm{F}n}^3 S_{np2}
\end{aligned}$$

$$\frac{\mathrm{d}\sigma_{ab}}{\mathrm{d}\Omega}(E_{\mathrm{lab}}, \theta_{\mathrm{cm}}) = \frac{m_N^2}{16\pi^2}(1 + \delta_{ab}) \mathcal{Q}_{ab}$$

## Effective cross-section [mb]

$$\begin{aligned}
S_{nn2}(p_{\mathrm{F}a}) &= \frac{m_N^2}{128\pi^2 p_{\mathrm{F}a}^3} \left\langle (q^2 + q'^2) \mathcal{Q}_{aa} \right\rangle \\
S_{np1} &= \frac{m_N^2}{64\pi^2 p_{\mathrm{F}n}} \langle \mathcal{Q}_{np} \rangle \\
S_{np2} &= \frac{m_N^2}{256\pi^2 p_{\mathrm{F}n}^3} \langle q^2 \mathcal{Q}_{np} \rangle \\
S_{nn4}(p_{\mathrm{F}a}) &= \frac{m_N^2}{16\pi^2} \frac{1}{(2p_{\mathrm{F}a})^5} \langle q^2 q'^2 \mathcal{Q}_{aa} \rangle \\
S_{np4} &= \frac{m_N^2}{16\pi^2} \frac{1}{2(2p_{\mathrm{F}n})^5} \langle q^4 \mathcal{Q}_{np} \rangle \\
S'_{np4} &= \frac{m_N^2}{16\pi^2} \frac{1}{32 p_{\mathrm{F}n}^5} \left\langle q^2 \left( (p_{\mathrm{F}n}^2 + p_{\mathrm{F}p}^2) - P^2 - \frac{q^2}{2} \right) \mathcal{Q}_{np} \right\rangle \\
S'_{np2} &= \frac{m_N^2}{16\pi^2} \frac{1}{8 p_{\mathrm{F}n}^3} \left\langle \left( (p_{\mathrm{F}n}^2 + p_{\mathrm{F}p}^2) - P^2 - \frac{q^2}{2} \right) \mathcal{Q}_{np} \right\rangle
\end{aligned}$$

# Transport matrix in the qualitative/poor man approach

$$k = \kappa,\ \eta$$

$$1 = \sum_b \left( \Lambda^k_{ab} \widetilde{\lambda}^k_a + \Lambda'^k_{ab} \widetilde{\lambda}^k_b \right) + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \widetilde{\lambda}^k_a$$

$$1 = \left( \Lambda^k_a + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \right) \widetilde{\lambda}^k_a + \sum_{b \neq a} \Lambda'^k_{ab} \widetilde{\lambda}^k_b$$

$$\Lambda^k_a = \sum_b \Lambda^k_{ab} + \Lambda'^k_{aa}.$$

Protons weakly affect neutrons

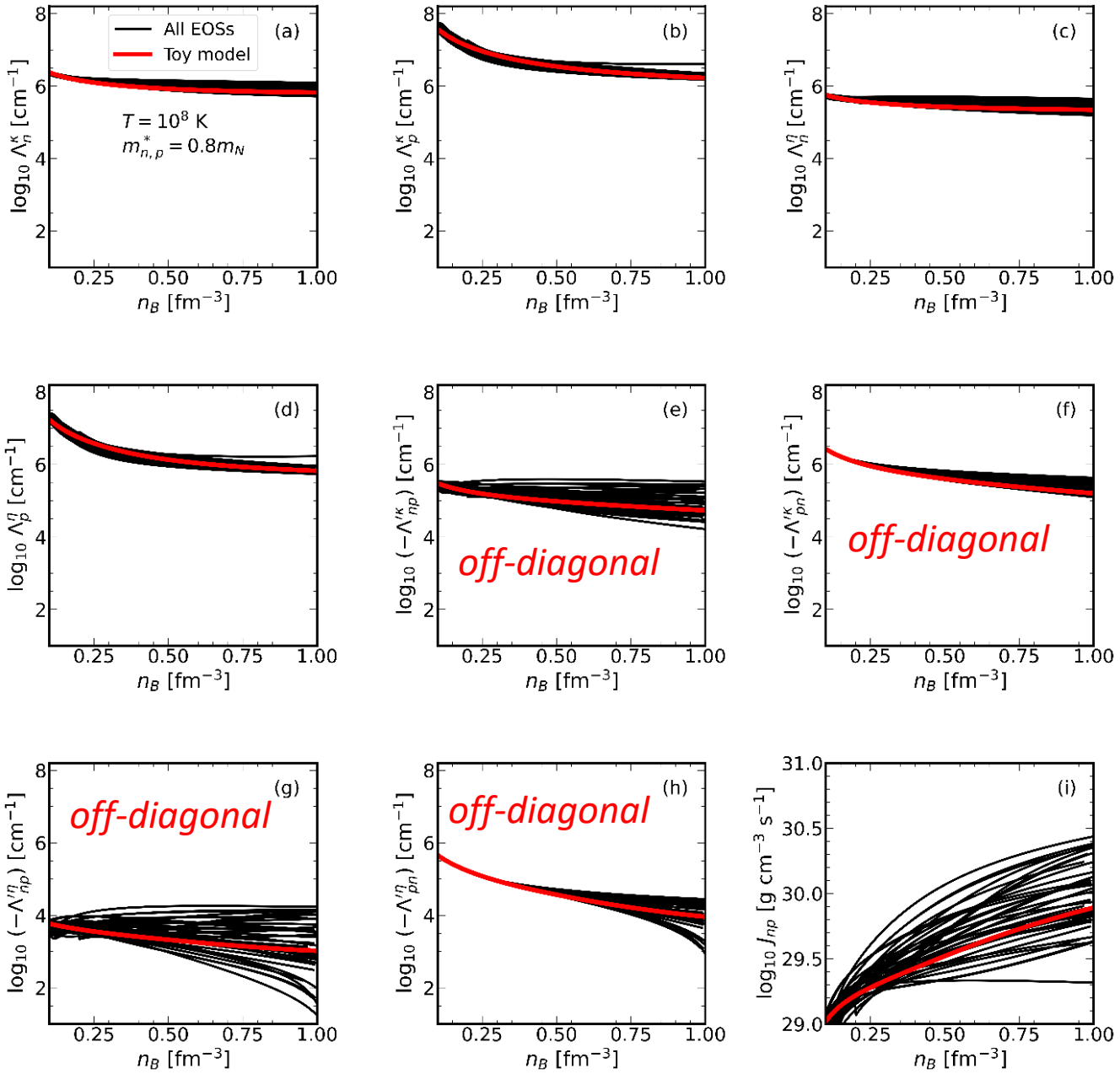
$$\Lambda'^k_{np} \mathrm{Re}\widetilde{\lambda}_p \ll 1$$

Results are again similar to r.t.a.

$$\mathrm{Re}\widetilde{\lambda}^k_n \approx \frac{1}{\Lambda^k_n}$$

Neutrons are magnetized due to interaction with protons

$$\mathrm{Im}\,\widetilde{\lambda}^k_n = -\frac{\Lambda'^k_{np}}{\Lambda^k_n}\ \mathrm{Im}\,\widetilde{\lambda}^k_p$$

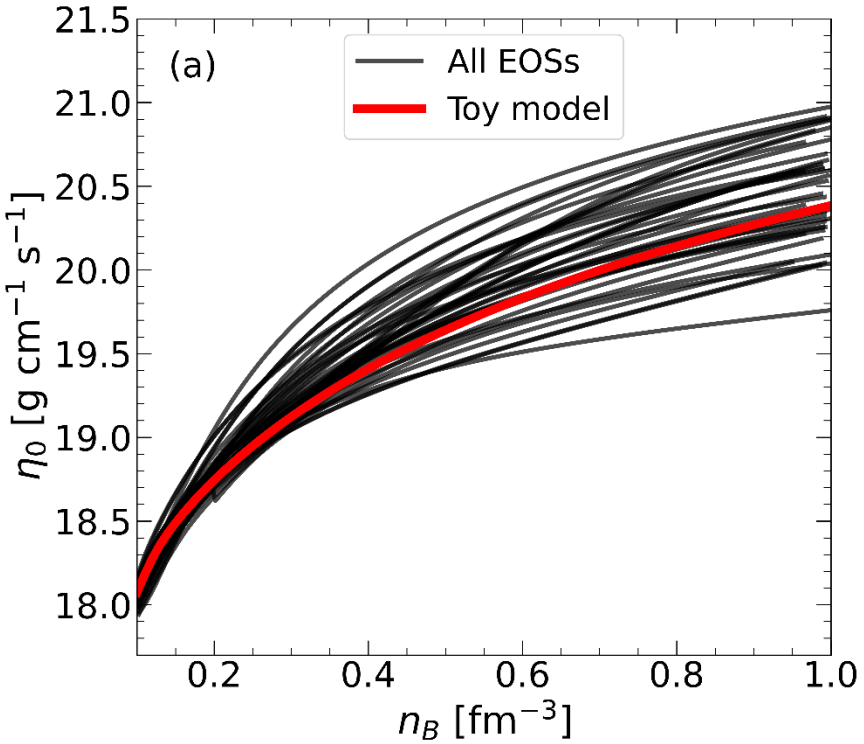


# Poor man estimates (B=0)

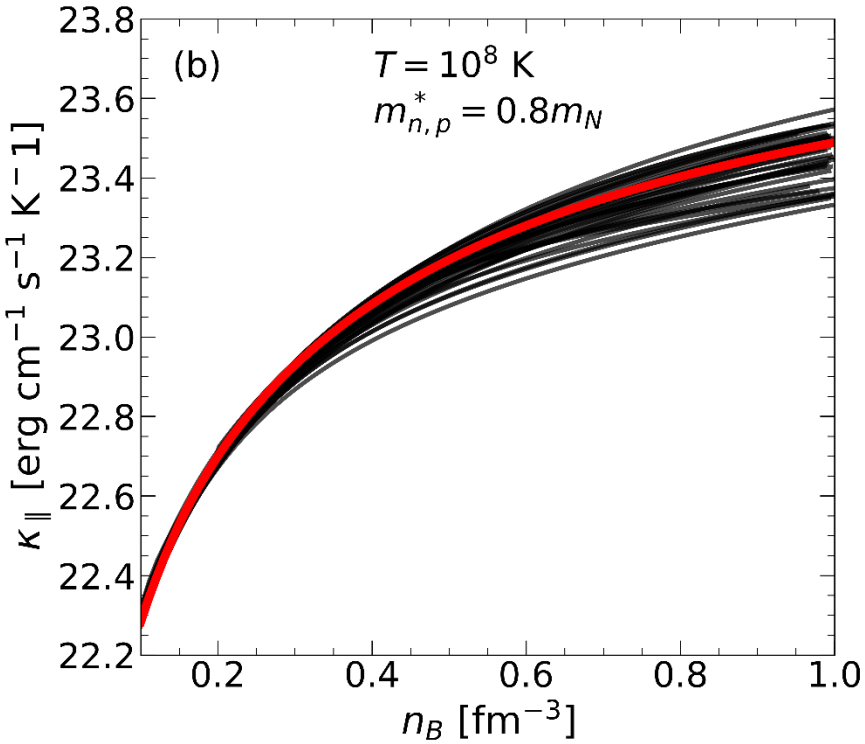
$$\widetilde{\eta}_a = \frac{n_a p_{\mathrm{F}a}}{5}\widetilde{\lambda}^{\eta}_a \qquad \widetilde{\kappa}_a = \frac{\pi^2 n_a T}{3 p_{\mathrm{F}a}}\widetilde{\lambda}^{\kappa}_a$$

$B$-parallel components (independent on $B$)

$\eta_0 = \widetilde{\eta}(B=0)$



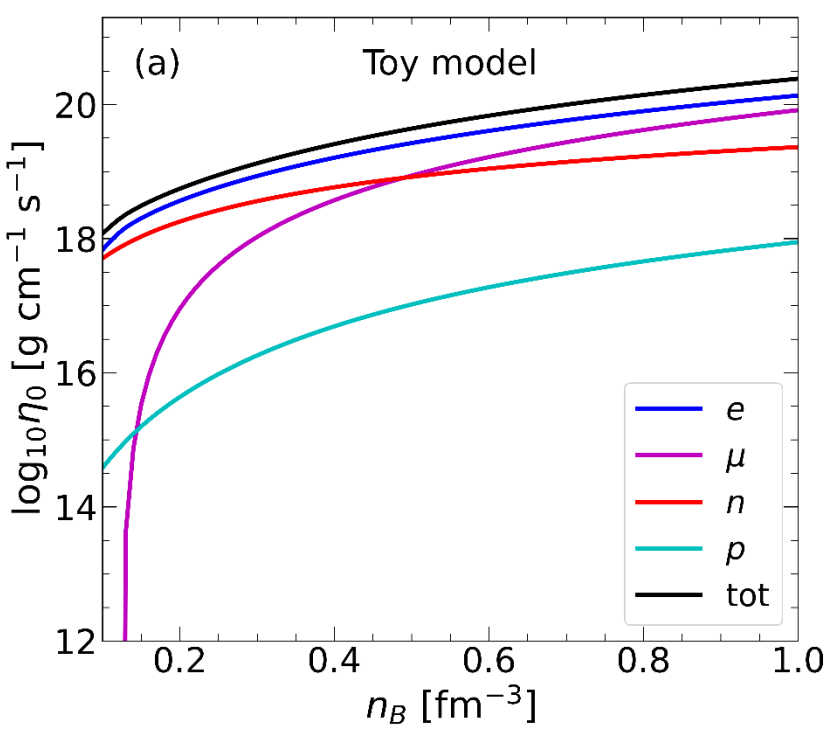

$\kappa_{\parallel} = \widetilde{\kappa}(B=0)$

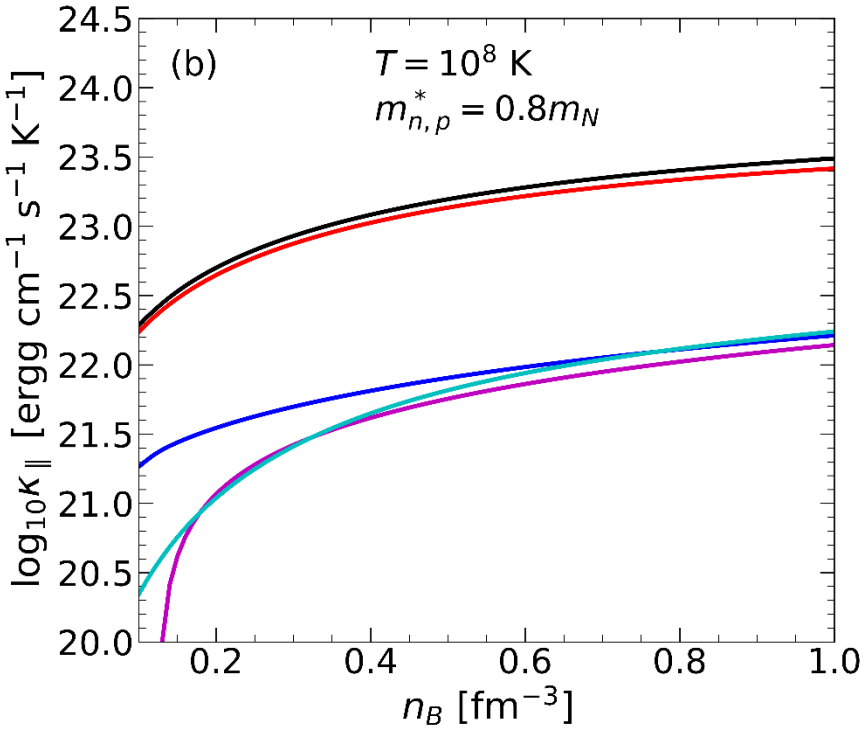


# Partial contributions computed by a poor man

$$\widetilde{\eta}_a = \frac{n_a p_{\mathrm{F}a}}{5}\widetilde{\lambda}^{\eta}_a \qquad \eta_0 = \widetilde{\eta}(B=0)$$

$$\widetilde{\kappa}_a = \frac{\pi^2 n_a T}{3p_{\mathrm{F}a}}\widetilde{\lambda}^{\kappa}_a \qquad \kappa_{\parallel} = \widetilde{\kappa}(B=0)$$




Leptons dominate

Neutrons dominate

# B-dependence

$$\widetilde{\kappa}_a = \frac{\pi^2 n_a T}{3 p_{\mathrm{F}a}} \widetilde{\lambda}_a^{\kappa}$$

$$\widetilde{\eta}_a = \frac{n_a p_{\mathrm{F}a}}{5} \widetilde{\lambda}_a^{\eta}$$

Magnetized species disappear from perpendicular transport

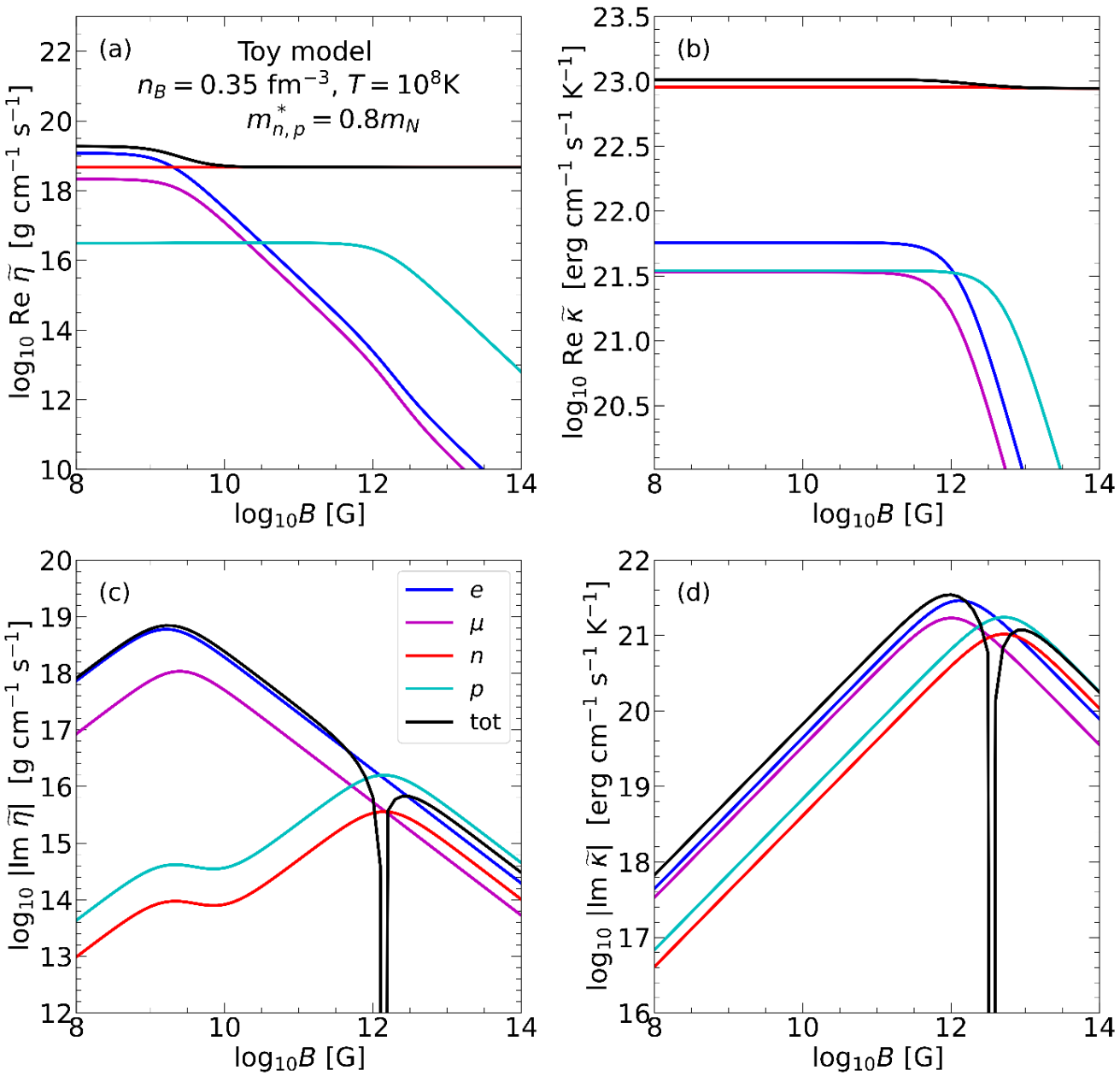
$$x_{\mathrm{Hall}} = \omega_{\mathrm{BF}a} \lambda_a / v_{Fa}$$

$$1 = \left( \Lambda_a^k + i \frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}} \right) \widetilde{\lambda}_a^k + \sum_{b \neq a} \Lambda'^k_{ab} \widetilde{\lambda}_b^k,$$

$$\widetilde{\lambda}_\ell^k = \frac{1}{\Lambda_\ell^k + i \omega_{\mathrm{BF}\ell} / v_{\mathrm{F}\ell}}$$

$$\mathrm{Re}\ \widetilde{\lambda}_n^k = \frac{1}{\Lambda_n^k}$$

$$\mathrm{Re}\ \widetilde{\lambda}_p^k = \mathrm{Re}\ \frac{1 - \Lambda'^k_{pn} / \Lambda_n^k}{\Lambda_p^k + i \omega_{\mathrm{BF}p} / v_{\mathrm{F}p}}$$



# Hall components

$$1 = \left(\Lambda_a^k + i\frac{\omega_{\mathrm{BF}a}}{v_{\mathrm{F}a}}\right)\widetilde{\lambda}_a^k + \sum_{b\neq a}\Lambda'^k_{ab}\widetilde{\lambda}_b^k,$$

**leptons**

$$\widetilde{\lambda}_\ell^k = \frac{1}{\Lambda_\ell^k + i\omega_{\mathrm{BF}\ell}/v_{\mathrm{F}\ell}}$$

**neutrons**

$$\mathrm{Im}\,\widetilde{\lambda}_n^k = -\frac{\Lambda'^k_{np}}{\Lambda_n^k}\,\mathrm{Im}\,\widetilde{\lambda}_p^k$$

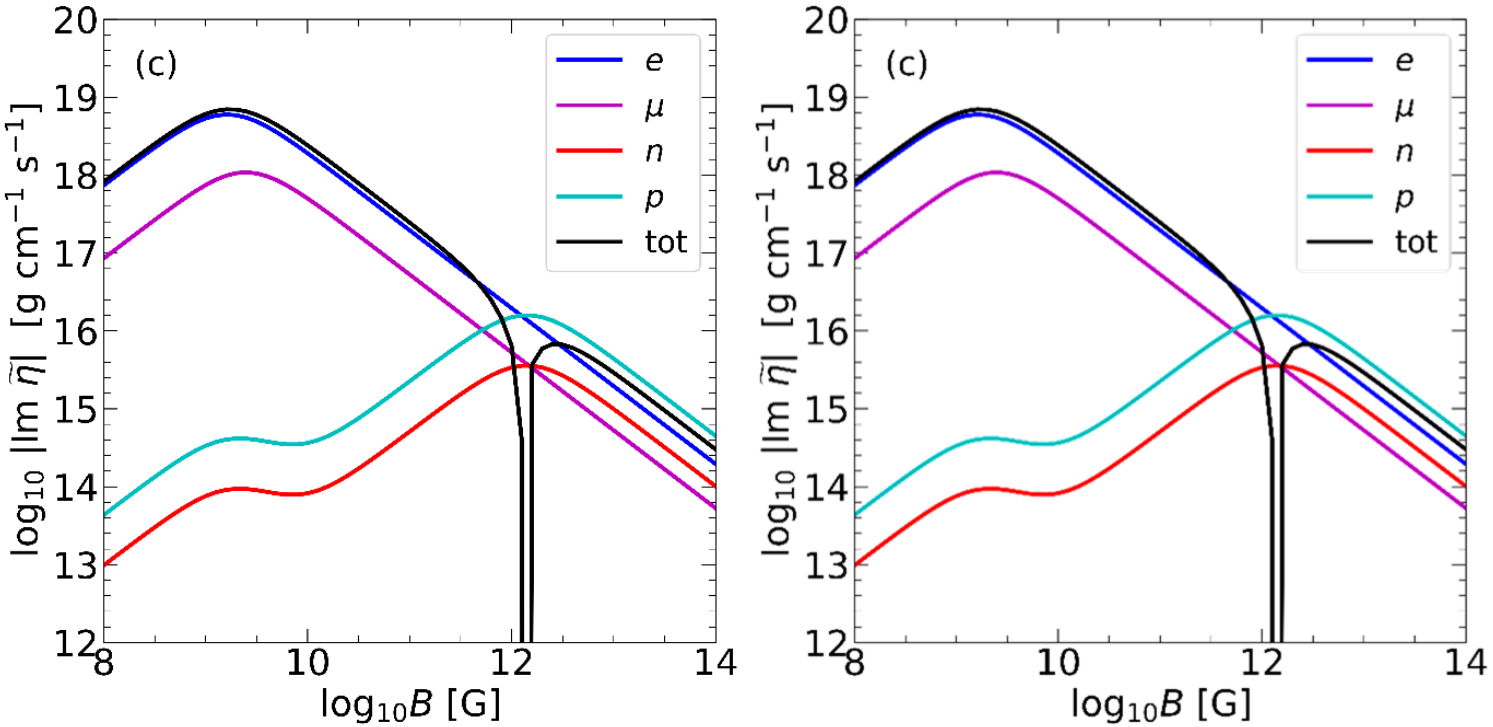
**protons**

$$\left(\Lambda_p^k - \frac{\Lambda'^k_{np}\Lambda'^k_{pn}}{\Lambda_n^k}\right)\mathrm{Im}\,\widetilde{\lambda}_p^k = -\frac{\omega_{\mathrm{BF}p}}{v_{\mathrm{F}p}}\,\mathrm{Re}\,\widetilde{\lambda}_p - \Lambda'^k_{pe}\,\mathrm{Im}\,\widetilde{\lambda}_e^k - \Lambda'^k_{p\mu}\,\mathrm{Im}\,\widetilde{\lambda}_\mu^k$$

$$x_{\mathrm{Halle},\mu} \gg 1 \ (\text{but } x_{\mathrm{Hall}p} \ll 1)$$

$$\mathrm{Im}\,\widetilde{\eta} \approx \mathrm{Im}\,\widetilde{\eta}_{e\mu} = -\frac{n_e p_{Fe}^2 + n_\mu p_{F\mu}^2}{5eB}$$

$$\mathrm{Im}\,\widetilde{\kappa} \approx \mathrm{Im}\,\widetilde{\kappa}_{e\mu} = -\frac{\pi^2 T}{3}\frac{n_e + n_\mu}{eB}$$

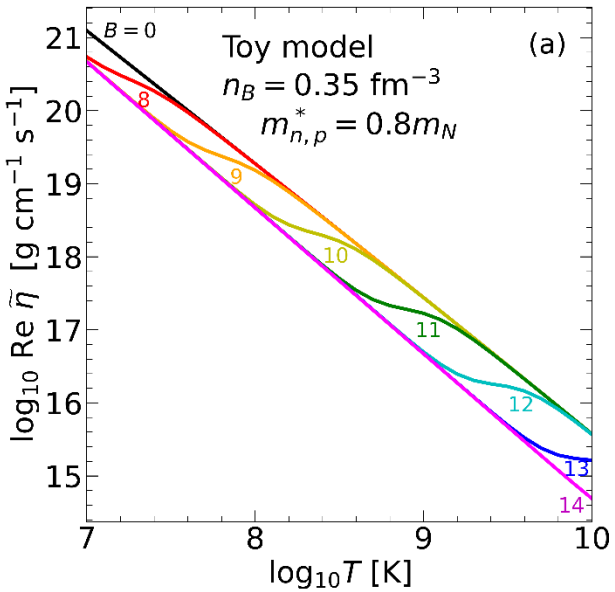


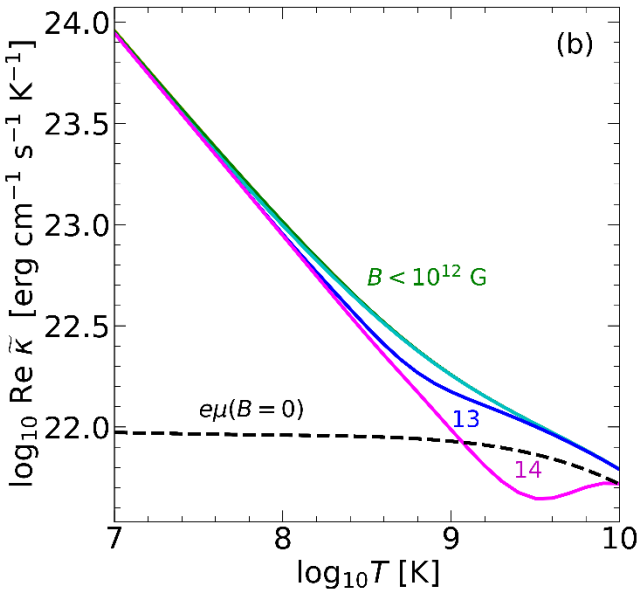
# Dependence on T

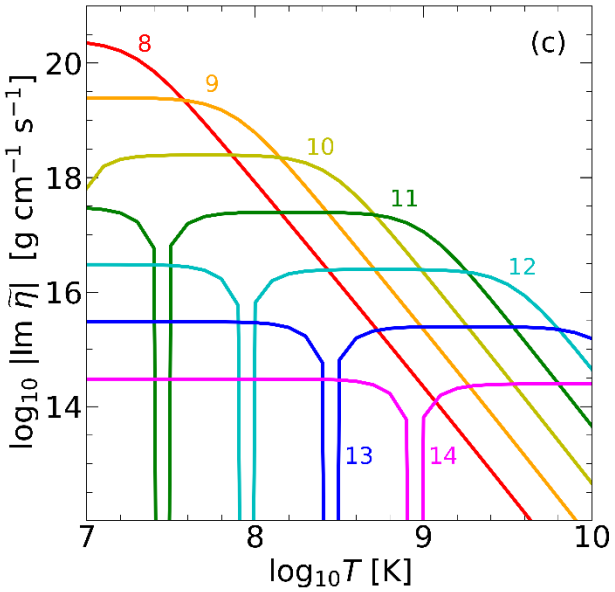
numbers show $\log_{10} B[\mathrm{G}]$

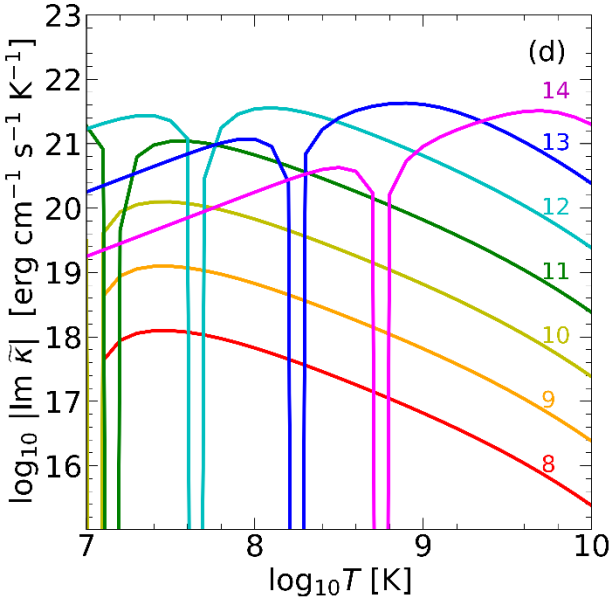
when leptons become magnetized, `perpendicular' shear viscosity switches from lepton-dominated to neutron-dominated regime
Thermal conductivity is always neutron-dominated

Hall components change sign when protons become magnetized



# Exact solution in NS cores

A note on the exact solutions. The dependence of electromagnetic interaction on transferred energy does not allow to construct the exact solutions in analytical form

Dimensionless system of kinetic equations

Electromagnetic interactions result in apparently singular terms

$$x(1-f(x)) = \int \mathrm{d}x'(1-f(x'))\frac{x'-x}{e^{x'-x}-1}\left[\left(C_i^{1l}+\frac{C_i^{1t}}{|w|}\right)(\Psi_i(x)-\Psi_i(x'))+C_i^{2t}\frac{\Psi_i(x')}{|w|^{1/3}}\right]-$$

$$-\sum_j \lambda_{ij}\int \mathrm{d}x'(1-f(x'))\frac{x'-x}{e^{x'-x}-1}\Psi_j(x')-\sum_j \lambda'_{ij}\int \mathrm{d}x'\frac{1-f(x')}{1-e^{x'-x}}F(x,x')\,\Psi_j(x')+if(-x)\omega_{B,i}\tau_i\Psi_i(x)$$

$$x=\frac{\varepsilon-\mu}{T} \qquad w=x'-x$$

Cylcotron term

Thermal conductivity

$$\kappa_i=\frac{\pi^2 n_i T}{3m_i^*}\tau_i\int \mathrm{d}x\, x f(x)f(-x)\Psi_i(x)$$

Shear viscosity

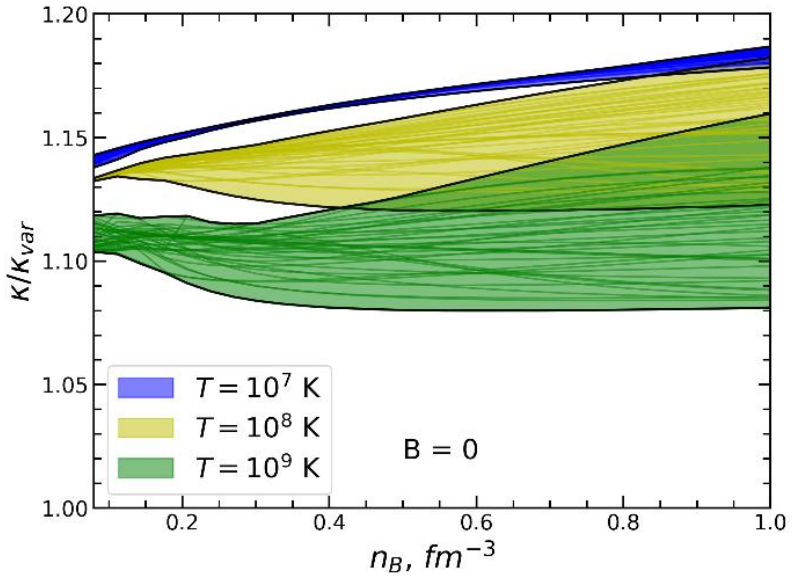
$$\eta_i=\frac{p_{Fi}^2 n_i}{5m_i^*}\tau_i\int \mathrm{d}x f(x)f(-x)\Psi_i(x)$$

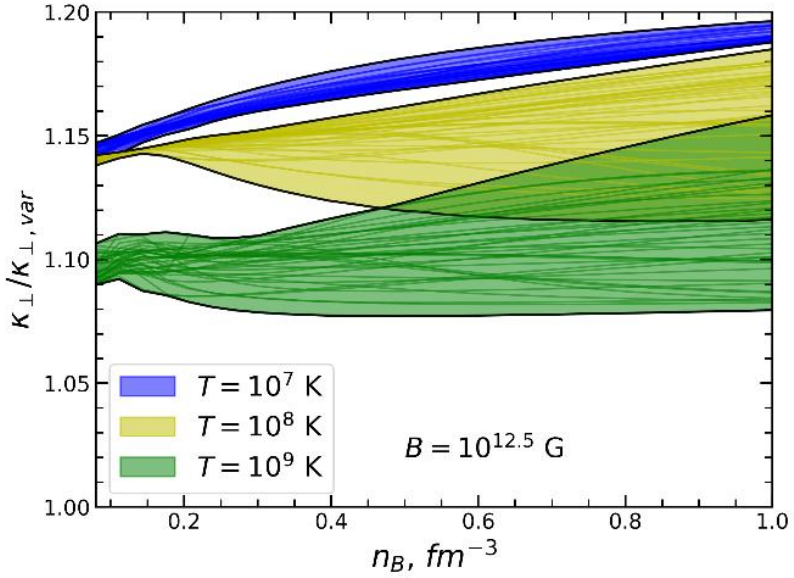
System of equations can be solved numerically, e.g. by the quadrature method

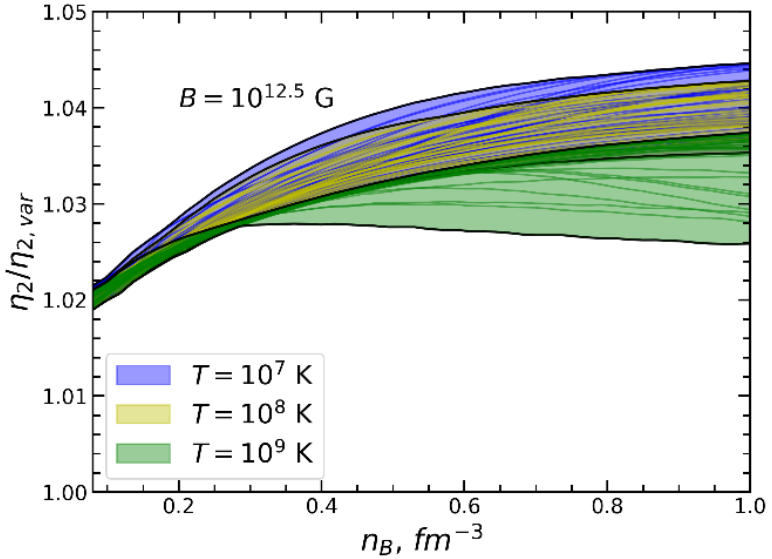
# Exact and variational solutions

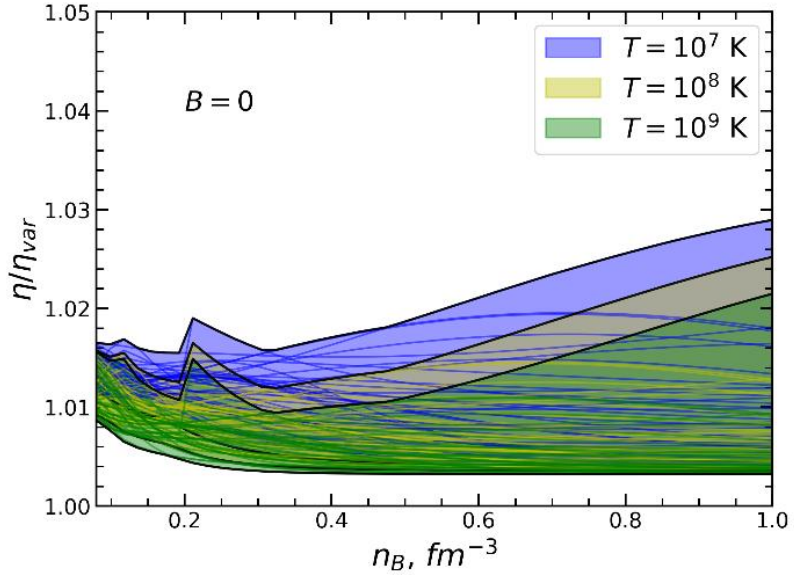
Vacuum cross-sections. Filled areas correspond to 39 EOSs

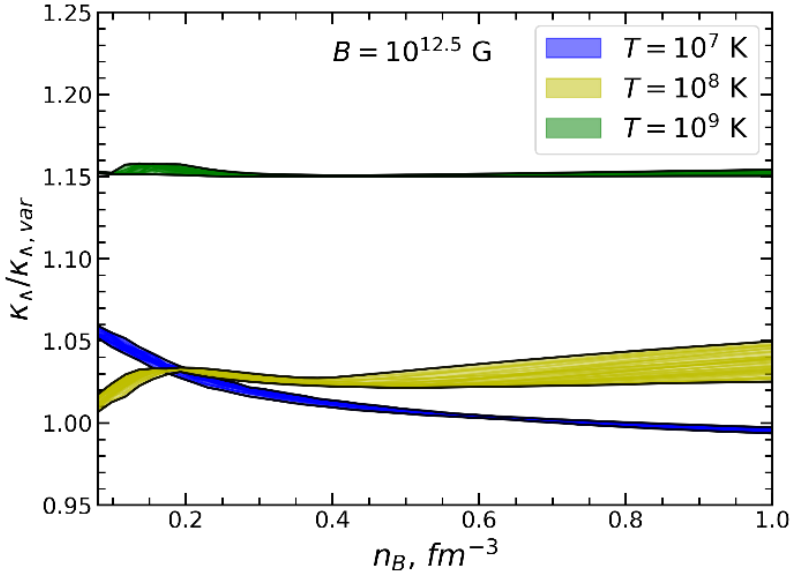
As expected, the corrections to the variational solutions are small

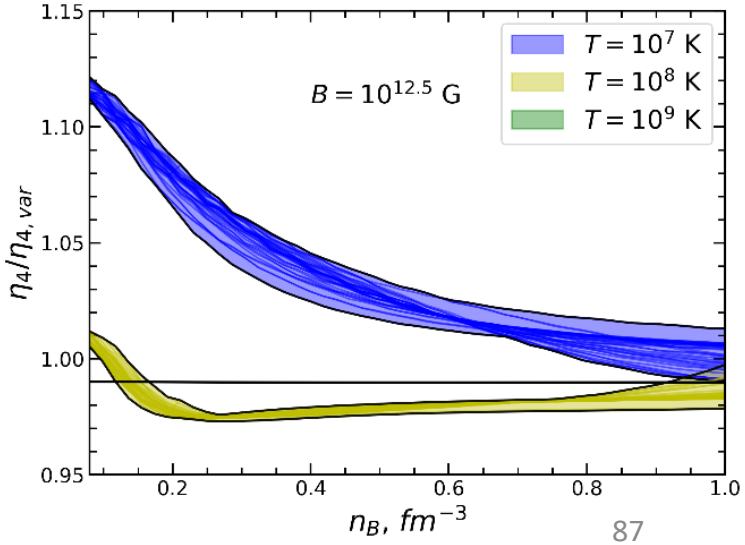


# Lepton transport in presence of pairing

# Bulk viscosity

# Lepton transport in presence of pairing

## Bulk viscosity

# Pairing in neutron star

Nucleon phaseshifts

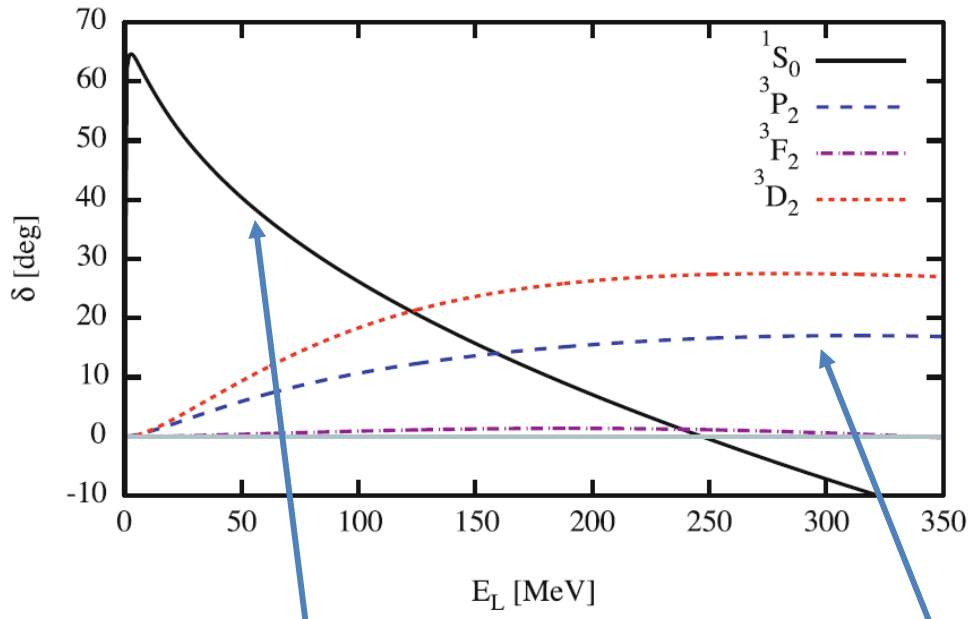
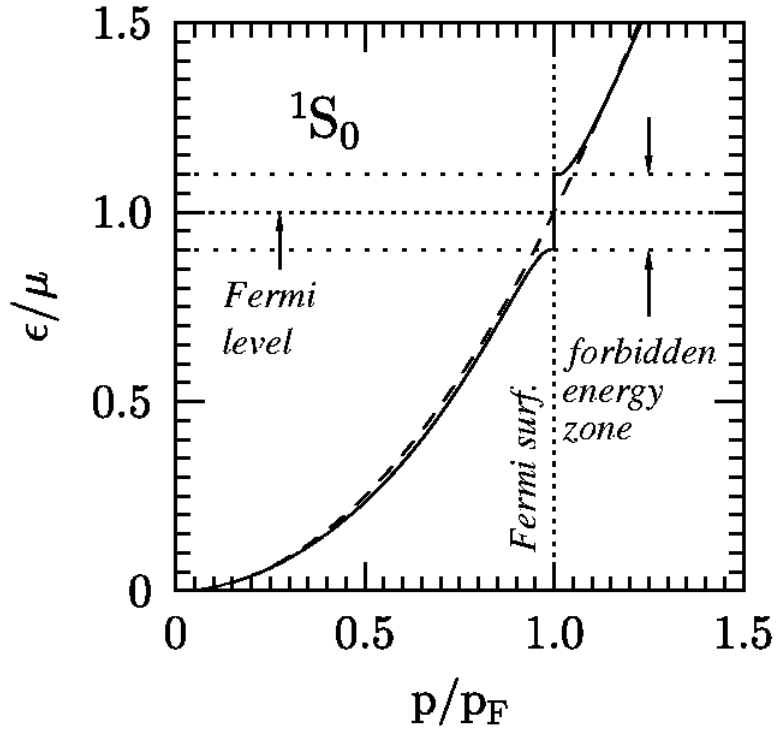



**Fig. 8.1** The nucleon-nucleon scattering phase shifts as a function of laboratory energy for the channels where pairing in neutron stars matter appears. The $S$ and $P$ wave scattering is responsible for neutron-neutron and proton-proton pairing, whereas the ${}^3D_2$ wave scattering can occur only between neutrons and protons. The $P$ and $F$ waves are coupled by the non-central tensor component of the nuclear interaction

$$\Delta_{TSL}(k) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}k' k'^2 \sum_{L'} \frac{V^{TS}_{LL'}(k,k')}{\sqrt{\epsilon(k')^2 + \Delta(k')^2}} \Delta_{TSL'}(k')$$

$$\Delta(k)^2 = \sum_{TSL} \Delta_{TSL}(k)^2$$

Gap in the energy spectrum




Important for n in NS crust and p in NS core
Spin-singlet pairing

$^1S_0$

Important for n in NS cores
Spin-triplet pairing

$^3PF_2$

# Zoo of NS superfluidity models

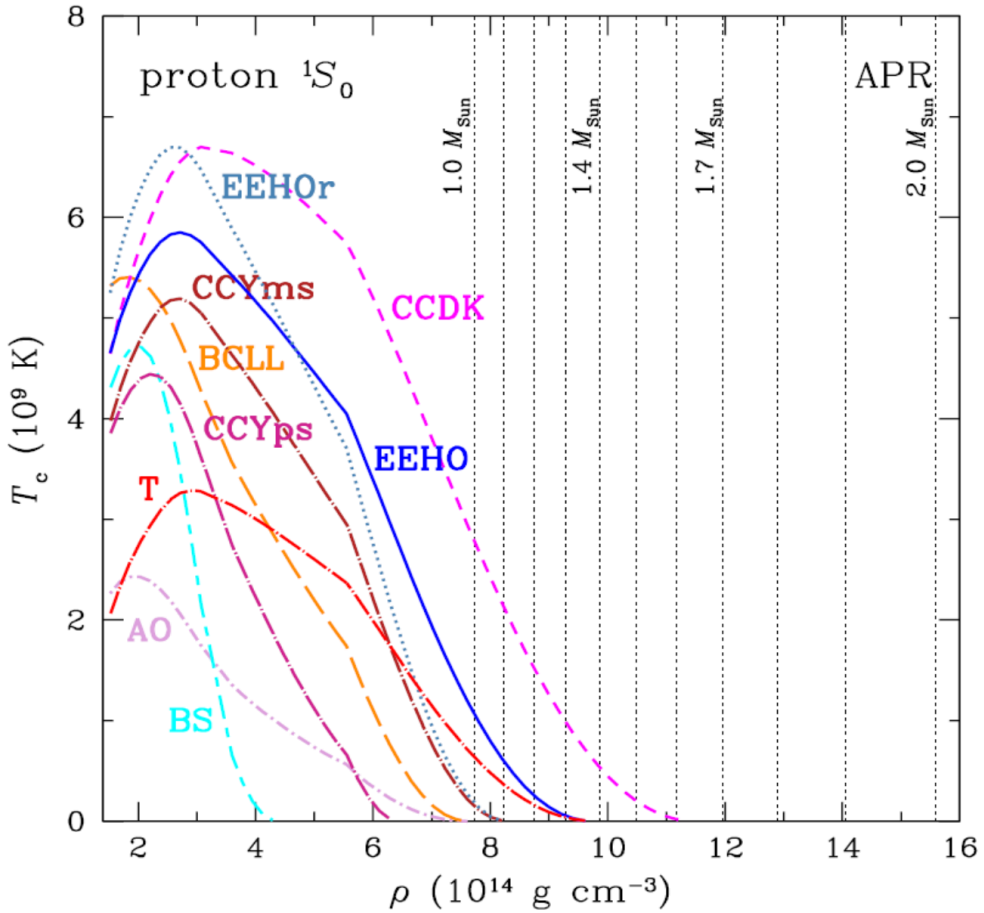
$^1S_0$ proton pairing

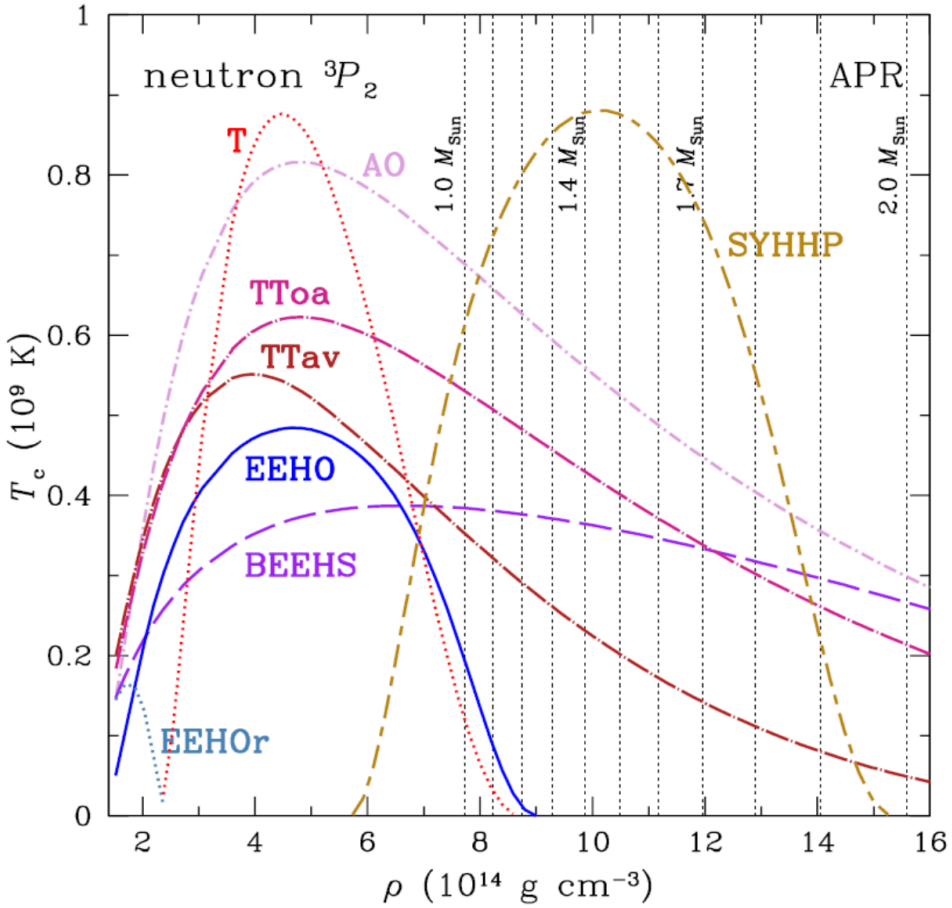
npe$\mu$

$^3P_2$ neutron pairing

$T_C \sim 5 \times 10^9$ K

$T_C \sim 0.5 \times 10^9$ K



# Transport in presence of pairing

Complete description in NS context is lacking

Transport by single-particle excitations $\Lambda_n = \Lambda_{nn} + \Lambda_{np}$

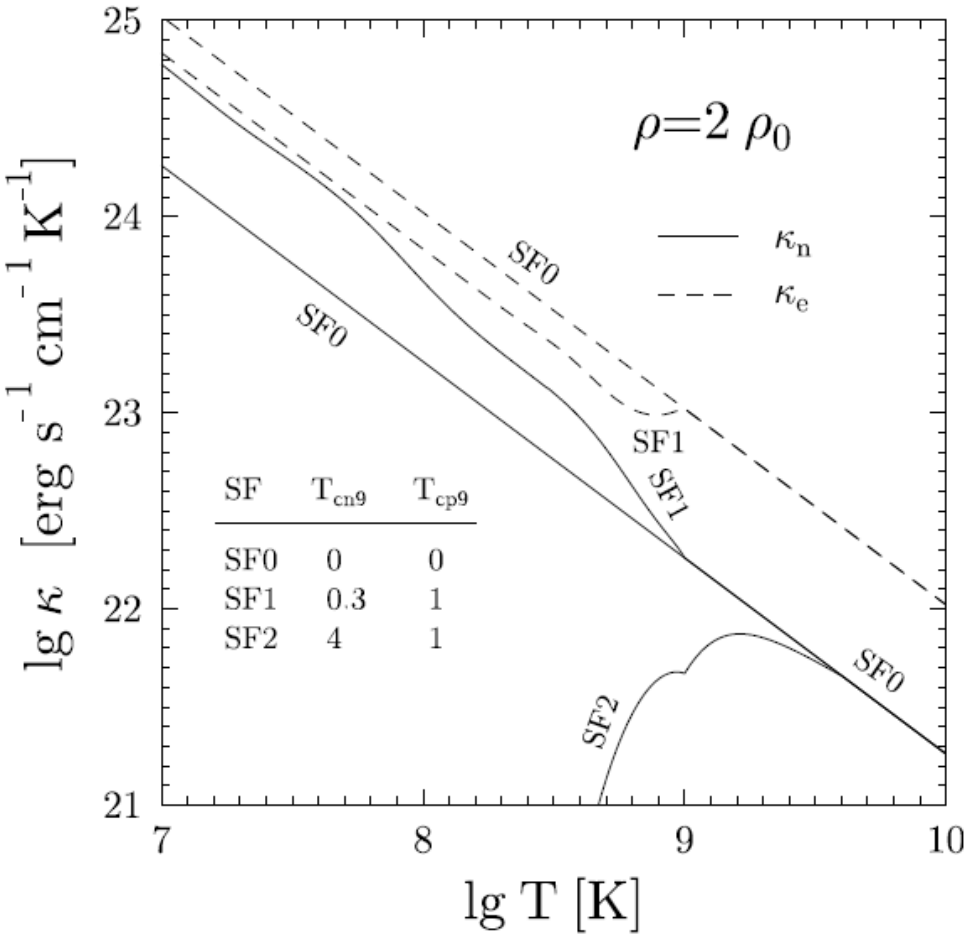
Collisions are damped $\Lambda \propto \exp(-\Delta/T)$

Usually only modifications due to the presence of gap in quasiparticle spectrum are considered

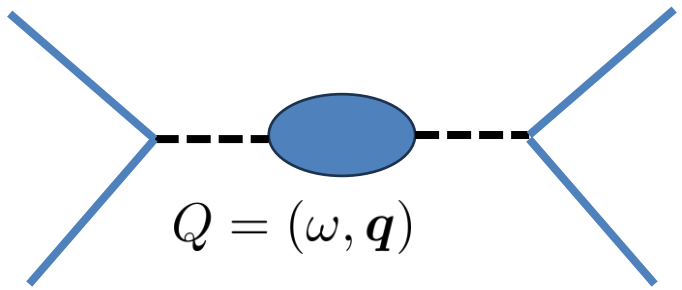
Not elaborated: collision integral modification including quasiparticle number noncoservation, modification of collision integral, anisotropic transport in case of the netron p-wave pairing

Lepton transport coefficients modify in presence due to modification of plasma screening by superconductivity

*Baiko et al, 2001*



# Electromagnetic interactions




**Photon propagator in isotropic media**

$$D_{\mu\nu}(\omega, \boldsymbol{q}) = \frac{(\mathcal{P}_L)_{\mu\nu}}{Q^2 - \Pi_L(\omega, \boldsymbol{q})} + \frac{(\mathcal{P}_T)_{\mu\nu}}{Q^2 - \Pi_T(\omega, \boldsymbol{q})} + \text{cal.}$$

**Contribution from one fermion species**

$$\Pi_L^{HDL}(x) = -\frac{Q^2}{|\boldsymbol{q}|^2} q_{TF}^2 \left[1 - x \ln \frac{x+1}{x-1}\right]$$

$$\Pi_T^{HDL}(k_0, k) = (q_{TF} v_F)^2 \left[x^2 + \frac{x(1-x^2)}{4} \ln \frac{x+1}{x-1}\right]$$

$$x = \frac{\omega}{q v_F}$$

$$q_{TF}^2 = \frac{4\alpha_f}{\pi} \frac{p_F^2}{v_F}$$

**Projectors**

$$\mathcal{P}_T^{00} = \mathcal{P}_T^{0i} = \mathcal{P}_T^{i0} = 0$$

$$\mathcal{P}_T^{ij} = \delta^{ij} - \frac{q^i q^j}{\boldsymbol{q}^2}$$

$$\mathcal{P}_L^{\mu\nu} = \frac{Q^\mu Q^\nu}{Q^2} - g^{\mu\nu} - \mathcal{P}_T^{\mu\nu}$$

**Importance of small-angle (long-range) collisions**

# Screening in presence of superconductivity

$^1S_0$ proton pairing

$$\frac{\Delta}{T} = \sqrt{1-t}\left(1.456 - \frac{0.157}{\sqrt{t}} + \frac{1.764}{t}\right) \qquad t = \frac{T}{T_{Cp}}$$

Low temperatures $\quad t < 0.3 \quad \Delta/T < 0.2$

Longitudinal screening does not change

Transverse screening becomes static which restores the Fermi-liquid behavior

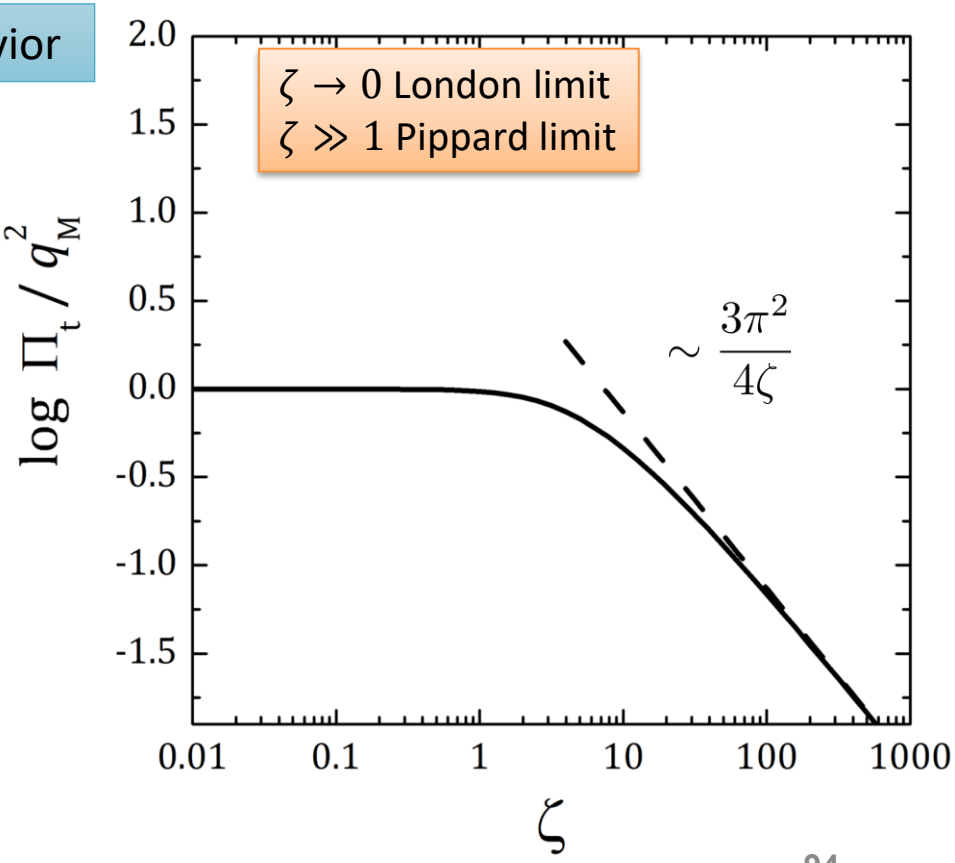
$$\Pi_T(\omega, \boldsymbol{k}) \approx \Pi_{Tp}(0,k) = q_M^2 J(\zeta) \quad \zeta = \frac{q v_{\mathrm{F}p}}{\Delta}$$

Meissner mass $\quad q_M^2 = \dfrac{4\alpha_f}{3\pi} p_{\mathrm{F}p}^2 v_{\mathrm{F}p}$

BCS $\quad J(\zeta) = \dfrac{3}{2}\displaystyle\int_{-1}^{1} dx\,(1-x^2)\,\frac{\mathrm{ArcSinh}(\zeta x/2)}{\zeta x\sqrt{1+(\zeta x)^2/4}}$

London limit $\quad J(0) = 1$

Pippard limit $\quad J(\zeta) \sim \dfrac{3\pi^2}{4\zeta}, \quad \zeta \to \infty$



# Screening in presence of superconductivity

$$\int_0^{q_{\max}} \mathrm{d}q \; \frac{q^n}{|q^2 + q_M^2 J(\zeta)|^2}$$

Governing parameter

$$A = \zeta(q = q_M) = \frac{v_{Fp} q_M}{\Delta}$$

London $A \ll 1$

$$q_M^2 = \frac{4\alpha_f}{3\pi} p_{\mathrm{F}p}^2 v_{\mathrm{F}p}$$

Pippard $A \gg 1$

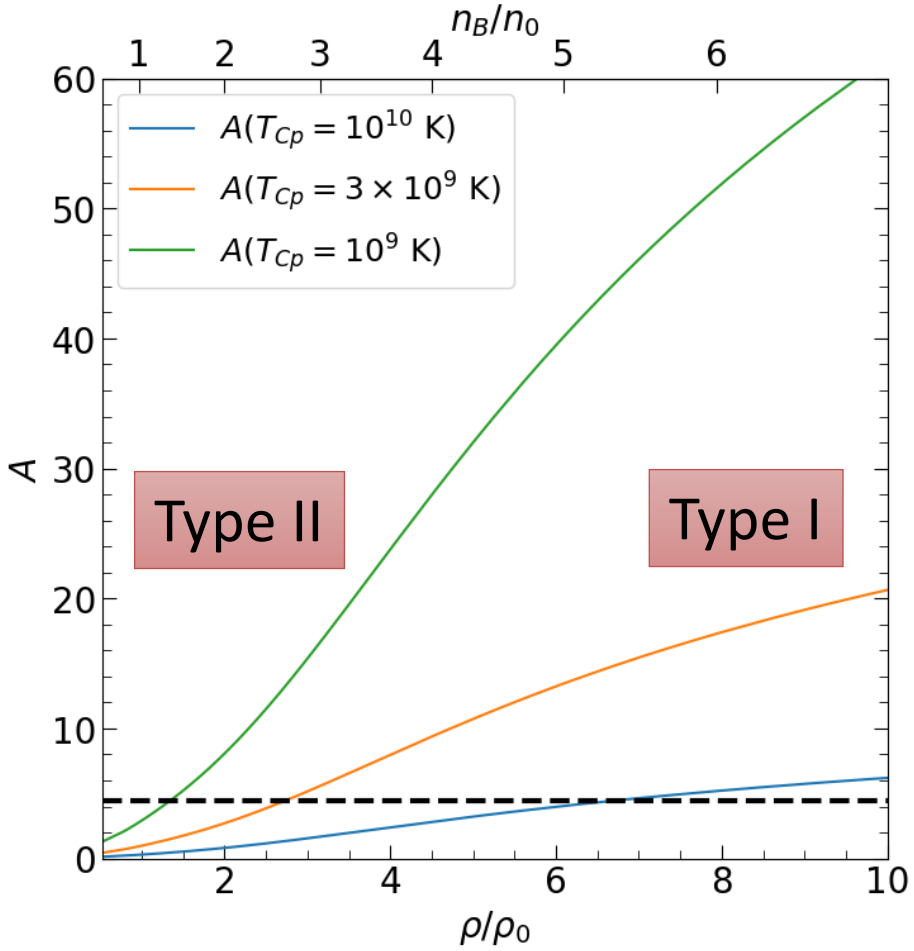
$$q_P = q_M \left(\frac{3\pi^2}{4A}\right)^{1/3} \propto \Delta^{-1/3}$$

Relation to SC type

$$A = \frac{v_{Fp} q_M}{\Delta} = \frac{\pi}{\varkappa}$$

$\varkappa$ – Ginzburg-Landau parameter

Type I $A > 4.4 \; (\varkappa < \sqrt{2})$



# Screening in presence of superconductivity

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Pippard $A \gg 1$

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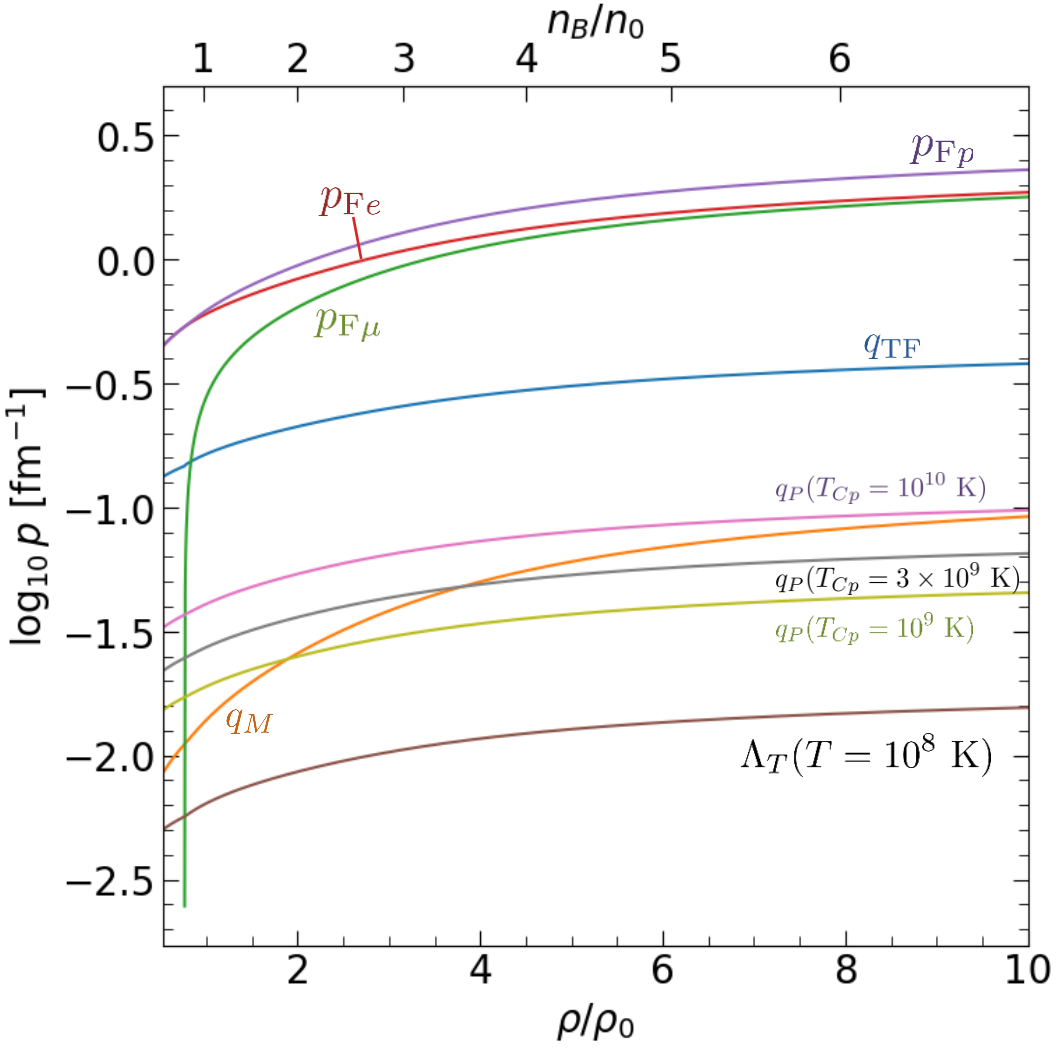


# Superconducting matter

Standard Fermi-liquid temperature scaling of transport coefficients is restored

SC of the II kind

$$\kappa_{e\mu}^{t,Lon} = \frac{5\ q_M^3}{18\pi^2\alpha_f^2 T}$$

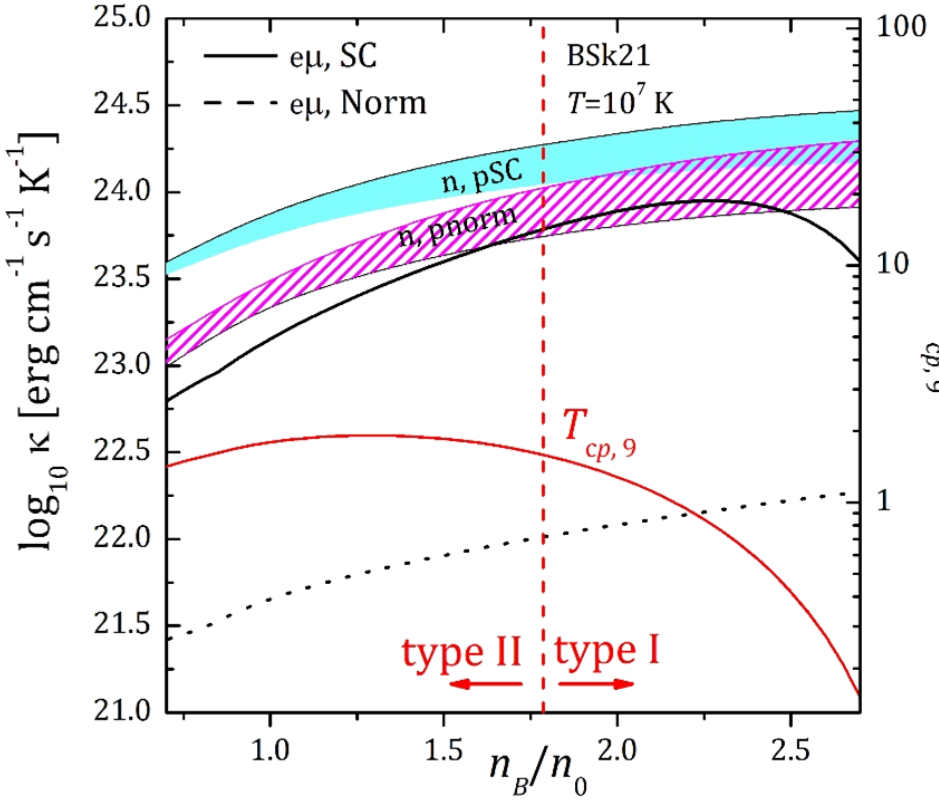
$$\eta_{e\mu}^{t,Lon} = \frac{6\ q_M}{5\alpha_f^2 (k_B T)^2} \frac{n_e^2 + n_\mu^2}{p_{Fe}^2 + p_{F\mu}^2}$$

Do not depend on gap

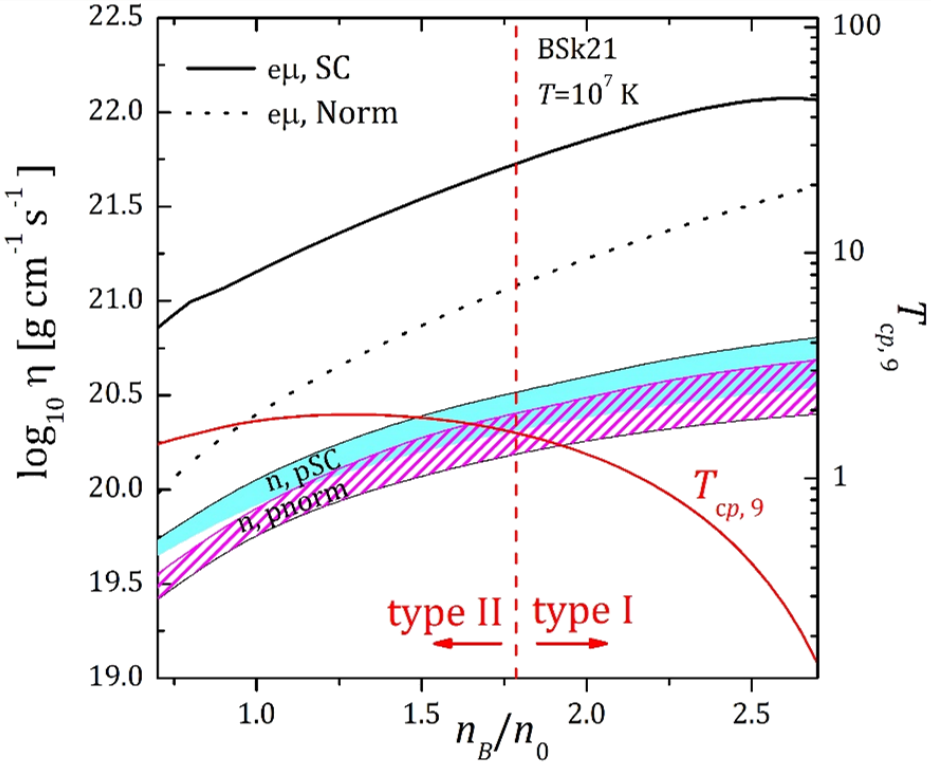
SC of the I kind

$$\kappa_{e\mu}^{t,Pip} = \frac{5 p_{Fp}^2}{24\alpha_f} \frac{\Delta}{T}$$

$$\eta_{e\mu}^{t,Pip} = \frac{1.71}{\alpha_f^{5/3}} \frac{p_{Fp}}{(k_B T)^2} \frac{n_e^2 + n_\mu^2}{p_{Fe}^2 + p_{F\mu}^2} \left(\frac{\Delta}{p_{Fp} c}\right)^{1/3}$$




$$\kappa_{e\mu} \ll \kappa_n [nn]$$




$$\eta_{e\mu} \gg \eta_n [nn]$$

Relation between lepton and baryon transport coefficients persists

# Lepton transport in presence of pairing

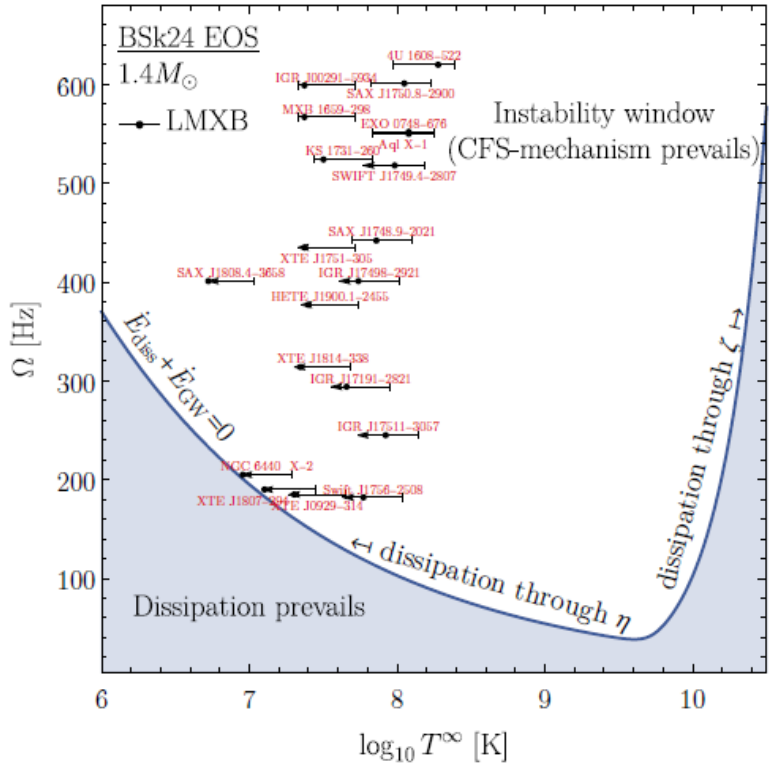
# Bulk viscosity

# Lepton transport in presence of pairing

## Bulk viscosity

Bulk viscosity are important for damping r-mode instabilities

$$\frac{1}{\tau_{GW}(f)} + \frac{1}{\tau_{\zeta}(f,T)} + \frac{1}{\tau_{\eta}(T)} > 0$$



# Bulk viscosity

Earlier we saw that the bulk viscosity of degenerate systems is **negligible** in comparison to the shear viscosity

The situation, however, changes dramatically if sufficiently slow chemical (nuclear) reactions can occur in the system (or other slow relaxation process). Bulk viscosity describes the response to the uniform compression (expansion) of the system which can drive it from the chemical equilibrium. If the characteristic chemical relaxation time is comparable to the oscillation timescale, the chemical re-equilibration will not be re-established during the compression. Therefore the compression-expansion cycles may be accompanied with significant energy dissipation, i.e. large effective bulk viscosity coefficient (Mandelstam and Leontovich 1937).

Let us see how this works in a simplest example. Consider that the star is subject to periodic oscillation in volume. Let us measure the volume per one baryon, $V = 1/n_B$ $\quad n_B(t) = n_{B0} + \delta n_B(t) = n_{B0} + \delta n_{B0}\cos\omega t$

Energy dissipation per unit volume – compression work

$$\langle \dot{E} \rangle = \frac{1}{\tau V_0}\int_0^\tau dt P(t)\frac{dV}{dt} = -\frac{1}{\tau n_{B0}}\int_0^\tau dt P(t)\frac{d\delta n_B}{dt}$$

On the other hand, the dissipation is

$$\langle \dot{E} \rangle = -\frac{\zeta}{\tau}\int_0^\tau (\mathrm{div}\boldsymbol{V})^2 = -\frac{\zeta}{\tau n_{B0}^2}\int_0^\tau \left(\frac{d\delta n_B}{dt}\right)^2 = -\zeta\frac{\omega^2\delta n_{B0}^2}{2n_{B0}^2}$$

Here we used the continuity equation

$$\frac{d\delta n_B}{dt} + n_B \mathrm{div}\boldsymbol{V} = 0$$

The non-zero bulk viscosity will emerge if pressure contains terms $\propto \sin\omega t$

# Bulk viscosity

*npe* matter

Direct Urca reactions $p + e \to n + \nu_e \qquad n \to p + e + \widetilde{\nu}_e$

In equilibrium the rates of forward and reverse equations are equal $\Gamma_{p+e\to n} = \Gamma_{n\to p+e} \qquad \mu_n = \mu_p + \mu_e$

Out of equilibrium is described by the reaction **affinity** $\delta\mu = \mu_p + \mu_e - \mu_n$

In the linear regime, the reaction rates are

$$\frac{\mathrm{d}\delta n_n}{\mathrm{d}t} = -\frac{\mathrm{d}\delta n_p}{\mathrm{d}t} = -\frac{\mathrm{d}\delta n_e}{\mathrm{d}t} = (\Gamma_{p+e\to n} - \Gamma_{n\to p+e}) = \lambda\delta\mu$$

$\lambda$ – transport coefficient corresponding to chemical equilibration

$$P(t) = P_0 + \frac{\partial P}{\partial n_B}\delta n_B(t) + \sum_{a=n,p,e}\frac{\partial P}{\partial n_a}\delta n_a(t) = P_0 + \frac{\partial P}{\partial n_B}\delta n_B(t) + B\delta n_e(t)$$

susceptibility $B = \dfrac{\partial P}{\partial n_n} - \dfrac{\partial P}{\partial n_p} - \dfrac{\partial P}{\partial n_e}$

Evolution of the affinity $\dfrac{\mathrm{d}\delta\mu}{\mathrm{d}t} = \dfrac{\partial\delta\mu_a}{\partial n_B}\dfrac{\mathrm{d}n_B}{\mathrm{d}t} + \sum_a \dfrac{\partial\delta\mu}{\partial n_a}\dfrac{\mathrm{d}n_a}{\mathrm{d}t}$

Thermodynamics $\mathcal{E} = \mathcal{E}(n_B, \{X_a \equiv n_a/n_B\}) \quad d\mathcal{E} = \dfrac{P}{n_B^2}\mathrm{d}n_B + \sum_a \mu_a \mathrm{d}X_a \qquad P = n_B^2\left(\dfrac{\partial\mathcal{E}}{\partial n_B}\right)_{X_a} \quad \mu_a = \left(\dfrac{\partial\mathcal{E}}{\partial X_a}\right)_{n_B}$

Then $\dfrac{\partial\mu_a}{\partial n_B} = \left(\dfrac{\partial^2\mathcal{E}}{\partial X_a \partial n_B}\right) = \dfrac{1}{n_B^2}\left(\dfrac{\partial P}{\partial X_a}\right)_{n_B} = \dfrac{1}{n_B}\dfrac{\partial P}{\partial n_a}$ thus $\dfrac{\partial\delta\mu_a}{\partial n_B} = \dfrac{1}{n_B}\left(\dfrac{\partial P}{\partial n_n} - \dfrac{\partial P}{\partial n_p} - \dfrac{\partial P}{\partial n_e}\right) = \dfrac{B}{n_B}$

Finally $\dfrac{\mathrm{d}\delta\mu}{\mathrm{d}t} = \dfrac{B}{n_{B0}}\dfrac{\mathrm{d}\delta n_B}{\mathrm{d}t} - \lambda C\delta\mu$

another susceptibility $C = \dfrac{\partial\delta\mu}{\partial n_p} + \dfrac{\partial\delta\mu}{\partial n_e} - \dfrac{\partial\delta\mu}{\partial n_n}$

# Bulk viscosity

Baryon density oscillations

$$n_B(t) = n_{B0} + \delta n_B(t) = n_{B0} + \delta n_{B0}\cos\omega t = n_{B0} + \delta n_{B0}\mathrm{Re}\,\mathrm{e}^{i\omega t}$$

Affinity and reaction coordinate

$$\delta\mu = \mathrm{Re}\,\delta\mu_0 \mathrm{e}^{i\omega t} \qquad \delta n_e = \mathrm{Re}\,\delta n_{e0}\mathrm{e}^{i\omega t}$$

$$\left.\begin{aligned} \frac{\mathrm{d}\delta\mu}{\mathrm{d}t} &= \frac{B}{n_{B0}}\frac{\mathrm{d}\delta n_B}{\mathrm{d}t} - \lambda C\delta\mu \\ -\frac{\mathrm{d}\delta n_e}{\mathrm{d}t} &= \lambda\delta\mu \end{aligned}\right\} \Rightarrow \left.\begin{aligned} i\omega\delta\mu_0 &= \frac{B}{n_{B0}} i\omega\delta n_{B0} - \lambda C\delta\mu_{B0} \\ -i\omega\delta n_{e0} &= \lambda\delta\mu_0 \end{aligned}\right\}$$

Solution

$$\delta n_{e0} = -\frac{B}{n_{B0}}\delta n_{B0}\frac{\lambda}{i\omega + \lambda C}$$

Plug into pressure to obtain out-of-phase terms

$$P(t) = P_0 + \frac{\partial P}{\partial n_B}\delta n_B(t) - \frac{B^2}{n_{B0}}\delta n_{B0}\,\mathrm{Re}\frac{\lambda \mathrm{e}^{i\omega t}}{i\omega + \lambda C}$$

Dissipation

$$\langle \dot{E}\rangle = -\frac{1}{\tau n_{B0}}\int_0^\tau dt P(t)\frac{d\delta n_B}{dt} = -\frac{B^2}{n_{B0}^2}\frac{\delta n_{B0}^2}{2}\frac{\lambda\omega^2}{(\lambda C)^2+\omega^2} = -\zeta\frac{\omega^2\delta n_{B0}^2}{2n_{B0}^2}$$

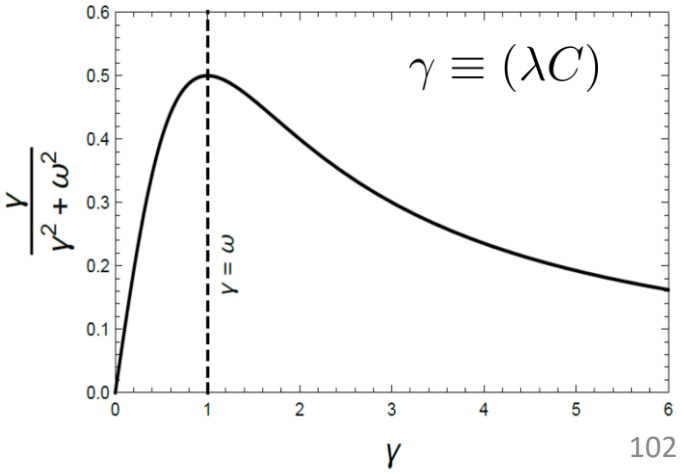
$$\zeta = \frac{\lambda B^2}{(\lambda C)^2 + \omega^2}$$

Typical resonance behavior

Need microscopic rate $\lambda C$ to be of the order of star oscillations

$\lambda C$ typically monotonically increase with T hence there is a maximum of $\zeta$ at a certain T

$$\max\zeta = \frac{B^2}{2\omega C}$$



# Bulk viscosity: Example

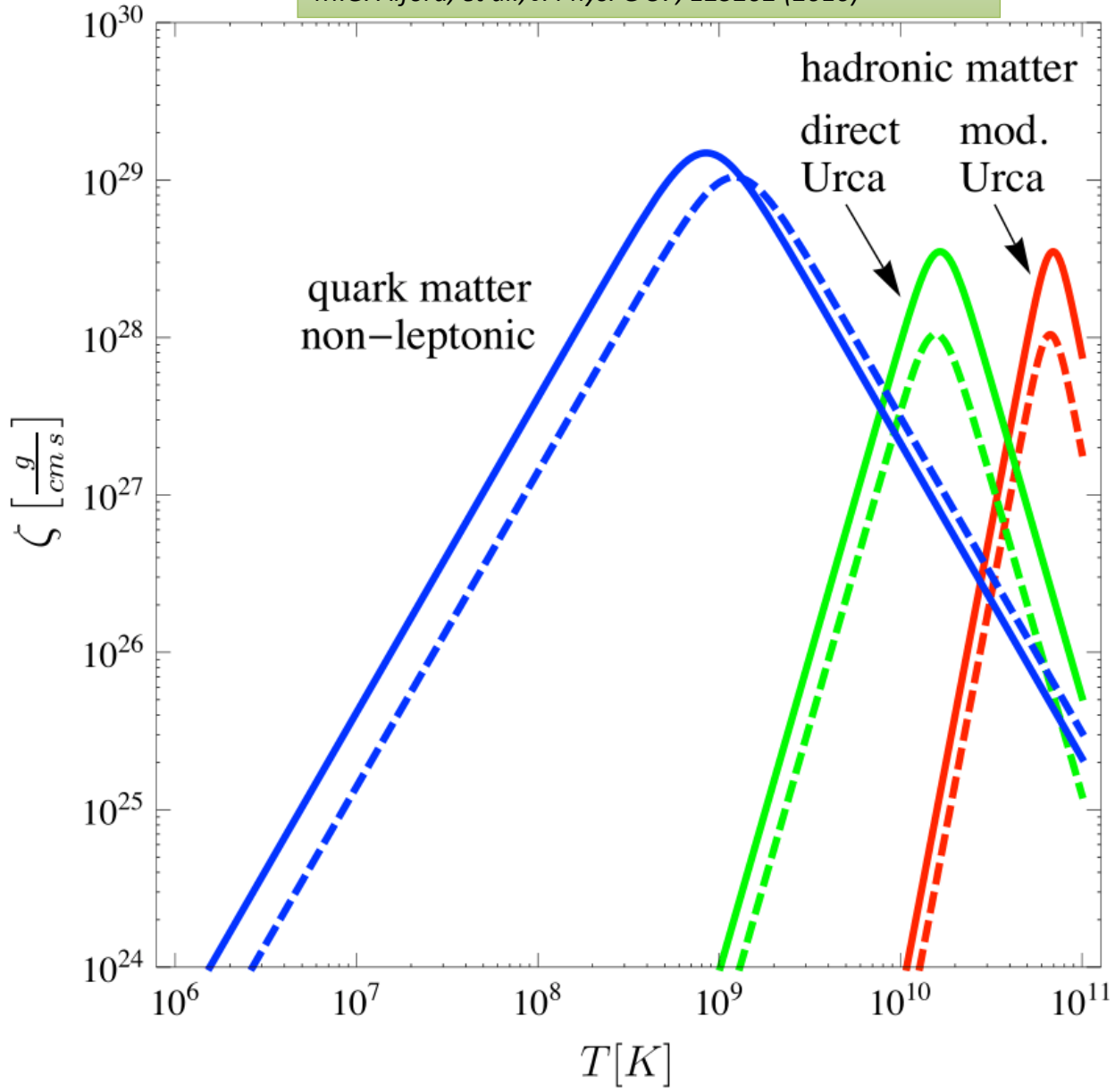
$$\zeta = \frac{\lambda B^2}{(\lambda C)^2 + \omega^2}$$

$$\max \zeta = \frac{B^2}{2\omega C} \qquad \lambda C = \omega$$

Different maximal values due to different susceptibilities (they do not depend on $\lambda$)

Location of the maximum shifts according to the reaction equilibration speed

*M.G. Alford, et al., J. Phys. G 37, 125202 (2010)*



# SUMMARY

Transport theory of multicomponent Fermi liquids in magnetic field was discussed in detail

The general approach was applied to dense matter in neutron star cores

Lepton transport coefficients demonstrate non-Fermi liquid behavior

In the absence of magnetic field, thermal conductivity is mediated by neutrons, while shear viscosity is mediated by leptons

In the magnetized matter thermal conductivity is weakly modified, however the shear viscosity is affected strongly even in the moderate magnetic field.

In ideal world, NS EOS databases should also contain information allowing the transport properties calculations.

# THAT'S ALL