Introduction to reactions with heavy ion

1. Introduction (experimental and theoretical aspects)
2. Systematics
3. Scattering and direct reactions
4. Deep-inelastic collisions (properties and description)
5. Pecularities of fusion reactions

1st nuclear reaction with *p* beam: 1931

p, α, *d* beams for study of nuclear structure

50-60th years – ion sourses

heavy ion wave length < 0.1 fm (classical particles)

50th : linear accelerators in USA

60th : cyclotron in Dubna

70-80th : linear accelerator at GSI, cyclotron at GANIL and ...

beam of light nuclei:

nuclear reactions / processes: elastic, inealastic scattering, nuclear transfer reactions, formation and decay of compound nucleus

beam of heavy ions:

the Coulomb fission of heavy nuclei and excitation of high-spin states, population of highly-deformed nuclear states, multinucleon transfer reactions, compound nucleus formation

Problems of synthesis of superheavy nuclei

Production of exotic nuclei, new isotopes

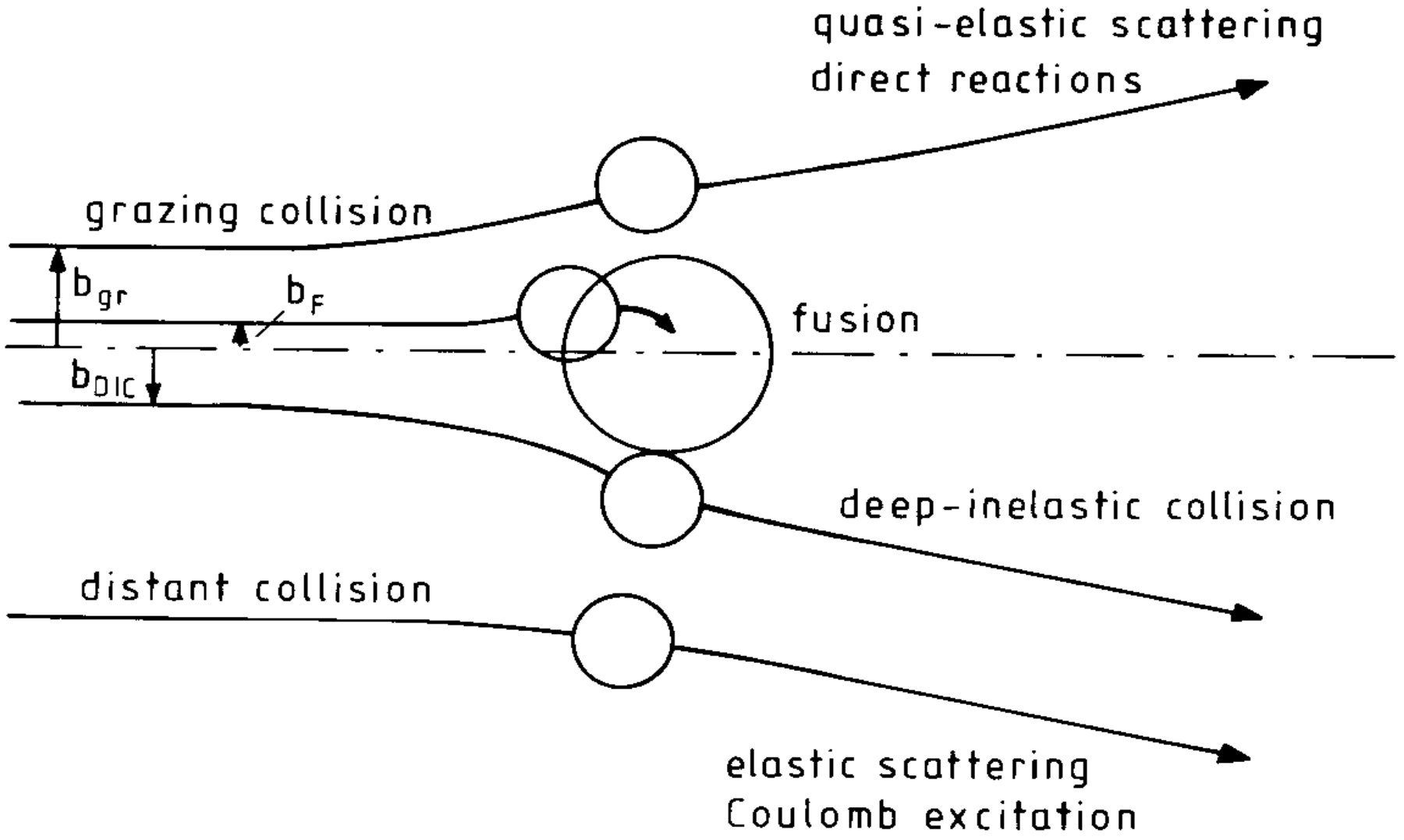
Study of various decay modes including fission, emission of delayed proton, *p* and 2*p* radioactivity

High-spin states

Highly excited compound nuclei

Sub-barrier processes, astrophysical reactions

Cluster or molecule states



Classification of reactions by impact parameter.

impact parameter $b=l/k$

Reaction cross section

$$\sigma_{\mathrm{r}} = \sum_{l} \sigma_{\mathrm{r}}(l),$$

$$\sigma_{\mathrm{r}}(l) = \frac{\pi}{k^2}(2l+1)T_l$$

For large angular momenta

$$\sigma_{\mathrm{r}} = \int_0^{\infty} dl\, \sigma_{\mathrm{r}}(l)$$

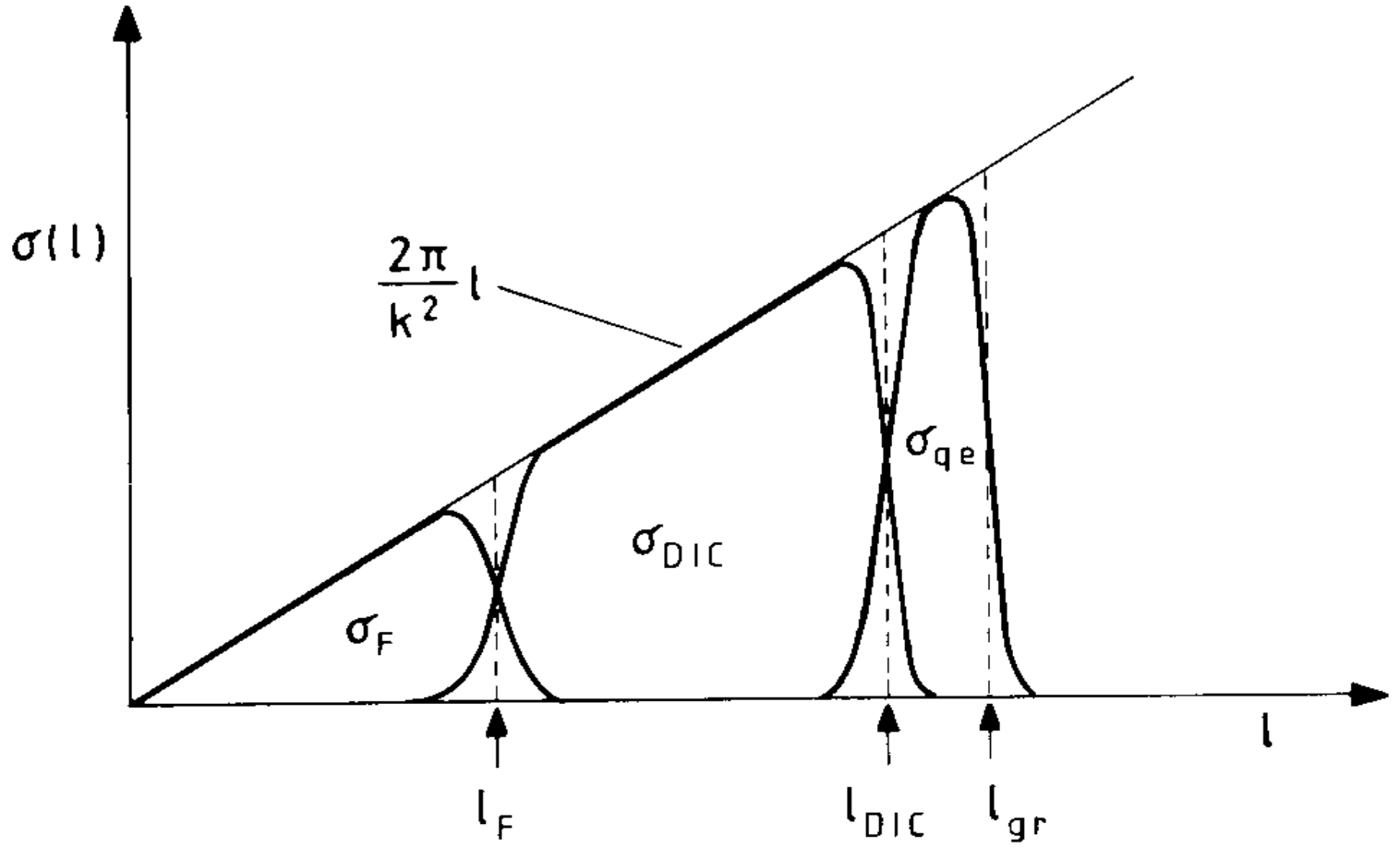
$$\sigma_{\mathrm{r}}(l) = \frac{2\pi}{k^2}\, l\, T(l).$$

$$T(l) = \begin{cases} 1 & \text{for } l < l_{\mathrm{gr}}, \\ 0 & \text{for } l > l_{\mathrm{gr}}, \end{cases} \qquad l_{\mathrm{gr}} = kb_{\mathrm{gr}}$$

$$\sigma_{\mathrm{r}}(l) = \frac{2\pi}{k^2} l$$

between the values $l = 0$ and $l = l_{\mathrm{gr}}$.

$$\begin{aligned} 0 &< l < l_{\mathrm{F}} && \text{fusion,} \\ l_{\mathrm{F}} &< l < l_{\mathrm{DIC}} && \text{deep-inelastic collisions,} \\ l_{\mathrm{DIC}} &< l < l_{\mathrm{gr}} && \text{quasi-elastic collisions.} \end{aligned}$$



$$\sigma_{\mathrm{F}} = \frac{2\pi}{k^2}\int_0^{l_{\mathrm{F}}} l\,dl = \frac{\pi}{k^2} l_{\mathrm{F}}^2,$$

$$\sigma_{\mathrm{DIC}} = \frac{2\pi}{k^2}\int_{l_{\mathrm{F}}}^{l_{\mathrm{DIC}}} l\,dl = \frac{\pi}{k^2}(l_{\mathrm{DIC}}^2 - l_{\mathrm{F}}^2)$$

Quantum scattering by short-range potential

$$H_{\text{tot}} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(r) = \frac{\mathbf{P}^2}{2M} + H$$

$$\mathbf{P} = \frac{\hbar}{i}\nabla_R, \quad H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r),$$

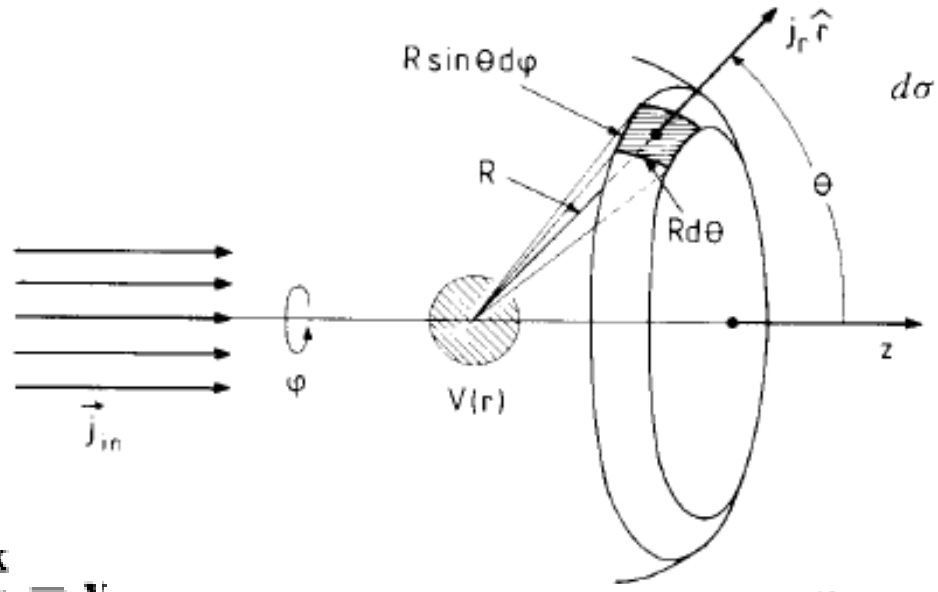
$M = m_1 + m_2$ is the total, and $\mu = m_1 m_2/(m_1 + m_2)$ the reduced mass.

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\mathbf{K}\cdot\mathbf{R}}\,\psi(\mathbf{r}),$$

$$H\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$\psi(\mathbf{r}) \;\to\; e^{i\mathbf{k}\cdot\mathbf{r}} + f(\Omega)\frac{e^{ikr}}{r} \quad \text{for } r \to \infty;$$

probability current density $\mathbf{j} = \dfrac{\hbar}{2\mu i}(\psi^*\nabla\psi - \psi\nabla\psi^*),$



$$d\sigma(\Omega) = \frac{\text{probability current into } d\Omega \text{ in the direction } \Omega}{\text{probability current density of the incident wave}}.$$

$$\mathbf{j} = \frac{\hbar}{2\mu i}(\psi^* \nabla \psi - \psi \nabla \psi^*),$$

$$\mathbf{j}_{\text{in}} = \frac{\hbar \mathbf{k}}{\mu} = \mathbf{v},$$

$$j_r r^2 d\Omega = \frac{\hbar}{2\mu i}\left[f^* \frac{e^{-ikr}}{r}\frac{\partial}{\partial r}\left(f\frac{e^{ikr}}{r}\right) - f\frac{e^{ikr}}{r}\frac{\partial}{\partial r}\left(f^*\frac{e^{-ikr}}{r}\right)\right] r^2 d\Omega$$
$$\to v|f(\Omega)|^2 d\Omega \quad \text{for } r \to \infty.$$

$$\frac{d\sigma}{d\Omega} = \frac{j_r R^2}{|\mathbf{j}_{\text{in}}|} = |f(\Omega)|^2 \qquad f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(S_l - 1)P_l(\cos\theta)$$

$$S_l = e^{2i\delta_l}$$

$$\sigma_{\text{el}} = 2\pi \int_{-1}^{1} d(\cos\theta)\,\frac{d\sigma}{d\Omega} = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_l - 1|^2,$$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1) f_l P_l(\cos\theta);$$

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1) i^l j_l(kr) P_l(\cos\theta).$$

reaction cross section

$$\psi(r,\theta) \to \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left((-)^{l+1}\frac{e^{-ikr}}{r}+S_l\frac{e^{ikr}}{r}\right)P_l(\cos\theta).$$

$$-\int d\mathbf{F}\cdot\mathbf{j} = -\frac{\hbar}{2\mu i}R^2 2\pi\int_{-1}^{1}d(\cos\theta)\left(\psi^*\frac{\partial\psi}{\partial r}-\psi\frac{\partial\psi^*}{\partial r}\right)_{r=R},$$

$$-\frac{\hbar}{2\mu i}R^2 2\pi\frac{1}{4k^2R^2}\sum_{l=0}^{\infty}2(2l+1)\left\{\left[(-)^{l+1}e^{ikR}+S_l^*e^{-ikR}\right]\right.$$

$$\left.\times\left[(-)^{l+1}(-ik)e^{-ikR}+ikS_le^{ikR}\right]-\text{c.c.}\right\}=\frac{\hbar\pi}{\mu k}\sum_{l=0}^{\infty}(2l+1)(1-|S_l|^2).$$

$$\sigma_r=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-|S_l|^2). \qquad \sigma_r=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)T_l. \qquad T_l=1-|S_l|^2,$$

total cross section

$$\sigma_{el}=2\pi\int_{-1}^{1}d(\cos\theta)\frac{d\sigma}{d\Omega}=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|S_l-1|^2,$$

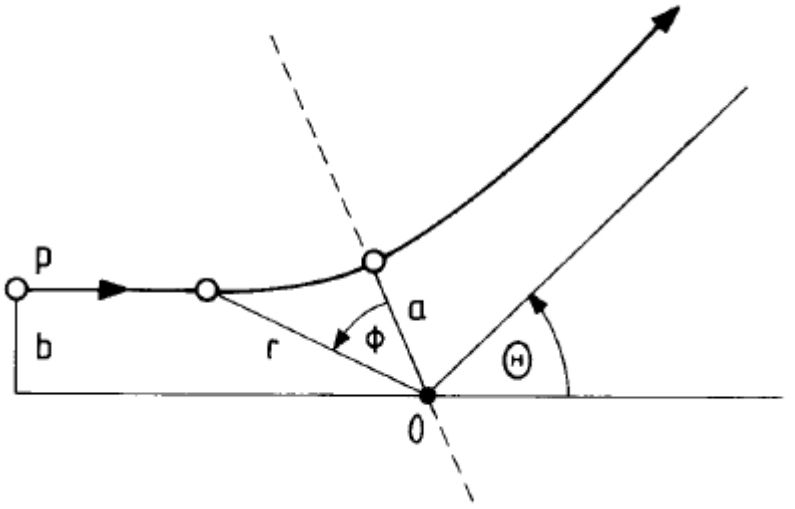
$$\sigma_{tot}=\sigma_{el}+\sigma_r.$$

$$\sigma_{tot}=\frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-\text{Re}\,S_l).$$

Classical scattering

$$E = \frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} + V(r)$$

$$L = \mu r^2 \frac{d\phi}{dt}.$$



The *deflection angle* Θ

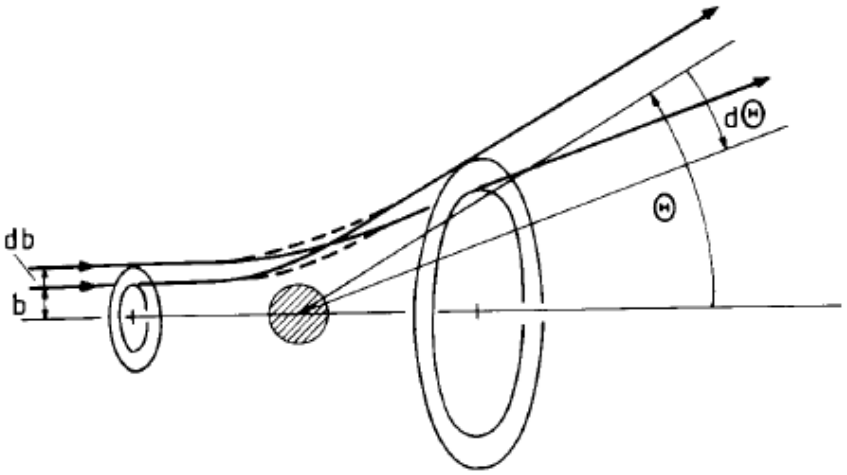
$$\Theta(b) = \pi - 2\int_a^\infty dr \frac{b}{r^2\sqrt{1 - V(r)/E - b^2/r^2}}.$$

$$d\sigma = \frac{\text{particle current into } d\Omega \text{ in the direction } \Omega}{\text{particle current density of the incident particles}}$$

$$d\sigma = \frac{J\, b db d\varphi}{J}.$$

$$d\Omega = |\sin\Theta d\Theta| d\varphi$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta |d\Theta/db|};$$

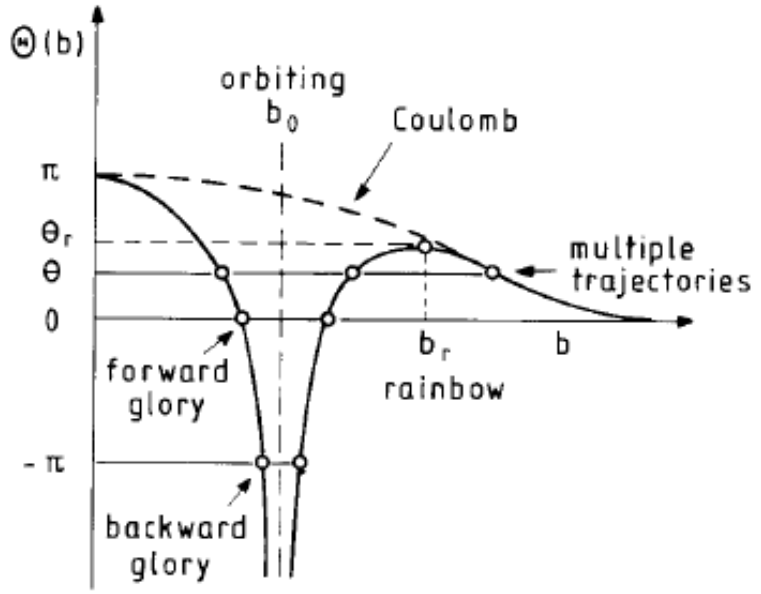


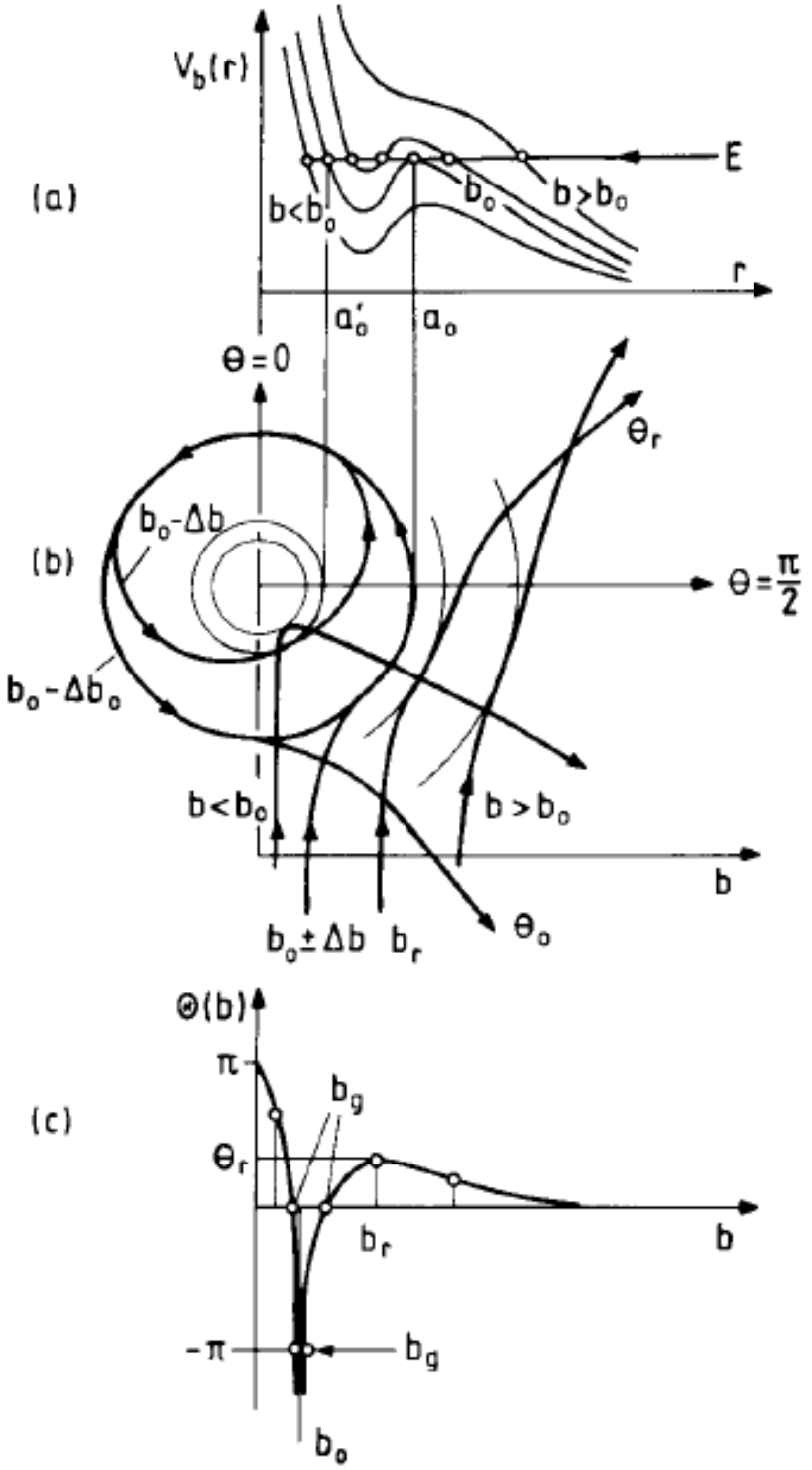
The Coulomb potential has the form

$$V(r) = \frac{Z_1 Z_2 e^2}{r} = \frac{2\eta E}{kr}, \qquad \eta = \frac{Z_1 Z_2 e^2}{\hbar v}$$

$$\frac{d\sigma}{d\Omega} = \frac{\eta^2}{4k^2 \sin^4(\theta/2)}.$$ *Rutherford cross section*

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta |d\Theta/db|};$$




(a)
$V_b(r)$
E
b_o
$b<b_o$
$b>b_o$
r
a_o'
a_o
(b)
$\Theta=0$
Θ_r
$b_o-\Delta b$
$\Theta=\frac{\pi}{2}$
$b_o-\Delta b_o$
$b<b_o$
$b>b_o$
b
$b_o\pm\Delta b$
b_r
Θ_o
(c)
$\Theta(b)$
π
b_g
Θ_r
b_r
b
$-\pi$
b_g
b_o

Formal theory of reactions

Two-body channels $x + A \rightarrow y + B,$

$$a = \{\alpha; a_x, a_A; \mathbf{k}_a\},$$

$$\mathbf{k}_a = \frac{m_A \mathbf{k}_x - m_x \mathbf{k}_A}{m_x + m_A}$$

$$\mathcal{E}_a = \frac{\hbar^2 k_a^2}{2\mu_\alpha},$$

$$H_\alpha = T_\alpha + h_\alpha, \qquad T_\alpha = -\frac{\hbar^2}{2\mu_\alpha}\nabla_\alpha^2 \qquad h_\alpha = h_x + h_A.$$

$$(H_\alpha - E_a)|a\rangle = 0$$

$$E_a = \mathcal{E}_a + \epsilon_a,$$

$$h_\alpha |a_x, a_A\rangle = \epsilon_a |a_x, a_A\rangle \text{ with } \epsilon_a = \epsilon_x + \epsilon_A,$$

$$i\hbar \frac{d}{dt}|\Phi_a(t)\rangle = H_\alpha |\Phi_a(t)\rangle$$

The total Hamiltonian H of the system is

$$H = T_\alpha + h_\alpha + V_\alpha = H_\alpha + V_\alpha$$

$$i\hbar \frac{d}{dt} |\Psi_a(t)\rangle = H |\Psi_a(t)\rangle$$

For $t = 0$ we have

$$|\Psi_a^+(0)\rangle = e^{\frac{i}{\hbar} H t_0} |\Phi_a(t_0)\rangle \quad \text{with } t_0 < -t_\infty,$$

the Lippmann-Schwinger equation

$$|a^\pm\rangle = |a\rangle + \frac{1}{E_a - H_\alpha \pm i\eta} V_\alpha |a^\pm\rangle,$$

Cross section

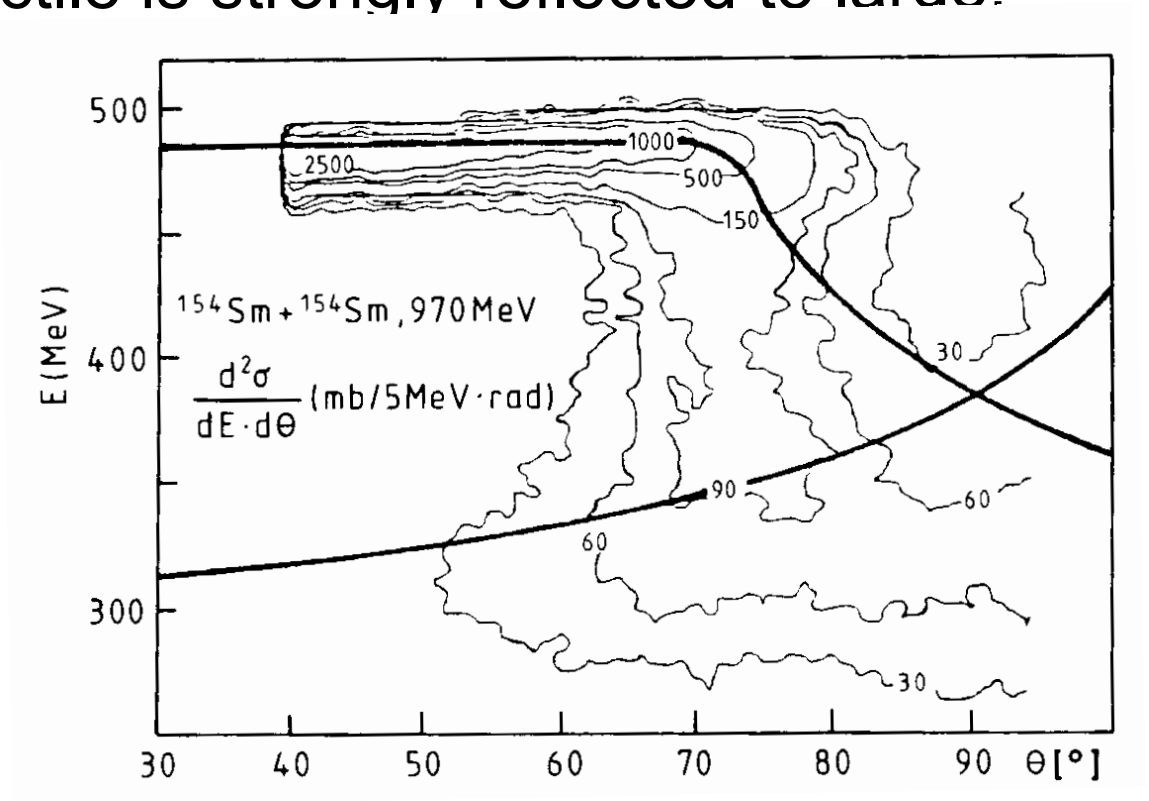
$$\frac{d\sigma_{a\to b}}{d\Omega_b} = \left(\frac{2\pi}{\hbar}\right)^4 \mu_\alpha \mu_\beta \frac{k_b}{k_a} |T_{a\to b}|^2.$$

$$T_{a\to b} = \langle b| \left(V_\alpha + V_\beta \frac{1}{E - H + i\eta} V_\alpha \right) |a\rangle,$$

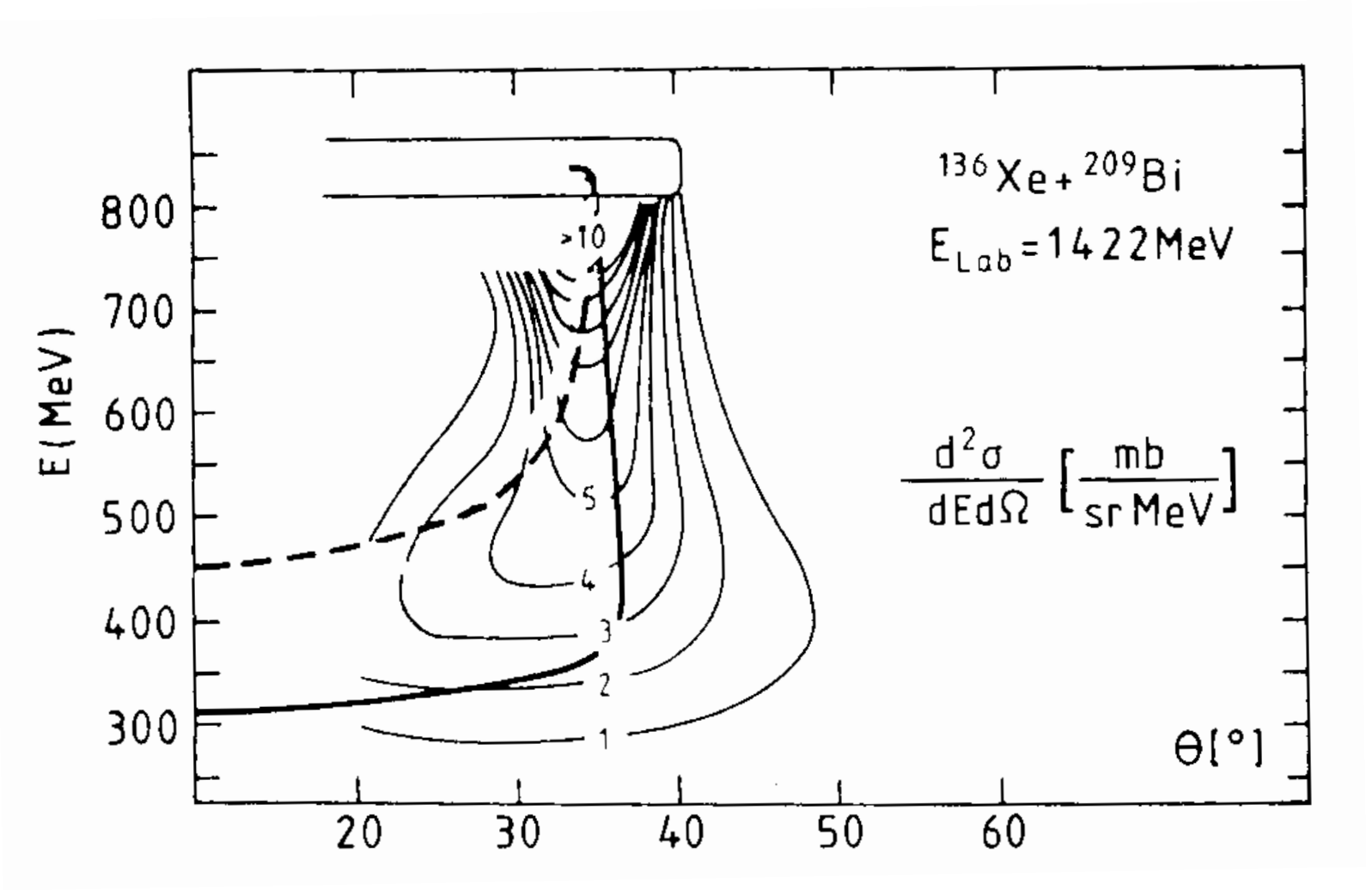
Characteristic features of deep-inelastic collisions (DIC)

Measurments of γ-multiplicities show that the γ-rays emitted after a DIC carry angular momentum, which is taken out of the relative motion of the collision partners. This shows that there is considerable transfer of angular momentum from the relative motion to the internal system.

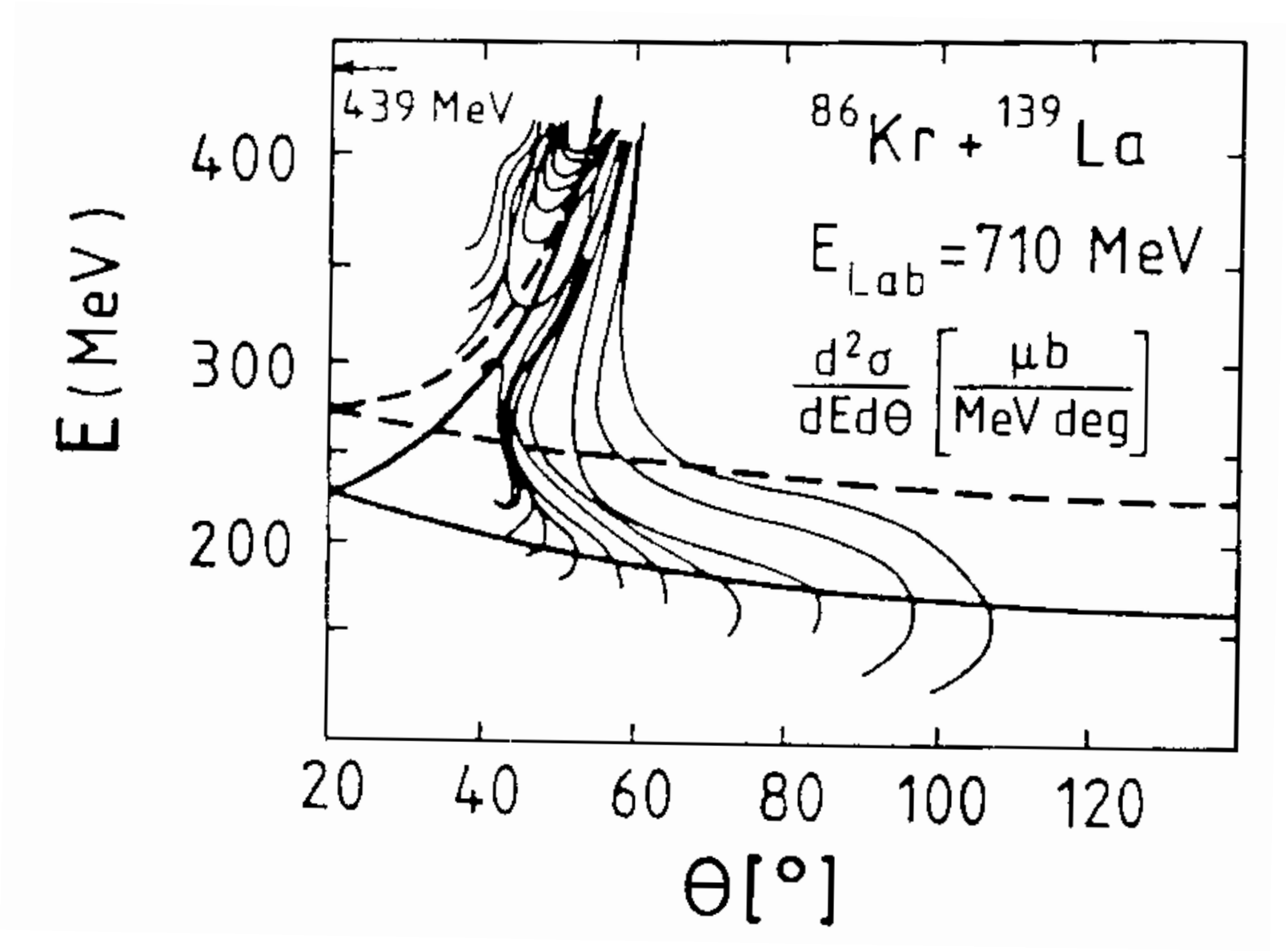
- Coulomb-like collisions. Collision partners are higly charged and the incident energy is relatively low. The Coulomb repulsion dominates and the projectile is strongly reflected to large backward angles.



- Focussing collisions. Higher energies or lighter nuclei. Scattering into a narrow angular region.



- Orbiting collisions. The attractive nuclear force dominates over the Coulomb force. This pulls the trajectory of the projectile around the target into the region of negative scattering angles.



Nuclear reactions at low energies

Two extreme scenarios

- Direct Nuclear Reactions:
  - fast and involve a ***single-nucleon interaction***.
  - the interaction time must be very short **(~10^{-22} s)**.
  - require incident particle energy ***larger than ~ 5 MeV/Ap***. (Ap is the atomic mass number of a projectile)
  - incident particles interact ***on the surface*** of a target nucleus rather than in the volume of a target nucleus.
  - products ***are not distributed isotropically in angle***, but they are forward-focused.
  - are of importance in measurements of nuclear structure.

- Compound Nucleus Reactions
  - involve ***many nucleon-nucleon interactions***.
  - a large number of collisions between the nucleons leads to a ***thermal equilibrium*** inside the compound nucleus.
  - the time scale of reactions is **10^{-18} s – 10^{-16} s**.
  - are usually created if the projectile has ***low energy***.
  - incident particles interact in the volume of a target nucleus.
  - products are distributed ***near isotropically in angle*** (the nucleus loses memory of how it was created – *Bohr's hypothesis of independence*).
  - ***the decay mode does not depend on how the compound nucleus is formed***.
  - ***resonances*** in the cross-section are typical for the compound nucleus reaction.

Types of direct reactions

- ✓ **Elastic scattering** in which a passing particle and targes stay in their ground states.

- ✓ **Inelastic scattering** in which a passing particle changes its energy state. For example, the (p, p') reaction.

- ✓ **Transfer reactions** in which one or more nucleons are transferred to the other nucleus. These reactions are further classified as:
  - **Stripping reaction** in which one or more nucleons are transferred to a target nucleus from passing particles. For example, the neutron stripping in the (d, p) reaction.
  - **Pick-up reaction** in which one or more nucleons are transferred from a target nucleus to a passing particle. For example, the neutron pick-up in the (p, d) reaction.

- ✓ **Break-up reaction** in which a breakup of a projectile into two or more fragments occurs.

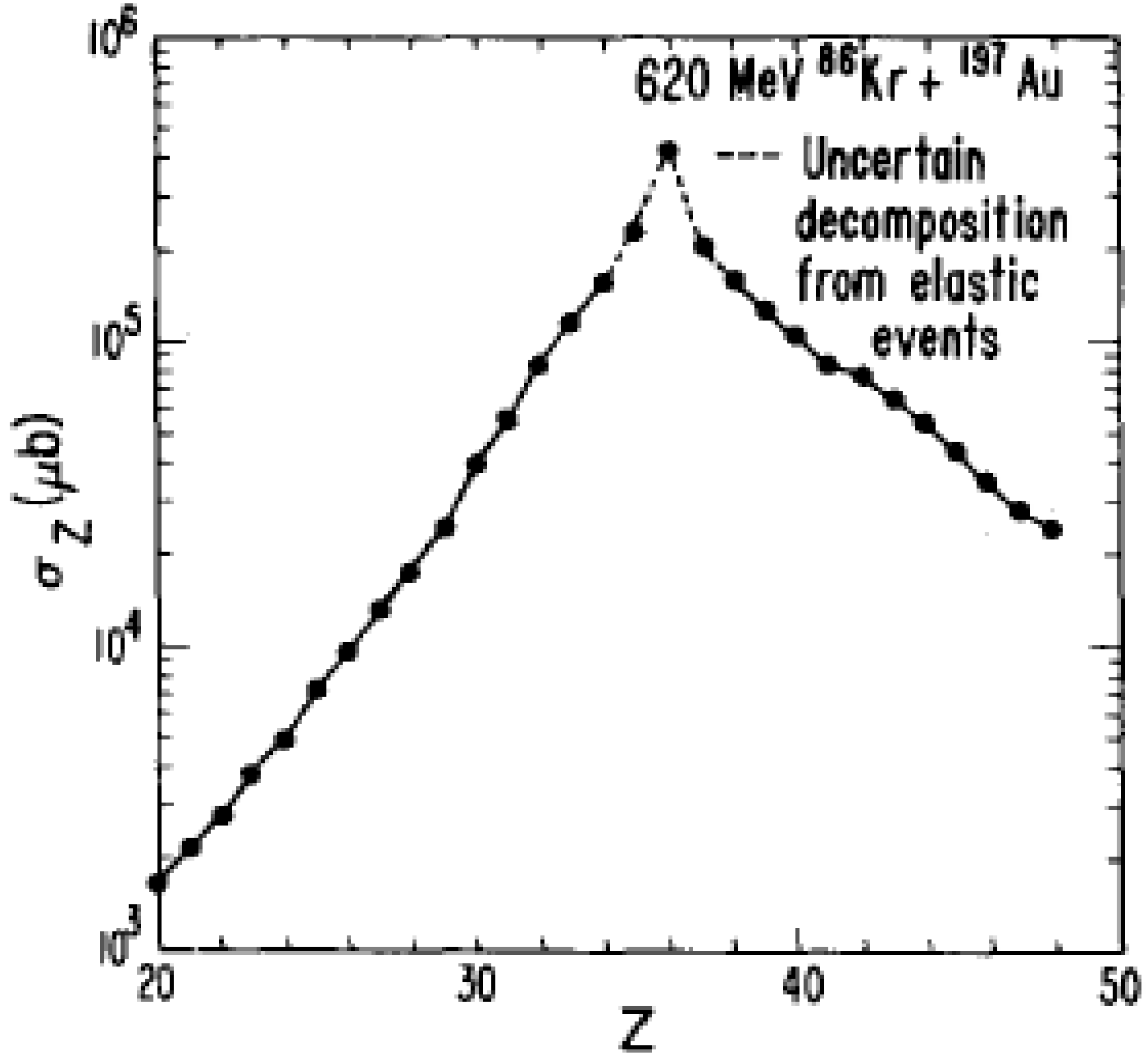
- ✓ **Knock-out reaction** in which a single nucleon or a light cluster is removed from the projectile by a collision with the target.

End of 60th – discovery of new type of nuclear reactions – DIC

Mechanism: dynamic & statistic pecularities
formation of DNS – result of nucler viscosity and microscopic effects

nuclear molecule ↔ DNS
quasistationary states dynamics

Study of DIC
identification of the products
scattering chamber
radiochemistry
ΔE-E detectors
time of flight
magnetic spectrometra
two-shoulder detectors
detectors for *n, p*, α, and γ


620 MeV ^{86}Kr + ^{197}Au
--- Uncertain decomposition from elastic events
σ_Z (μb)
10^6
10^5
10^4
10^3
20
30
40
50
Z

Characteristics of DIC

- total dissipation of kinetic energy ? energy distribution has maxima at V_b for the fragments, independent on $E_{c.m.}$
- angular distributions have maxima at forward angles

  decrease of anisotropy with increasing number of transfered nucleon
- large variation of mass (charge) distributions (max. at A_p (A_t) and Z_p (Z_t))
- N/Z ratio
- sharing of excitation energy and angular momentum

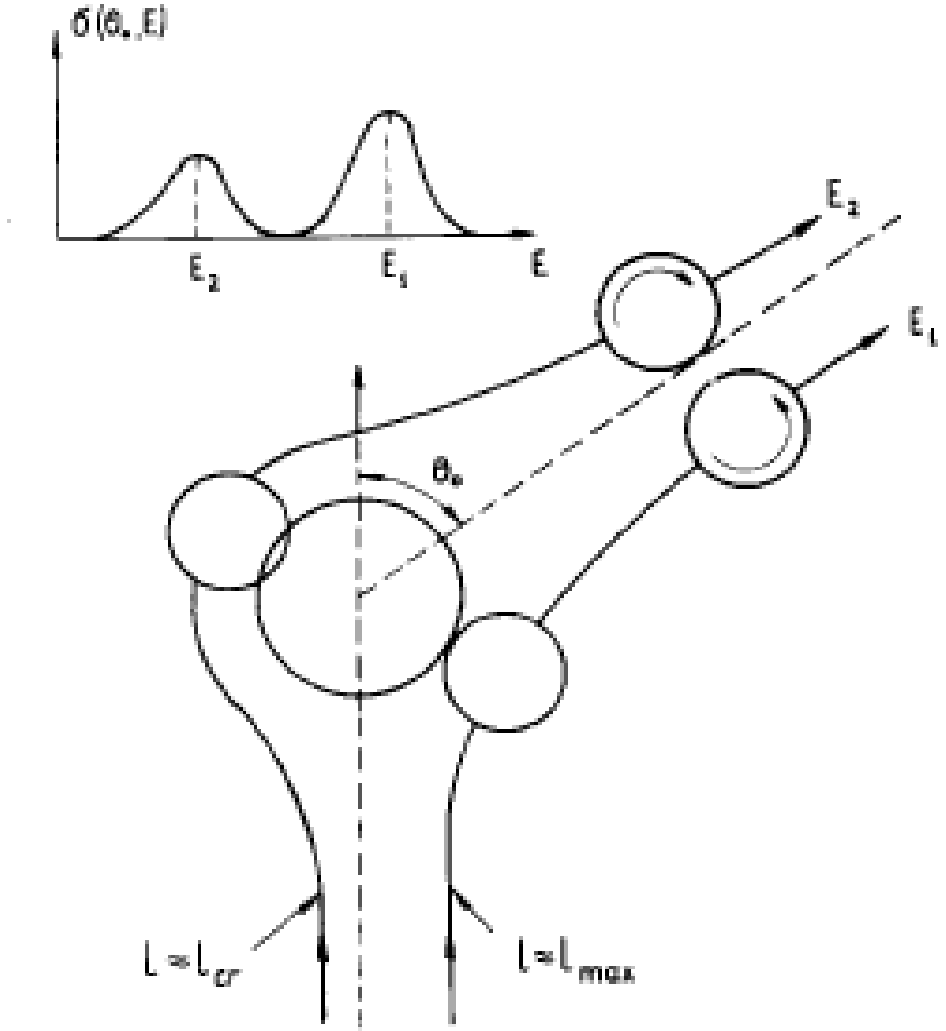
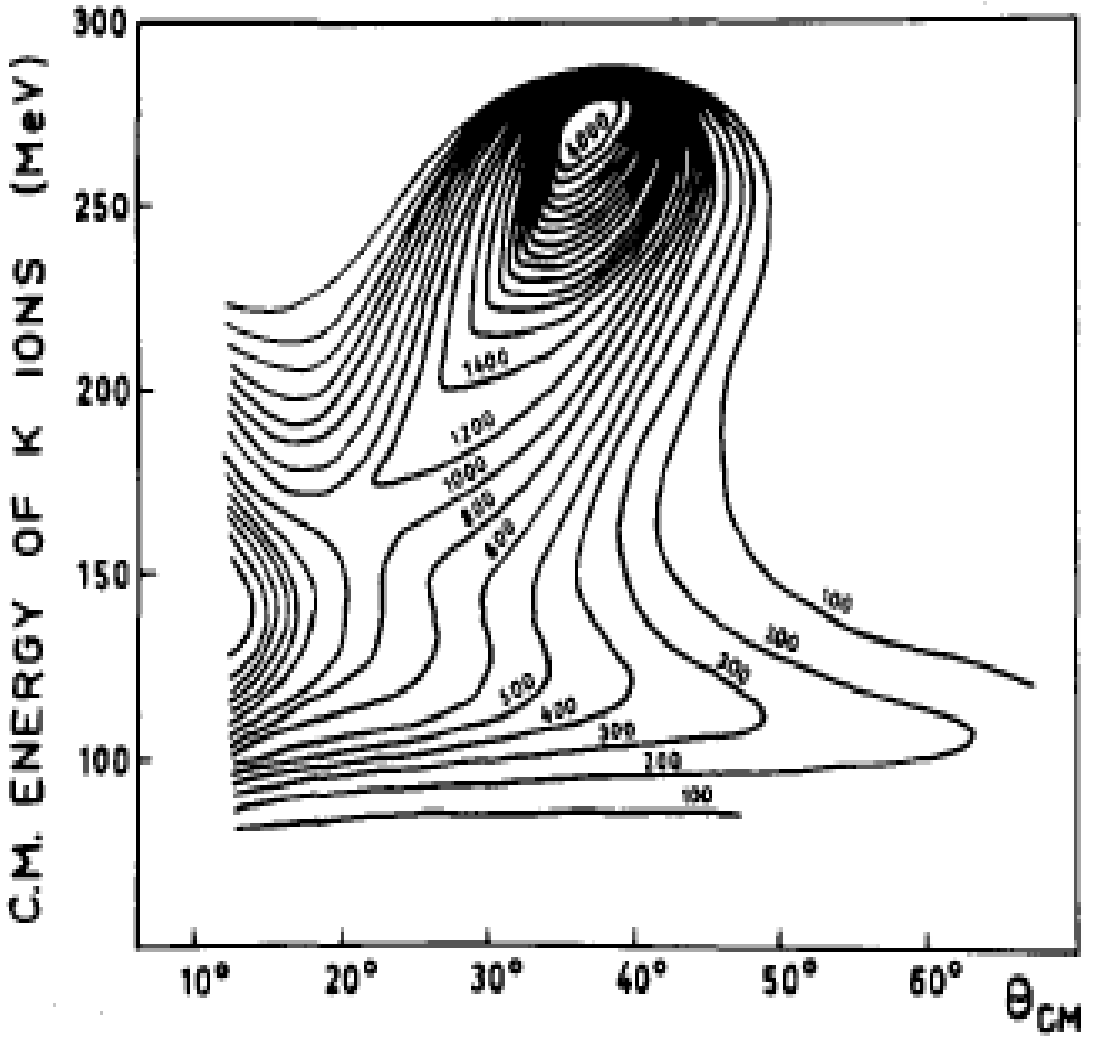


Illustration of the formation of two peaks in the energy spectrum



Contour diagram representing the transfer reaction data for ^{232}Th(^{40}Ar,K) at 388 MeV

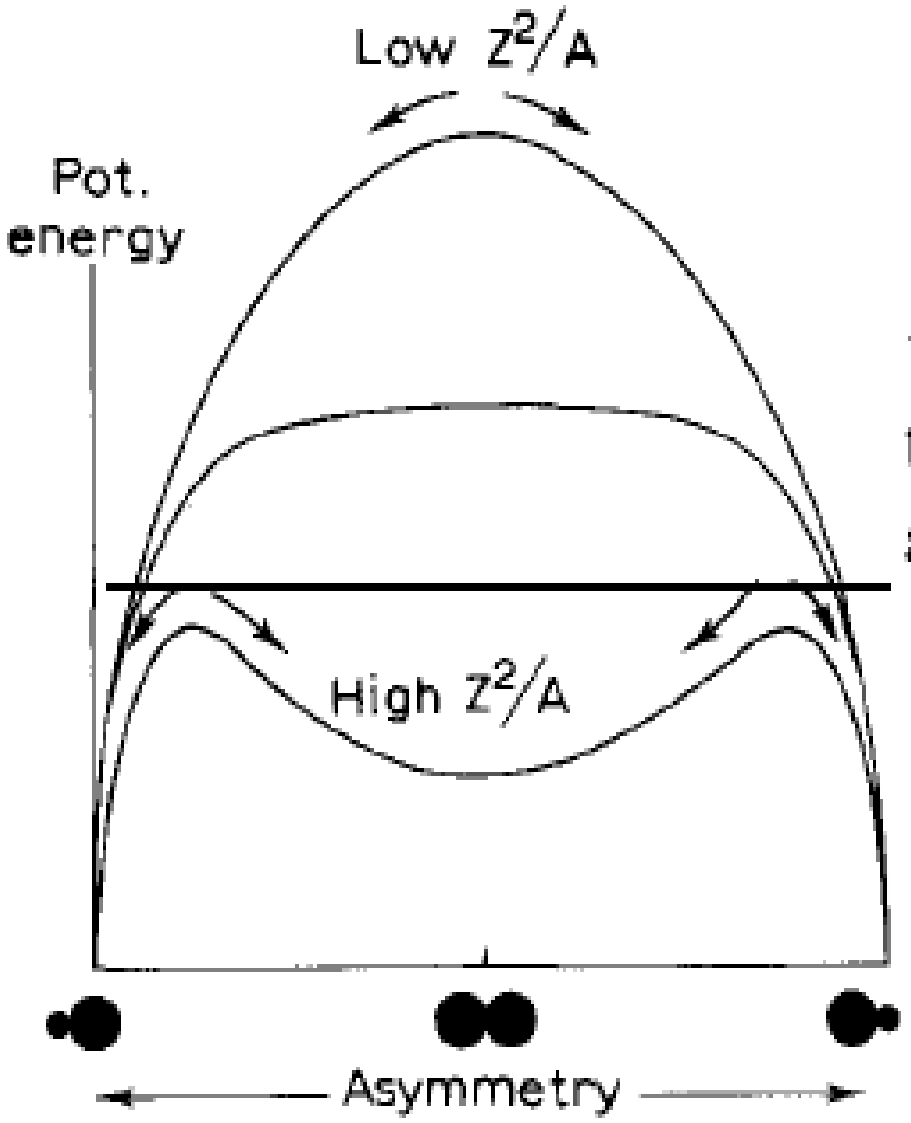


Illustration of the dependence of the potential energy of the system of two touching nuclear drops on mass asymmetry and parameter $(Z_1 + Z_2)^2/(A_1 + A_2)$.

Set of coordinates for the description of DNS evolution:

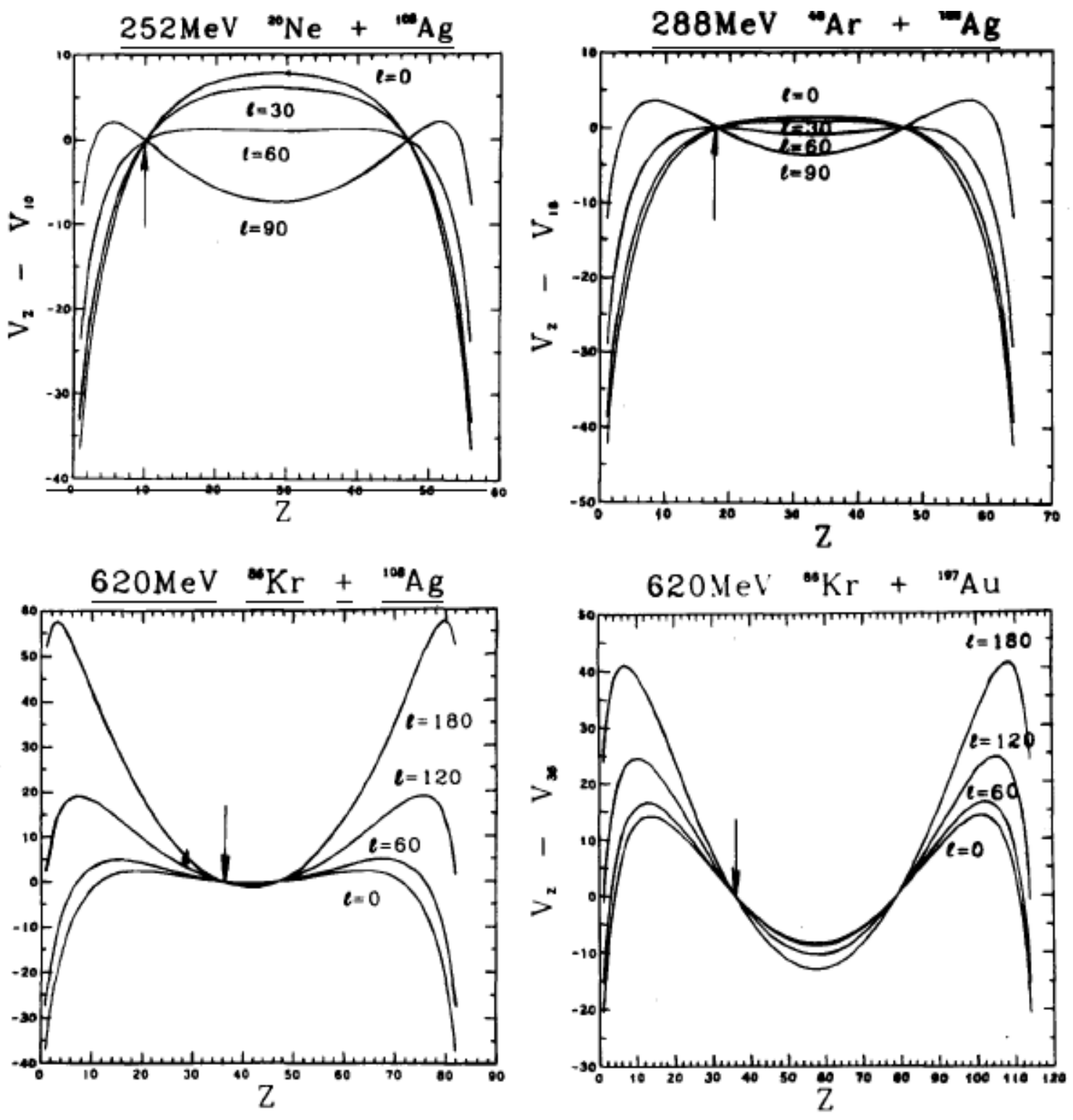
$$\eta_Z=(Z_1-Z_2)/(Z_1+Z_2)\ ,\ \ \eta=(A_1-A_2)/(A_1+A_2)\ ,\ \ R$$

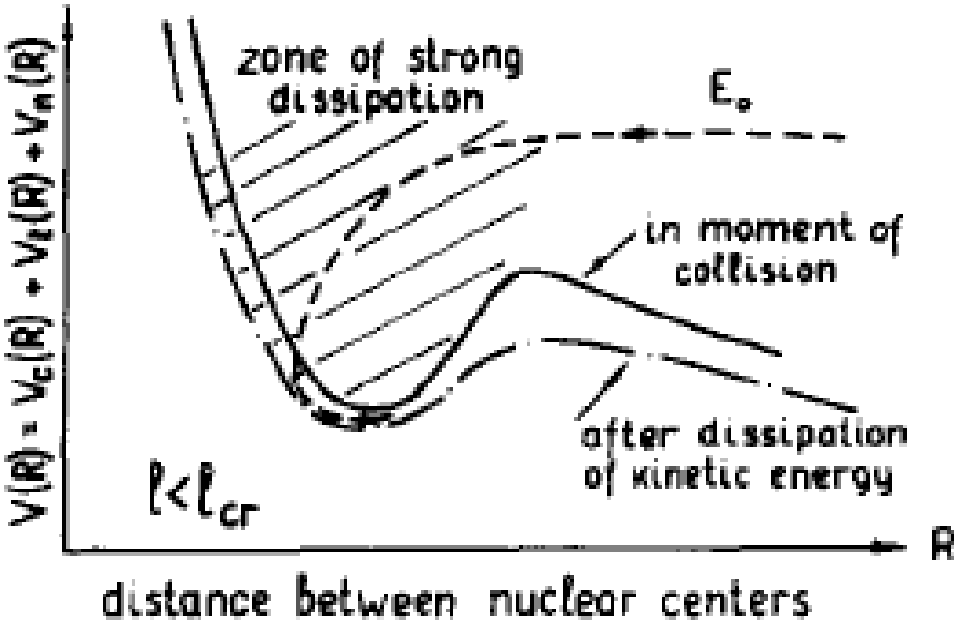
The potential energy of DNS:

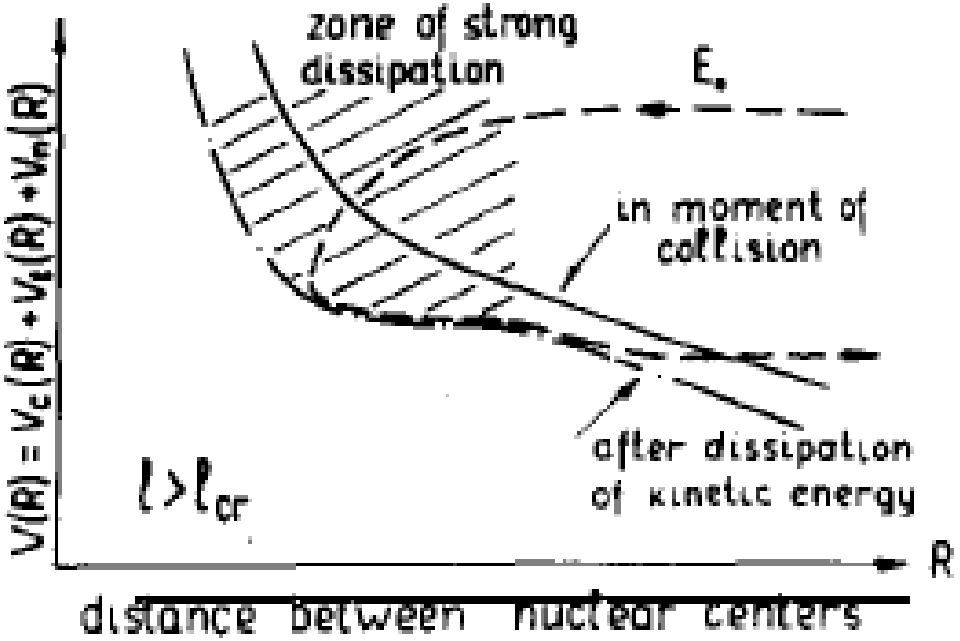
$$U(R,\eta,\eta_Z,\beta_1,\beta_2,J)=B_1+B_2+V(R,\eta,\eta_Z,\beta_1,\beta_2,J)$$

The nucleus-nucleus potential:

$$V(R,\eta,\eta_Z,\beta_1,\beta_2,J)=V_C(R,\eta_Z,\beta_1,\beta_2)+V_N(R,\eta,\beta_1,\beta_2)+V_{rot}(\eta,\beta_1,\beta_2,J)$$


252MeV ^{20}Ne + ^{108}Ag
$V_z - V_{10}$
Z
ℓ=0
ℓ=30
ℓ=60
ℓ=90
288MeV ^{40}Ar + ^{108}Ag
$V_z - V_{10}$
Z
ℓ=0
ℓ=30
ℓ=60
ℓ=90
620MeV ^{86}Kr + ^{108}Ag
$V_z - V_{30}$
Z
ℓ=180
ℓ=120
ℓ=60
ℓ=0
620MeV ^{86}Kr + ^{197}Au
$V_z - V_{30}$
Z
ℓ=180
ℓ=120
ℓ=60
ℓ=0


zone of strong dissipation
E_0
in moment of collision
after dissipation of kinetic energy
$V(R) = V_c(R) + V_{\ell}(R) + V_n(R)$
$\ell < \ell_{cr}$
R
distance between nuclear centers


zone of strong dissipation
E_0
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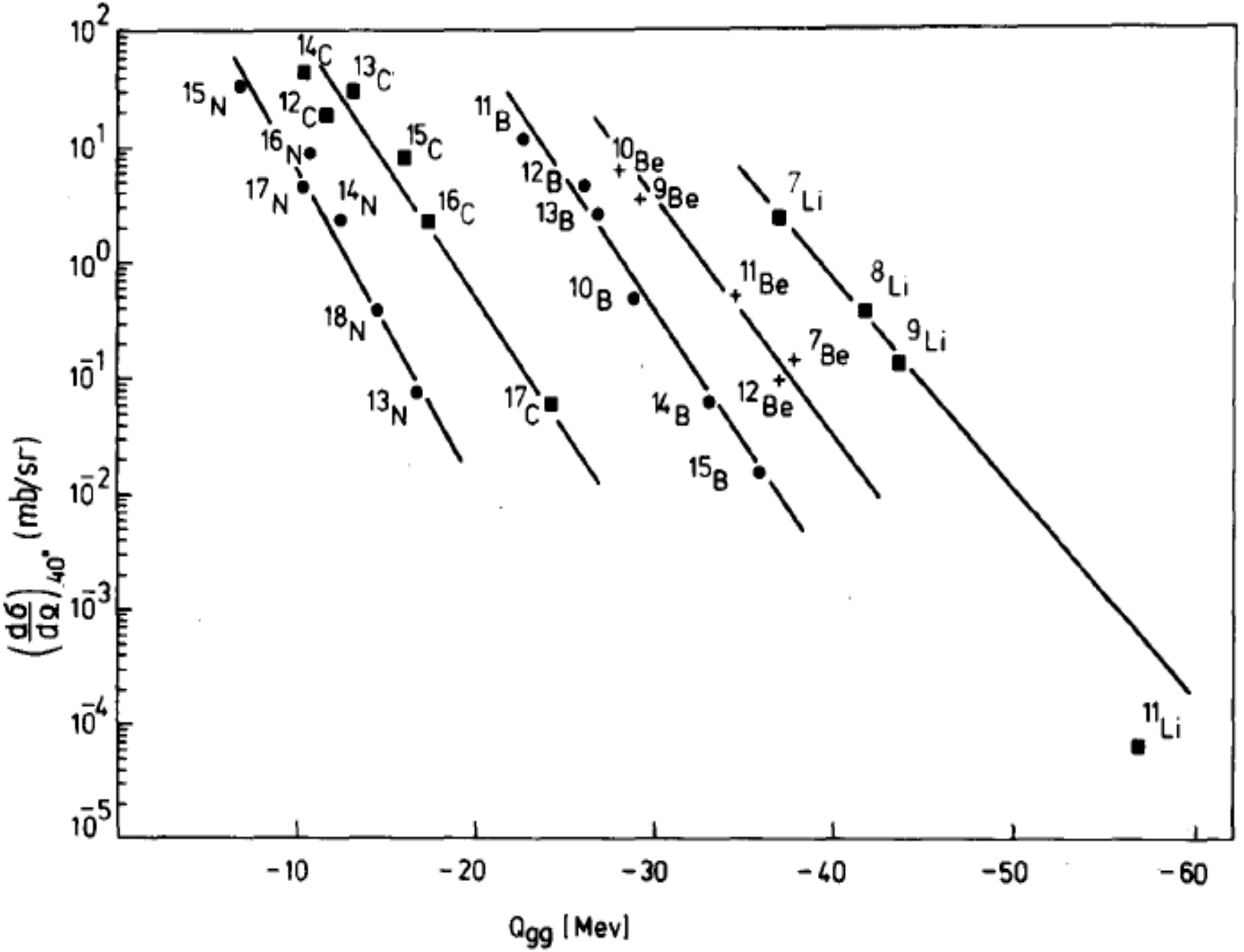
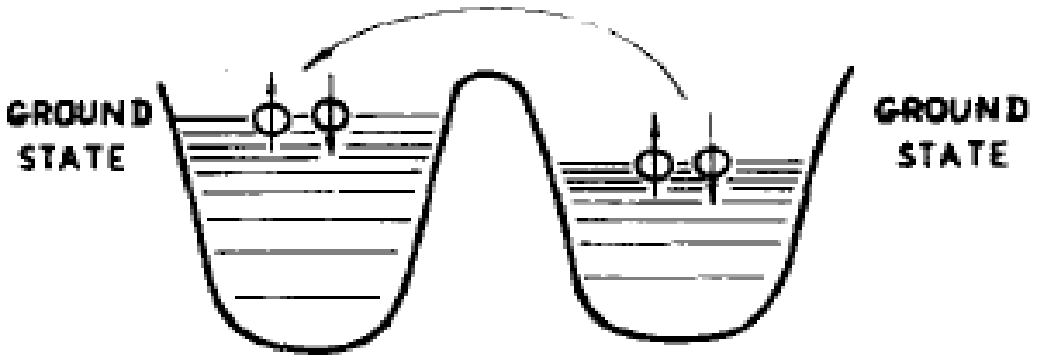
$\ell > \ell_{cr}$
R
distance between nuclear centers



$$Q_{gg} = (M_1 + M_2) - (M_3 + M_4)$$

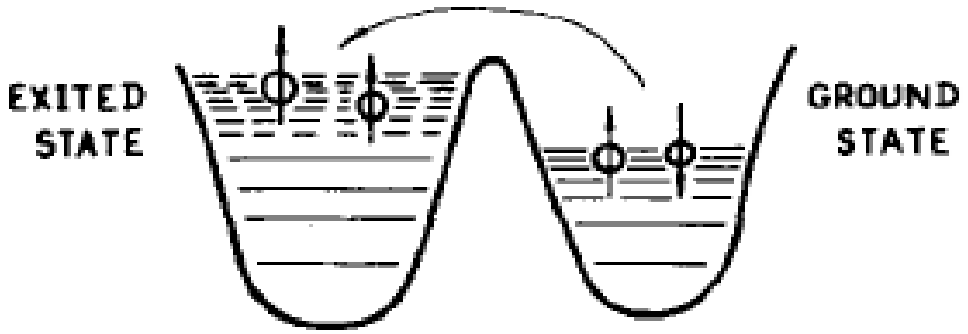
^{232}Th + ^{16}O 137 MeV

NON-PAIRING CORRECTIONS



$Q_{gg} = (M_1 + M_2) - (M_3 + M_4)$ corresponds to transition $(G.S.)_{donor} \rightarrow (G.S.)_{acceptor}$

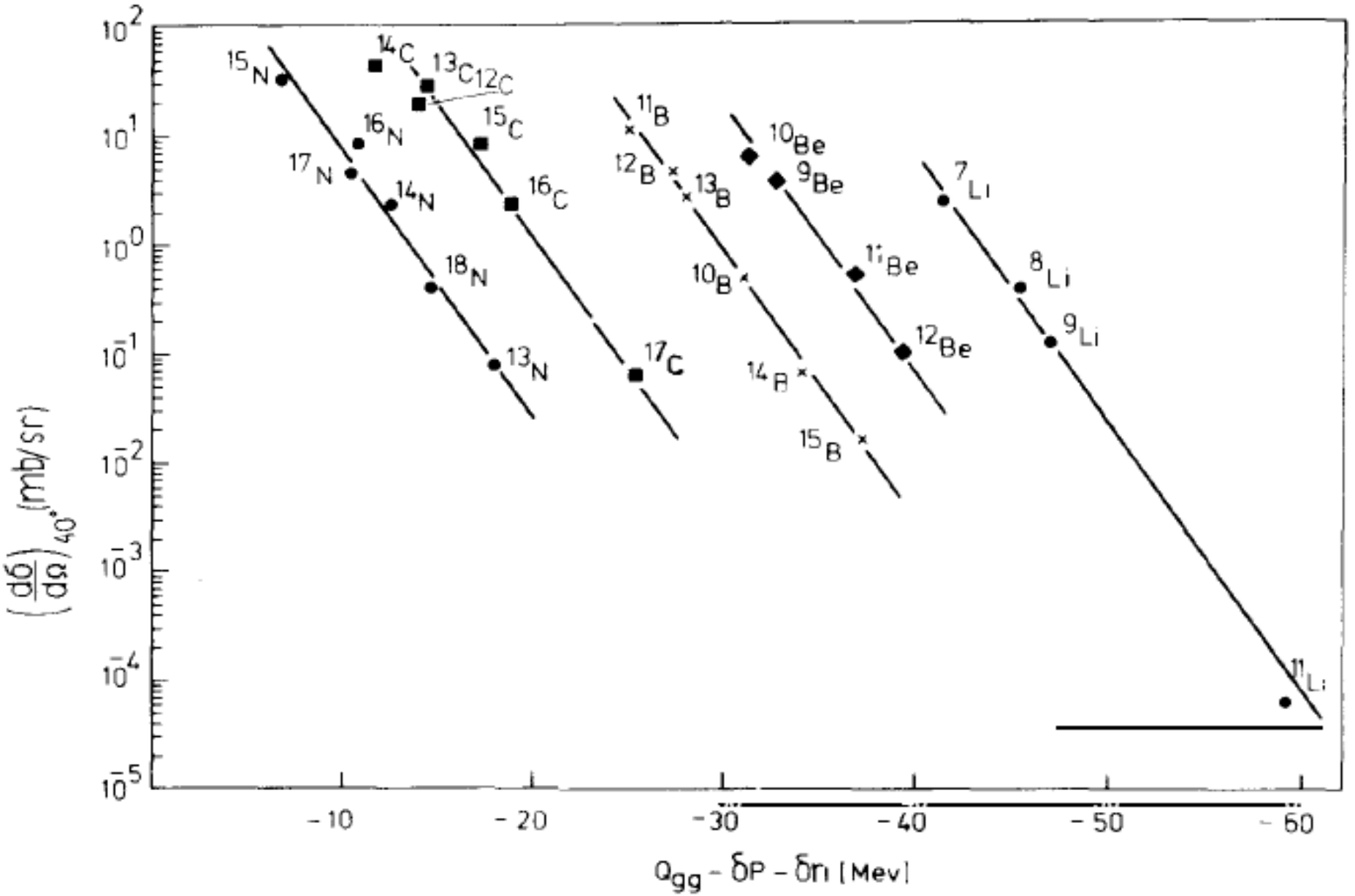
DEEP INELASTIC TRANSFERS



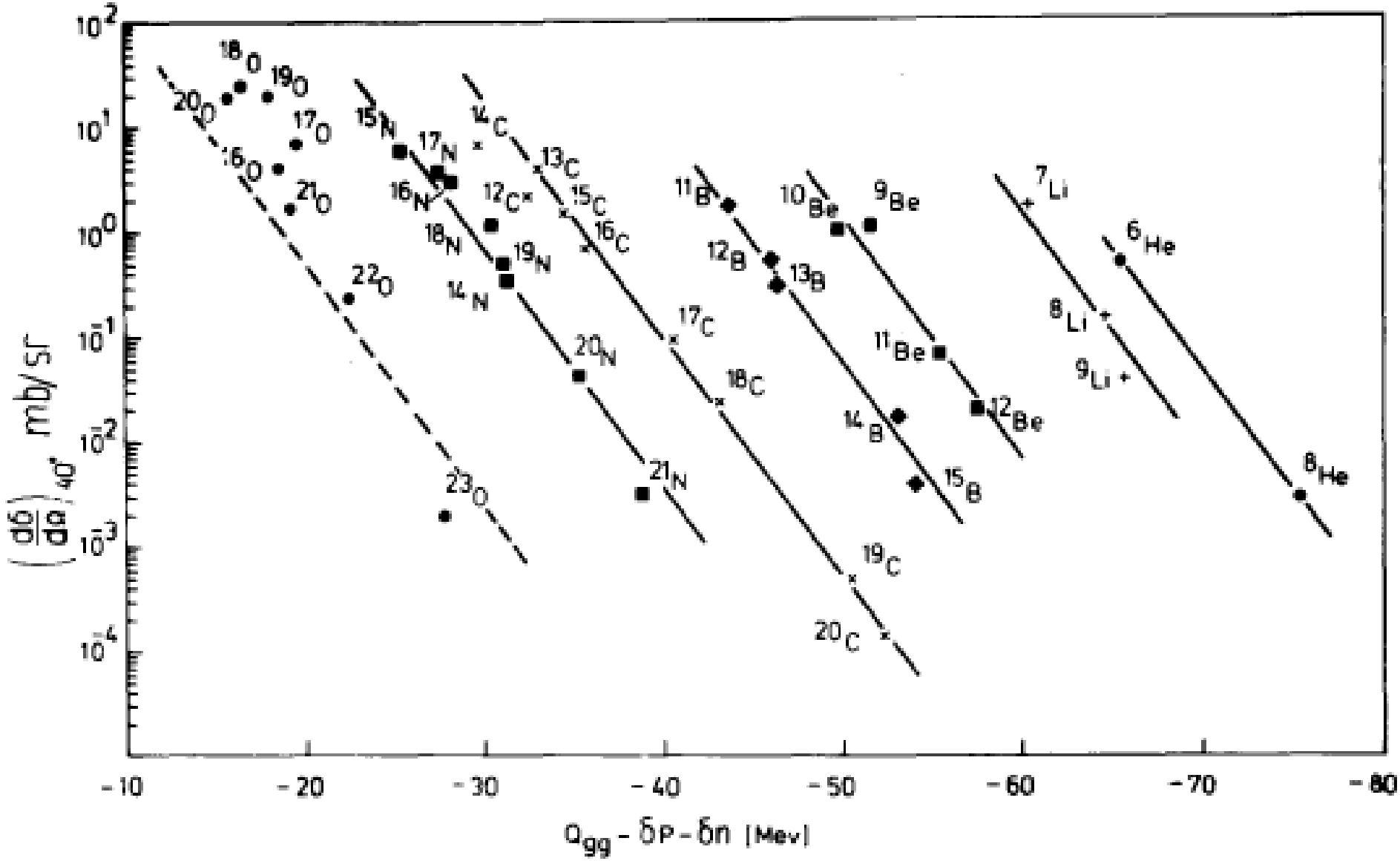
transitions $(G.S.)_{donor} \rightarrow (E.S.)_{acceptor}$

$$\delta(p) + \delta(n) = \sum \begin{array}{l}\text{pairing energy in acceptor nucleus} \\ \text{of transferred nucleon pairs}\end{array}$$

Illustration of the necessity of introducing corrections for non-pairing.


10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\left(\frac{d\sigma}{d\Omega}\right)_{40^\circ}$ (mb/sr)
^{15}N
^{14}C
^{13}C
^{12}C
^{16}N
^{15}C
^{17}N
^{16}C
^{14}N
^{18}N
^{13}N
^{17}C
^{11}B
^{12}B
^{13}B
^{10}Be
^{9}Be
^{10}B
^{11}Be
^{14}B
^{12}Be
^{15}B
^{7}Li
^{8}Li
^{9}Li
^{11}Li
-10
-20
-30
-40
-50
-60
$Q_{gg} - \delta P - \delta n$ (Mev)

^{232}Th + ^{16}O

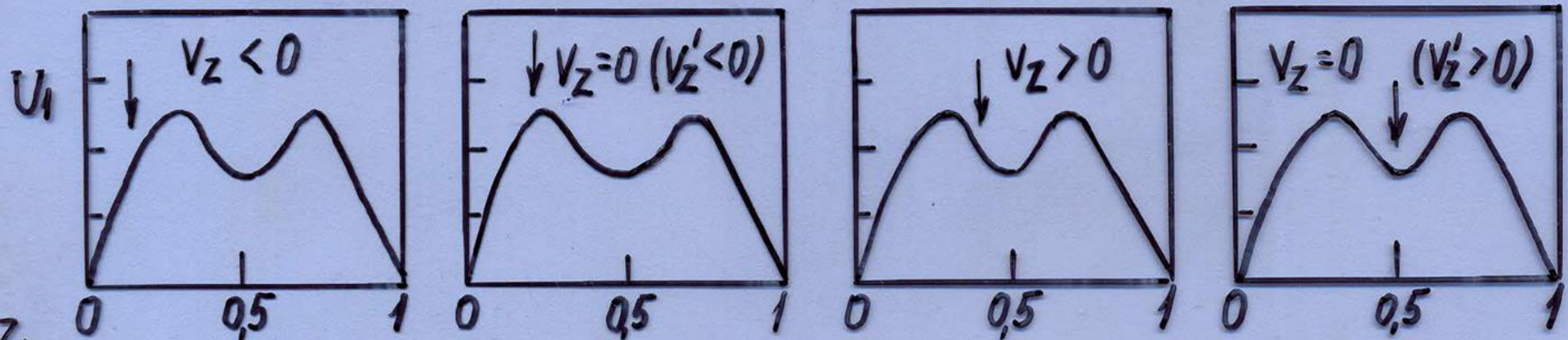


The Q_{gg} systematics for the reaction ^{232}Th + ^{22}Ne, corrected for non-pairing.

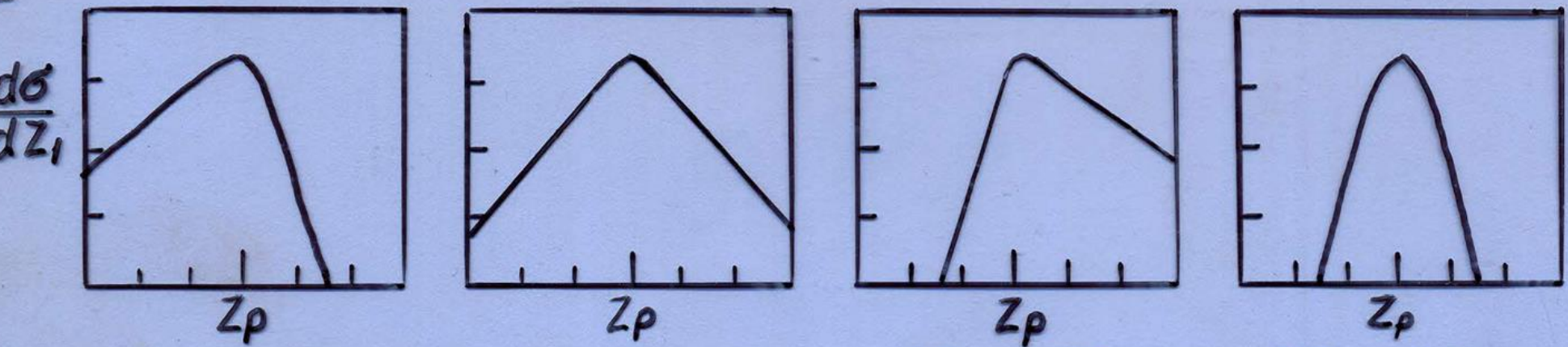
$$\sigma \sim \exp\{(Q_{gg} + \Delta E_c)/T\}$$

$$\frac{\partial}{\partial t}P(Z_1,t) = -\frac{\partial}{\partial Z_1}(V_Z P) + \frac{\partial^2}{\partial Z_1^2}(D_Z P)$$

$V_Z \sim \frac{\partial U_e}{\partial Z_1}$ $\quad \langle Z_1 \rangle = Z_p + V_Z \cdot t$ $\quad \sigma_Z^2 = 2D_Z \cdot t$



$X = \frac{Z_1}{Z}$



$^{20}Ne + ^{107}Ag$ (252 MeV)

$^{40}Ar + ^{237}Th$ (388 MeV)

$^{86}Kr + ^{166}Er$ (515 MeV)

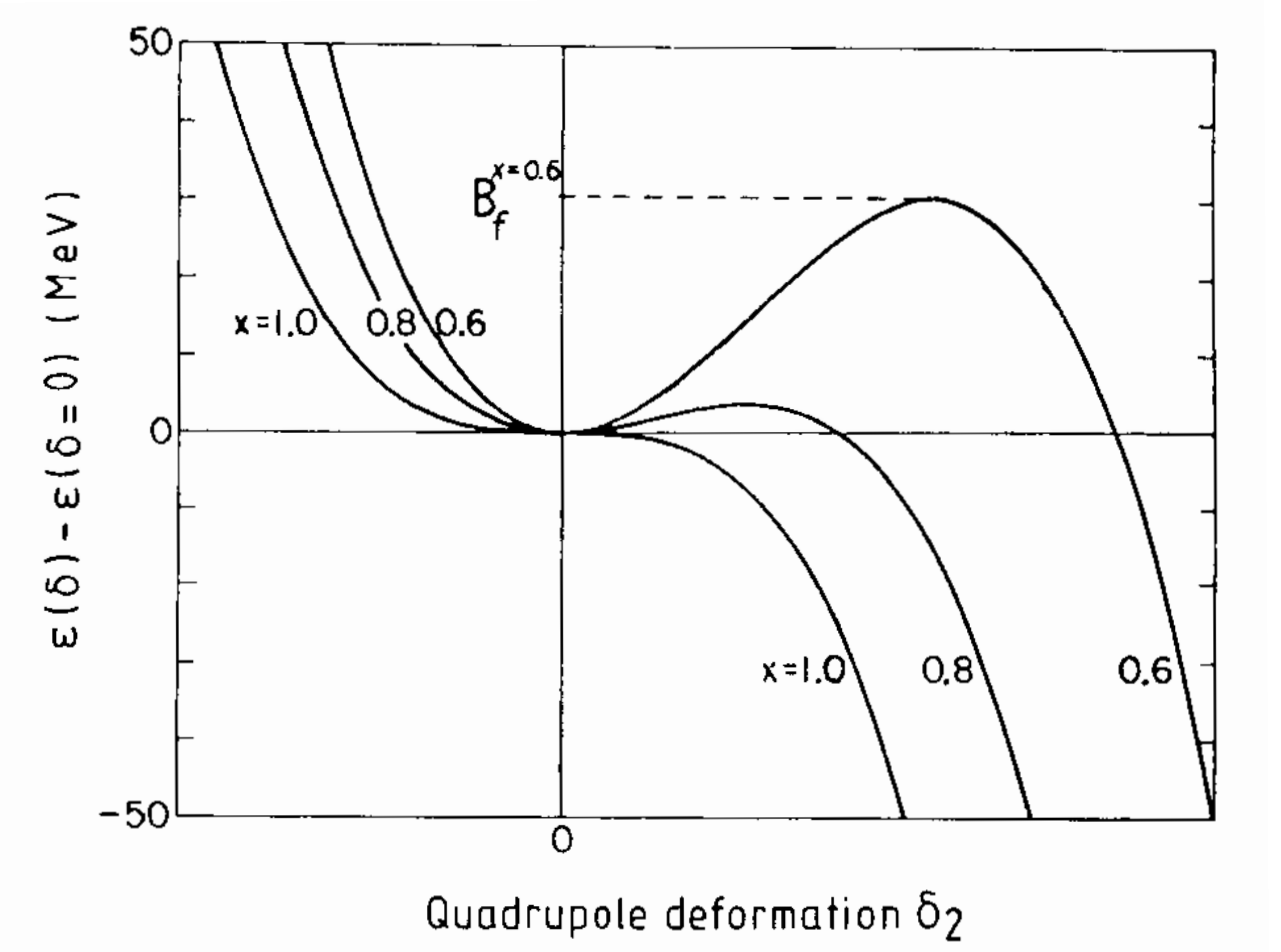
$^{238}U + ^{238}U$ (1766 MeV)

Fusion

stability of the formed compound nuclei
fisility parameter

$$x \approx \frac{1}{50}\frac{Z^2}{A} \approx \frac{Z}{120}.$$

x>1 – unstable
x<1 - stable



The projectile moves in the field of the Coulomb-plus-nuclear potential *V(r)*. For a given impact parameter *b* the radial motion is governed by the potential

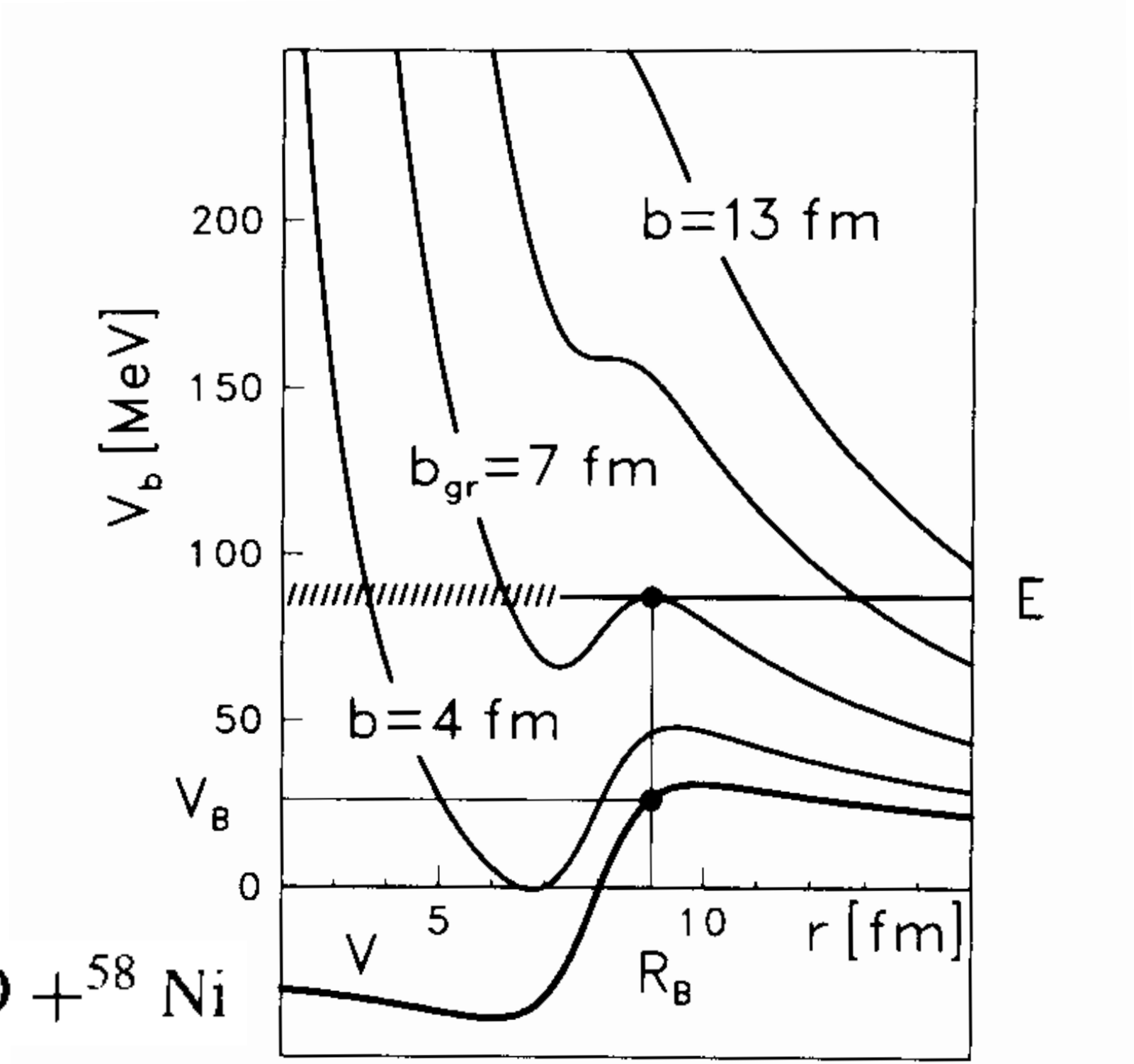
$$b = L/\sqrt{2\mu E}$$

$$V_b(r) = V(r) + E\frac{b^2}{r^2},$$

$$V_{\mathrm{B}} = V(R_{\mathrm{B}}) = V_{b=0}(R_{\mathrm{B}}),$$

$$V_{\mathrm{B}} + E\frac{b_{\mathrm{gr}}^2}{R_{\mathrm{B}}^2} = E,$$

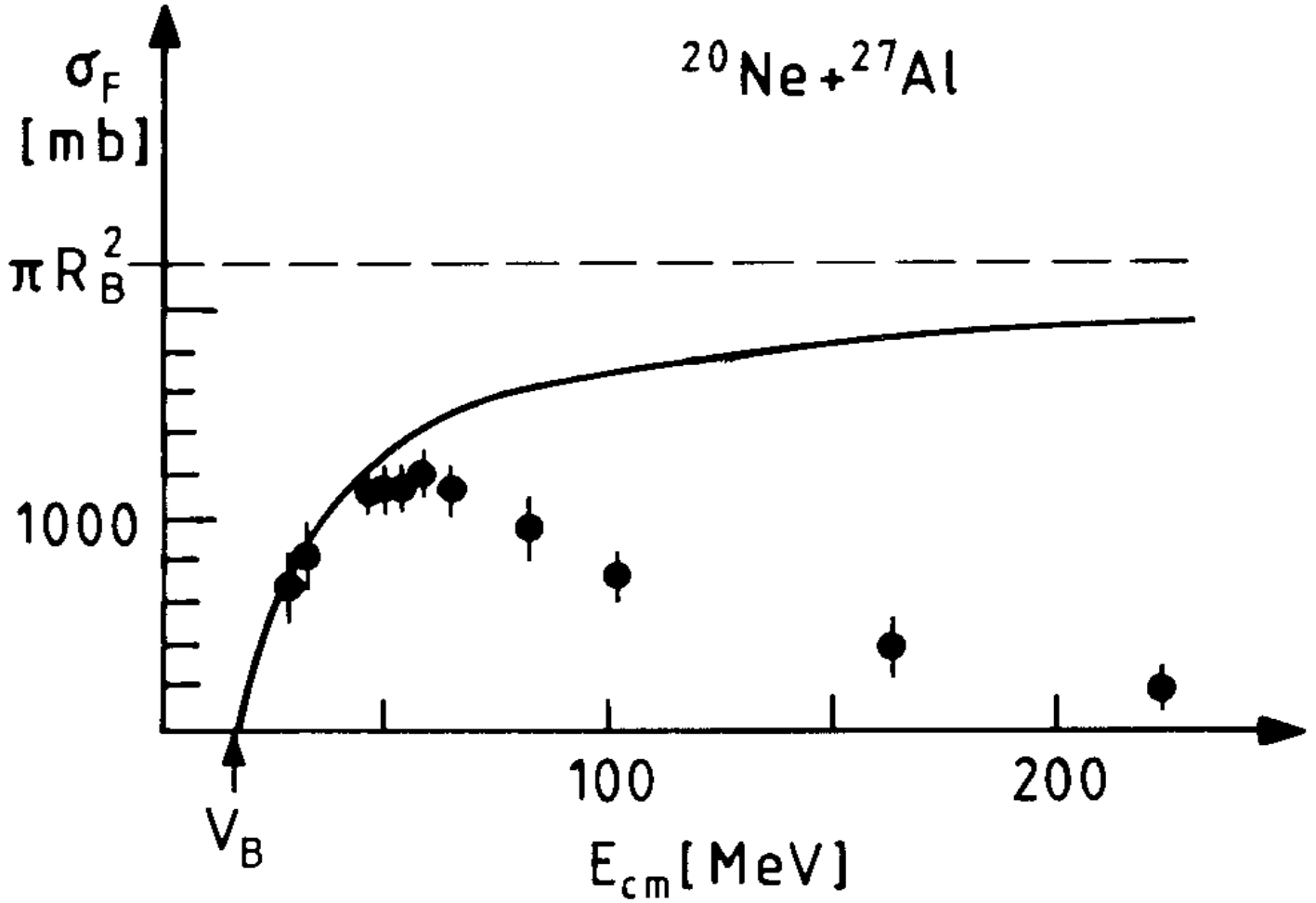
$$b_{\mathrm{gr}} = R_{\mathrm{B}}\sqrt{1 - \frac{V_{\mathrm{B}}}{E}}.$$


200
150
100
50
0
V_b [MeV]
b=13 fm
b_{gr}=7 fm
b=4 fm
E
V_B
V
5
10
r [fm]
R_B
$^{18}O + ^{58}Ni$

Total fusion cross section

$$\sigma_F = \pi b_{gr}^2.$$

$$\sigma_F(E) = \pi R_B^2 \left(1 - \frac{V_B}{E}\right).$$

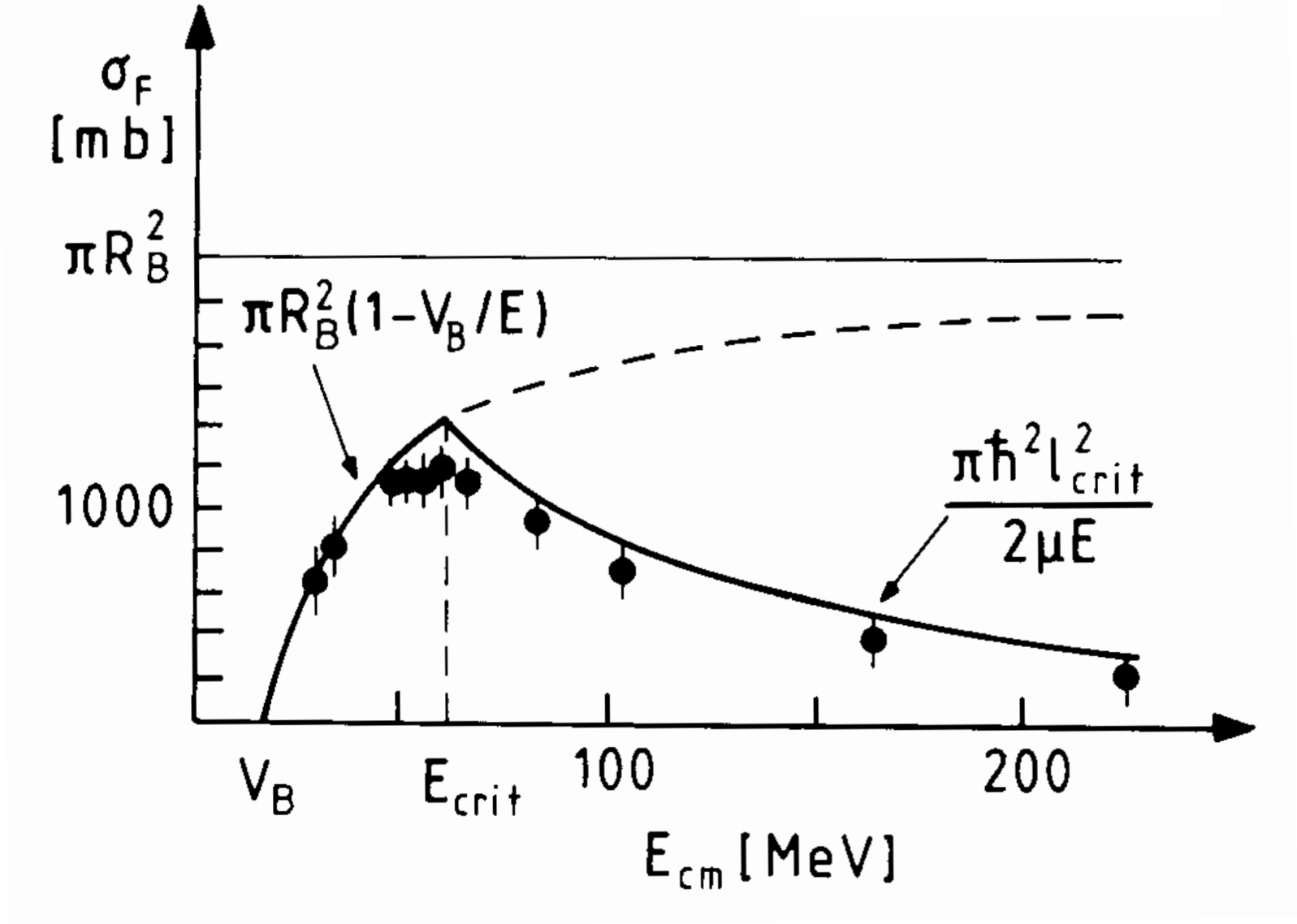


Limitation by angular momentum

The compound nucleus becomes unstable against fission above the certain value of angular momentum l_{crit}=l_{crit}^{f}, $b_{\mathrm{crit}} = l_{\mathrm{crit}}/k.$

$$\sigma_{\mathrm{F}} = \begin{cases} \pi b_{\mathrm{gr}}^2 & \text{for } b_{\mathrm{gr}} < b_{\mathrm{crit}}, \\ \pi b_{\mathrm{crit}}^2 & \text{for } b_{\mathrm{gr}} > b_{\mathrm{crit}}. \end{cases}$$

$$\sigma_{\mathrm{F}} = \begin{cases} \pi R_{\mathrm{B}}^2 (1 - V_{\mathrm{B}}/E) & \text{for } E < E_{\mathrm{crit}}, \\ \pi \hbar^2 l_{\mathrm{crit}}^2 / 2\mu E & \text{for } E > E_{\mathrm{crit}}. \end{cases}$$


$^{20}Ne + ^{27}Al$
σ_F [mb]
πR_B^2
$\pi R_B^2(1-V_B/E)$
$\frac{\pi\hbar^2 l_{crit}^2}{2\mu E}$
1000
V_B
E_{crit}
100
200
E_{cm} [MeV]

Sub-barrier fusion

transmission coefficient in the WKB approximation

$$T = \exp\left(-\frac{2}{\hbar}\int_b^a |p(x')|dx'\right),$$

$$p(x) = \sqrt{2\mu[E - V(x)]},$$

For the parabolic barrier, Hill-Wheeler formula

$$T = T(E) = \frac{1}{1 + \exp[2\pi(V_B - E)/\hbar\omega]}.$$

$$T_l(E) = \frac{1}{1 + \exp\{2\pi[V_B + \hbar^2 l(l+1)/2\mu R_B^2 - E]/\hbar\omega_B\}},$$

$$\omega_B^2 = \left| \frac{1}{\mu} \frac{d^2}{dr^2} \left(V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2} \right) \right|_{R_B}$$

$$\sigma_F(E) = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) T_l(E)$$

$$\approx \frac{2\pi}{k^2} \int_0^{\infty} \frac{l dl}{1 + \exp\{2\pi[V_B + \hbar^2 l^2/2\mu R_B^2 - E]/\hbar\omega_B\}}.$$

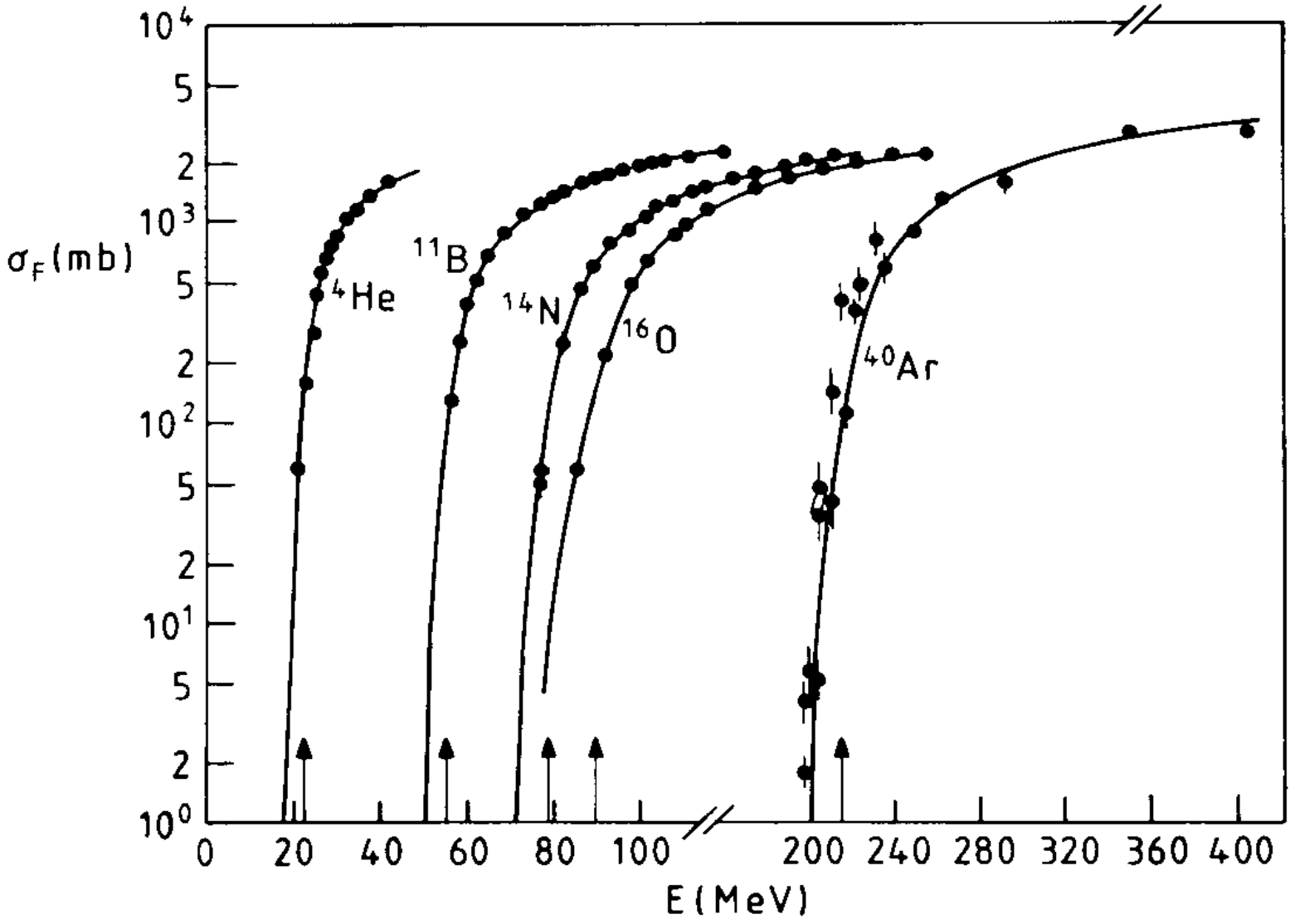
With the substitutions $y = l^2$, $a = \exp[2\pi(V_B - E)/\hbar\omega_B]$ and $b = \pi\hbar/\mu R_B^2\omega_B$ we obtain

$$\sigma_F(E) = \frac{\pi}{k^2}\int_0^\infty \frac{dy}{1 + a\exp(by)} = \frac{\pi}{k^2}\frac{1}{b}\ln\left(1 + \frac{1}{a}\right).$$

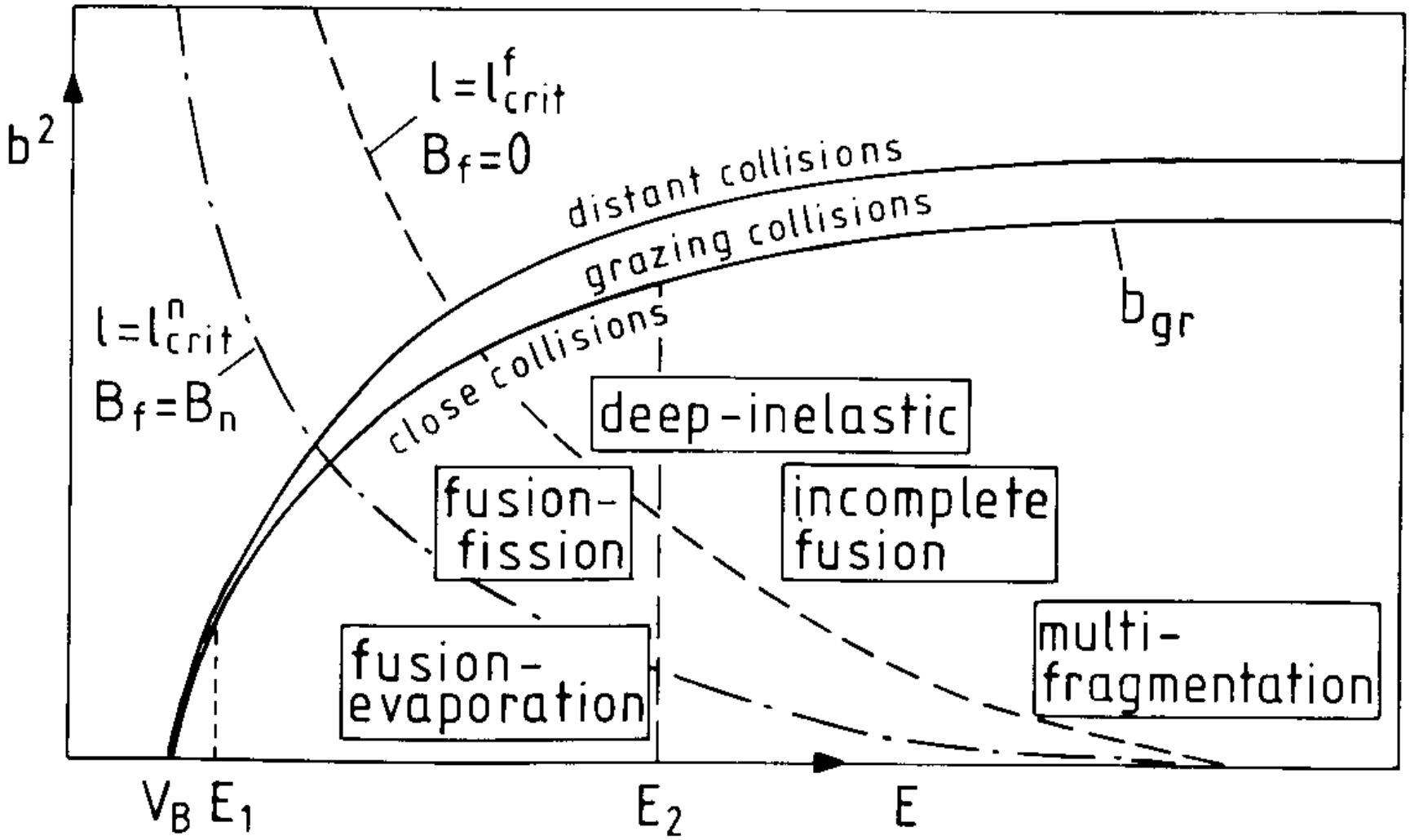
Going back to the original parameters, we arrive at the *Wong formula* for the fusion cross section

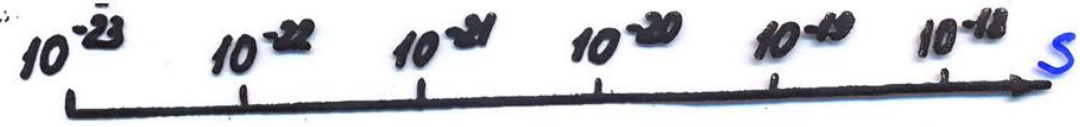
$$\sigma_F(E) = \frac{\hbar\omega_B R_B^2}{2E}\ln\{1 + \exp[2\pi(E - V_B)/\hbar\omega_B]\}.$$

$$\sigma_F(E) = \begin{cases} \pi R_B^2[1 - (V_B/E)] & \text{for } E > V_B, \\ \left(\hbar\omega_B R_B^2/2E\right)\exp[-2\pi(V_B - E)/\hbar\omega_B] & \text{for } E < V_B. \end{cases}$$


10^4
5
2
10^3
5
2
10^2
5
2
10^1
5
2
10^0
σ_F (mb)
^{4}He
^{11}B
^{14}N
^{16}O
^{40}Ar
0
20
40
60
80
100
200
240
280
320
360
400
E (MeV)

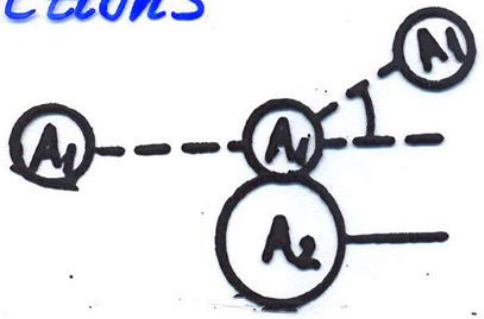
+^{238}U


b²
l = l^f_crit
B_f = 0
distant collisions
grazing collisions
close collisions
b_gr
l = l^n_crit
B_f = B_n
deep-inelastic
fusion-fission
incomplete fusion
multi-fragmentation
fusion-evaporation
V_B E_1
E_2
E



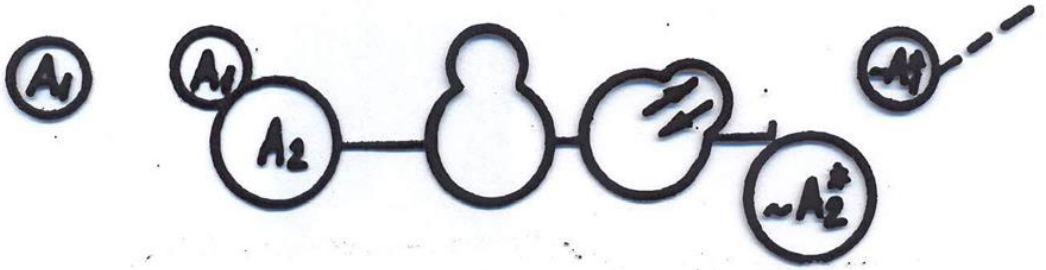
Direct reactions

$b \geq b_{gr}$



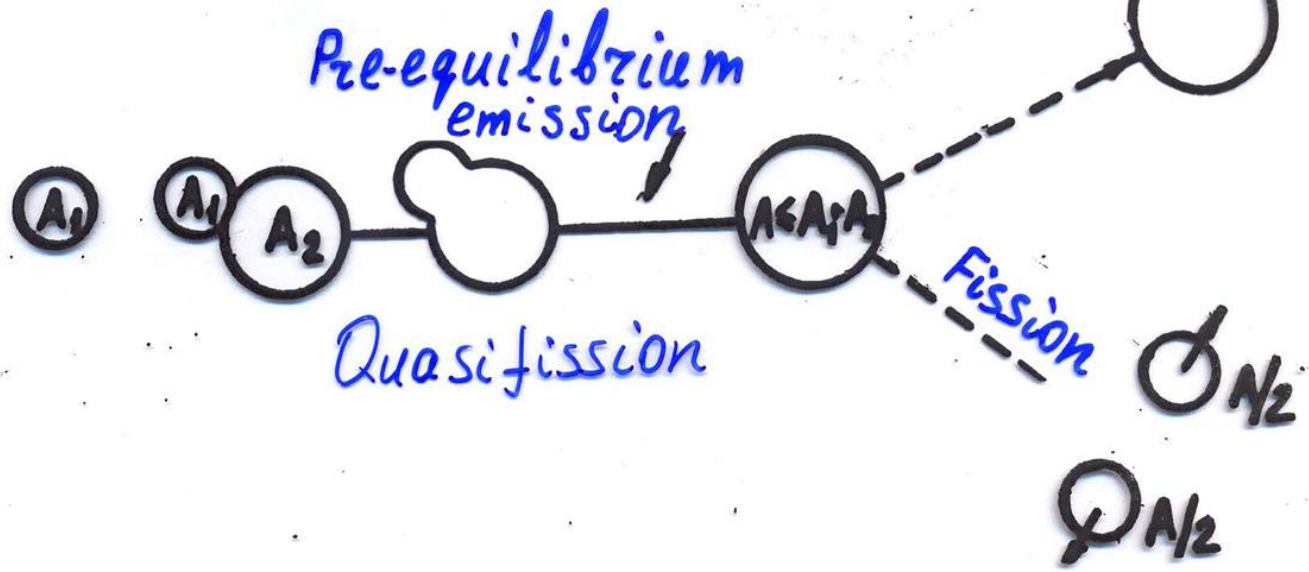
DIC

$b_{cr} \leq b \leq b_{gr}$

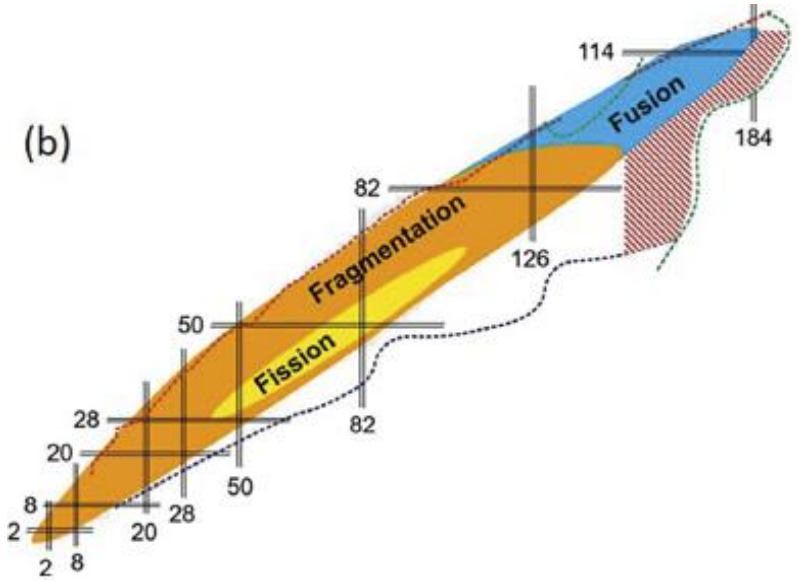


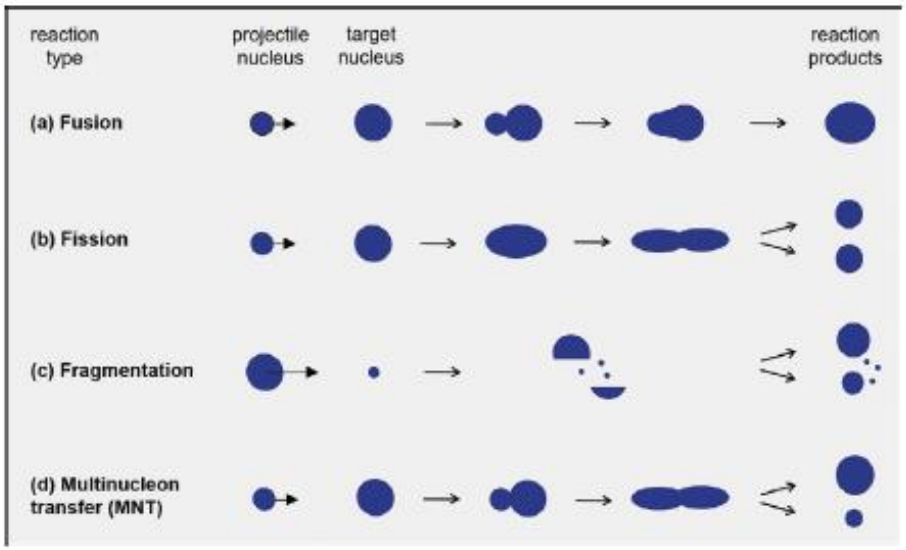
Fusion

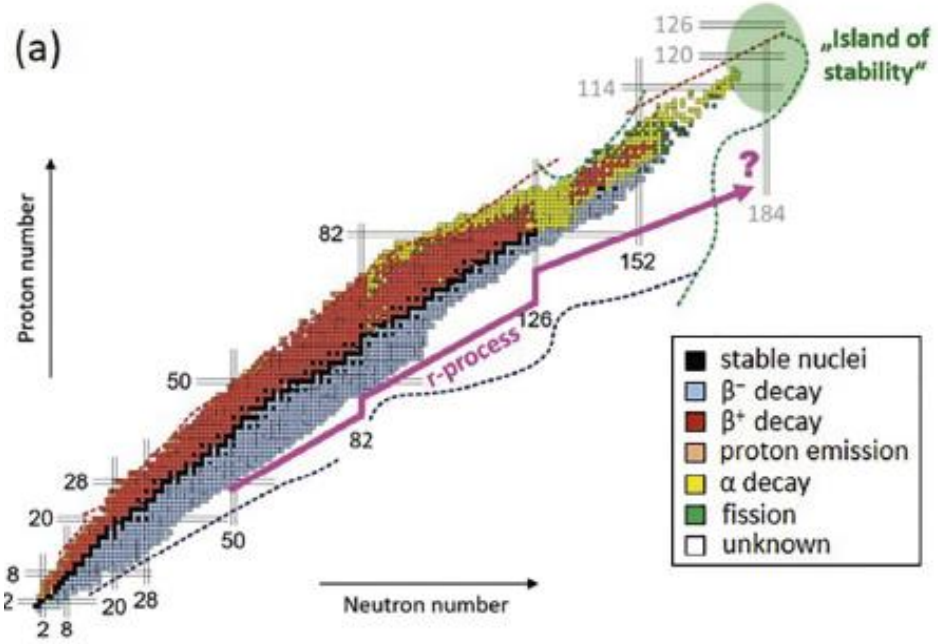
$b \leq b_{cr}$

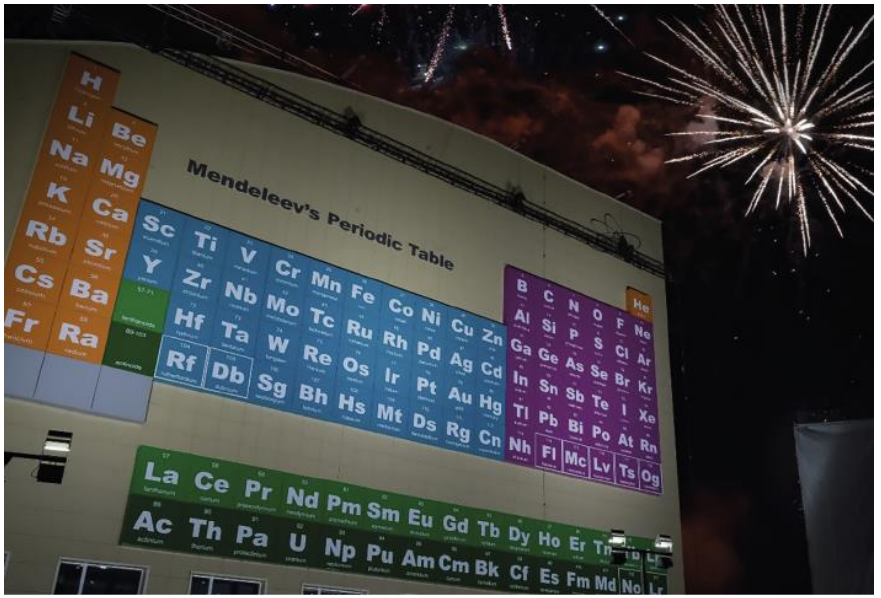


$Z_1 Z_2$


(b)
Fusion
Fragmentation
Fission
114
184
126
82
50
28
20
8
2
2
8
20
28
50
82


reaction type
projectile nucleus
target nucleus
reaction products
(a) Fusion
(b) Fission
(c) Fragmentation
(d) Multinucleon transfer (MNT)


(a)
„Island of stability"
126
120
114
184
?
82
152
126
r-process
50
82
28
20
50
8
2
20
28
2
8
Proton number
Neutron number
stable nuclei
β⁻ decay
β⁺ decay
proton emission
α decay
fission
unknown


Mendeleev's Periodic Table

Fusion reactions leading to formation of superheavy elements

Forces acting between the nuclei

- Conservative forces:

  Coulomb, nuclear and centrifugal forces create an entrance potential barrier. The projectile ion must have sufficient kinetic energy to overcome the barrier.

- Dissipative forces:

  Induce an irreversible transition of the kinetic energy of a collision into internal excitations of the nuclear.

Nucleus-nucleus interaction potential

$$V(R, Z_i, A_i, J) = V_{Nucl} + V_{Coul} + V_{rot}$$

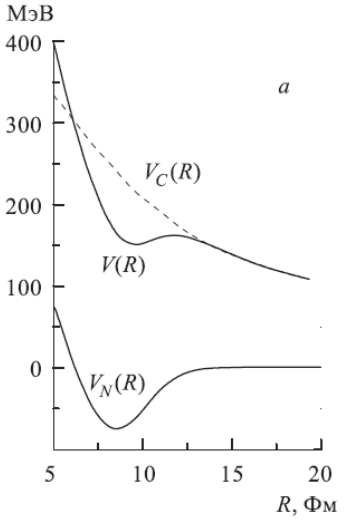
Coulomb energy: $V_{Coul}(R) = \dfrac{z_1 z_2 e^2}{R}$

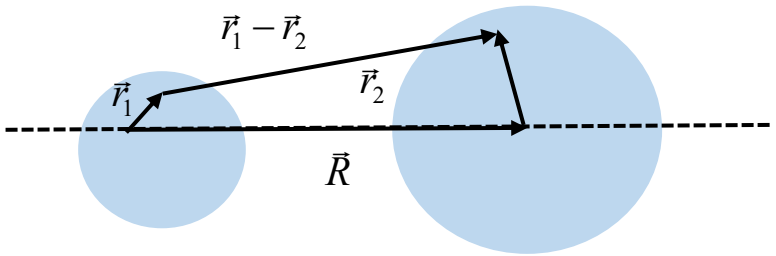
Rotational energy: $V_{rot} = \dfrac{l(l+1)}{2\mu R^2}$

Double folding: $V_{nucl}(R) = \int \rho_1(\vec{r}_1)\rho_2(\vec{R} - \vec{r}_2)F(\vec{r}_1 - \vec{r}_2)d\vec{r}_1 d\vec{r}_2$



Рис. 2.





Nucleus-nucleus potential:

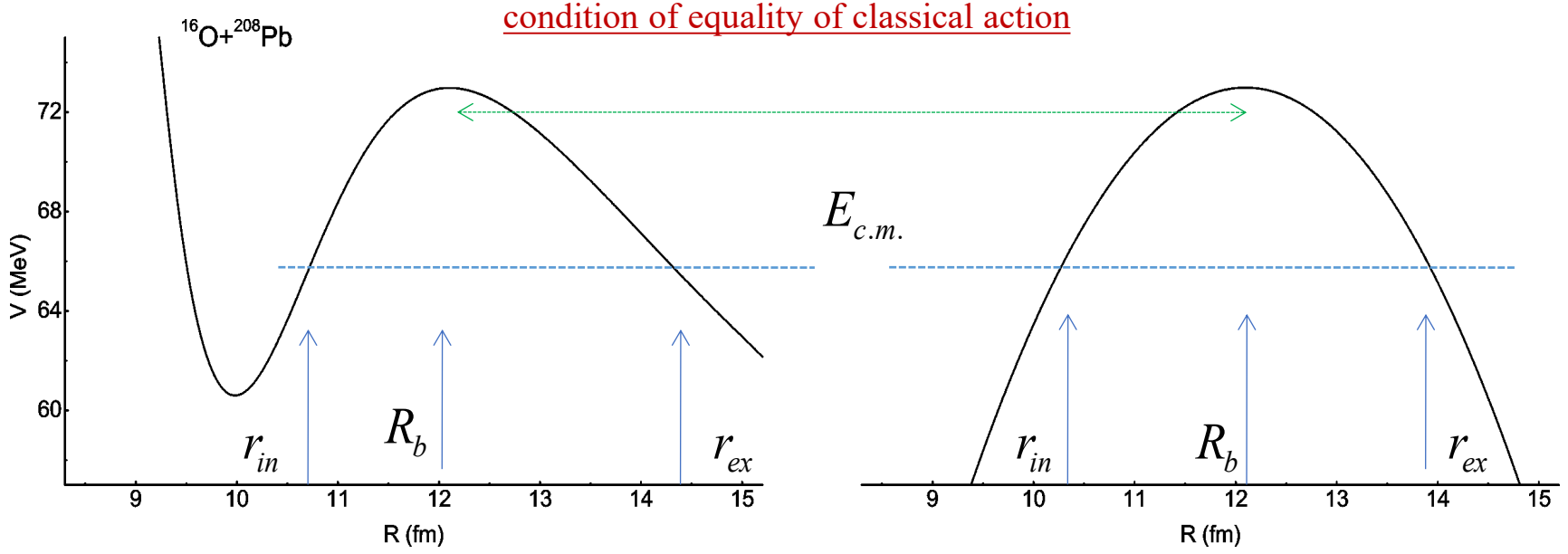
- ✓ **density - dependent effective nucleon-nucleon interaction**
- ✓ **Woods-Saxon parameterization for nucleus density**

Potential approximation:

realistic nucleus-nucleus potential → inverted oscillator

- ✓ **The real interaction between nuclei can be approximated by the inverted oscillator.**

The frequency of oscillator is found from the condition of equality of classical action



Dinuclear system concept

- At the *capture stage*, after complete dissipation of the kinetic energy of the collision, a *dinuclear system is formed*.

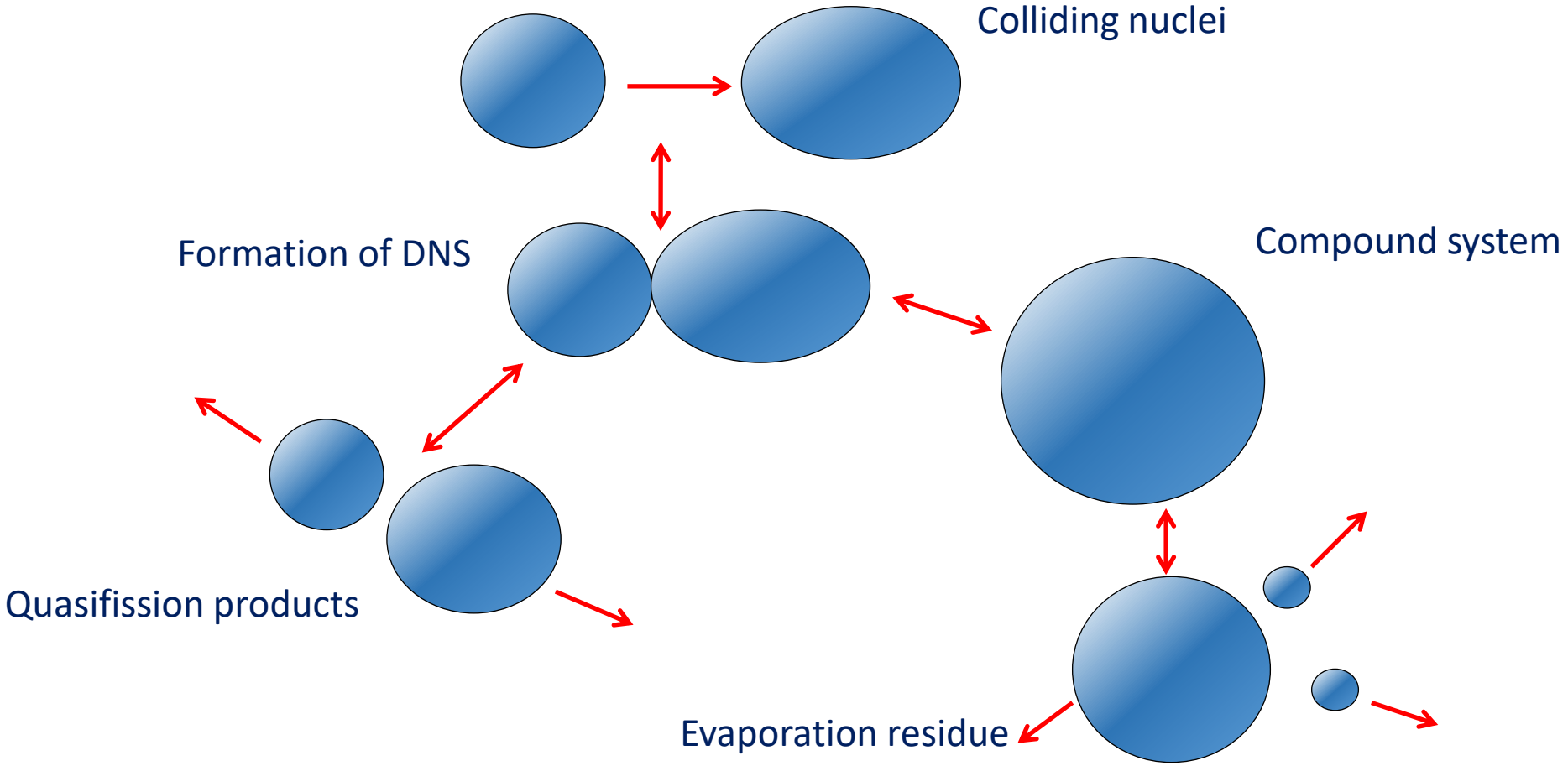
- The process of complete fusion is realized as a result of the *evolution of the DNS* in the direction of increasing its charge and mass asymmetries: the nucleons of the lighter nucleus are transferred to the heavier nucleus.

- In reactions between massive nuclei, evolution in the opposite direction is energetically favorable: nucleons from a heavier nucleus are transferred to a lighter nucleus, which leads to *symmetrization of the system*.

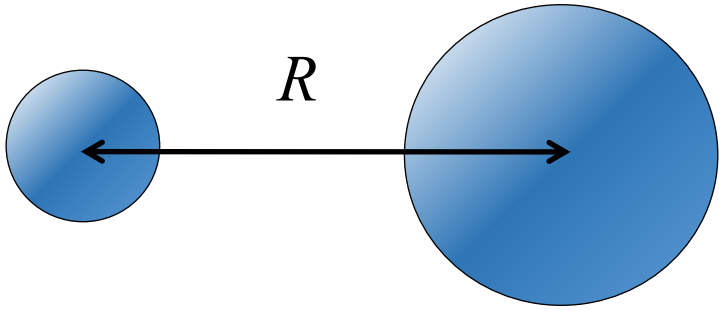
- In symmetric DNS, the Coulomb repulsion between the nuclei reaches the maximum value, and the system splits into two fragments that are close in mass — a *quasifission* occurs.

- Due to the statistical nature of evolution in massive DNS, *competition arises between channels of complete fusion and quasifission*, which leads to a decrease in the cross section for the formation of a compound nucleus compared to the capture cross section.

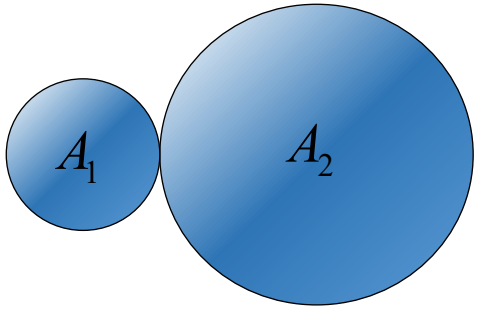
The mechanism of the reaction



Collective coordinates



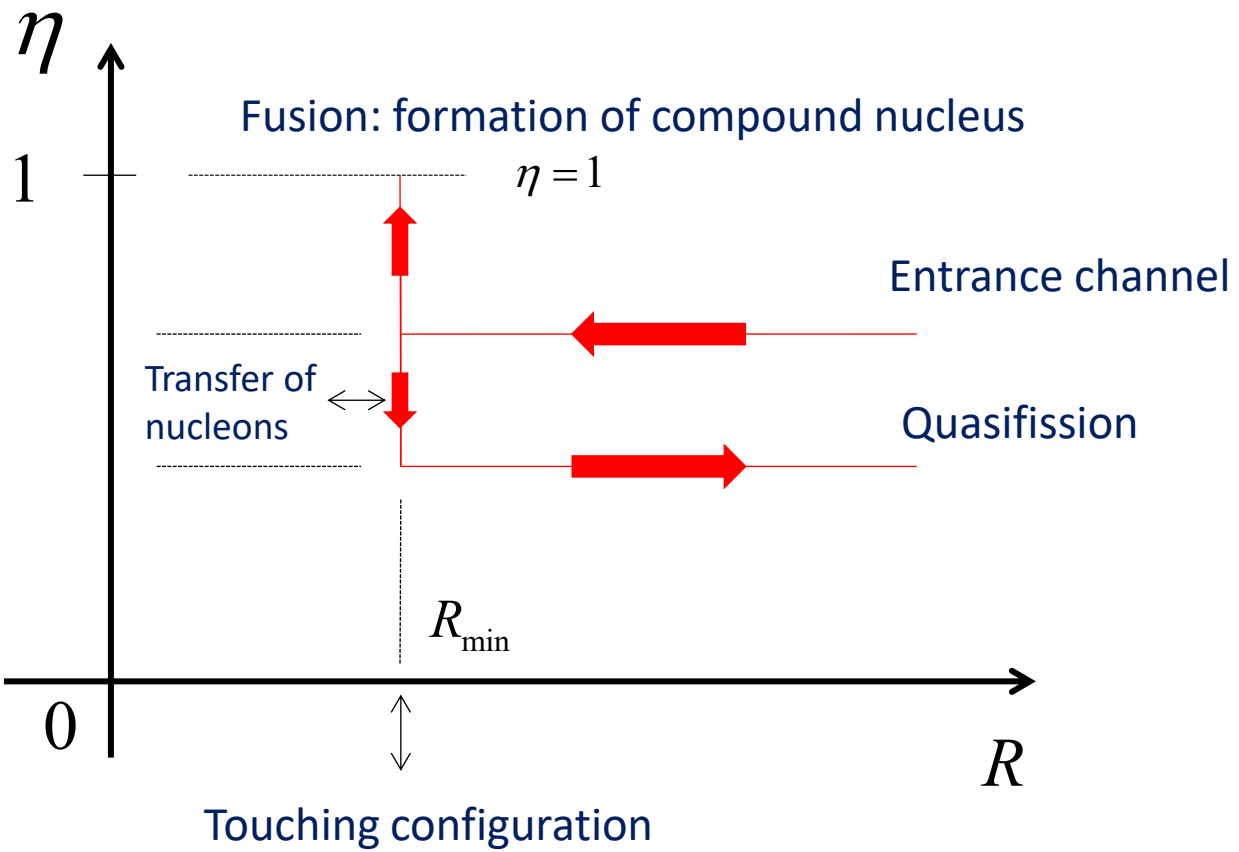
R --- the relative distance between the colliding nuclei



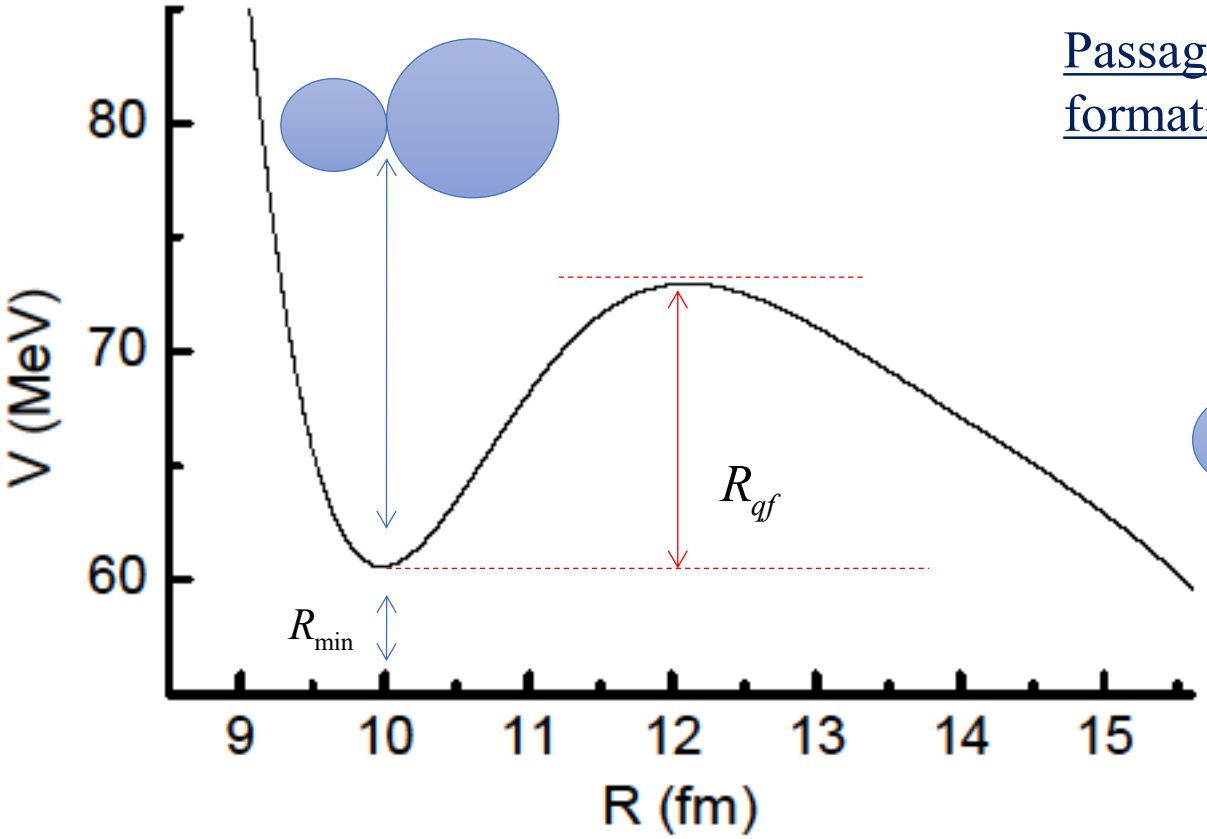
$$\eta = \frac{A_1 - A_2}{A_1 + A_2}$$ --- Mass asymmetry coordinate

$$\eta = 0 \quad \text{for} \quad A_1 = A_2, \quad \eta = \pm 1 \quad \text{for} \quad A_1 \text{ or } A_2 = 0$$

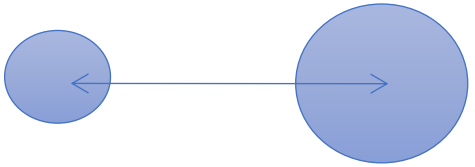
Schematic picture of the reaction mechanism



First stage: Capture

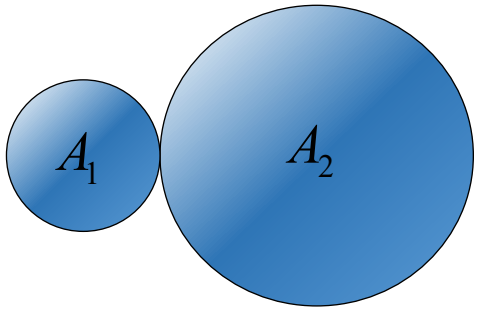


Passage through the Coulomb barrier and the formation of a DNS



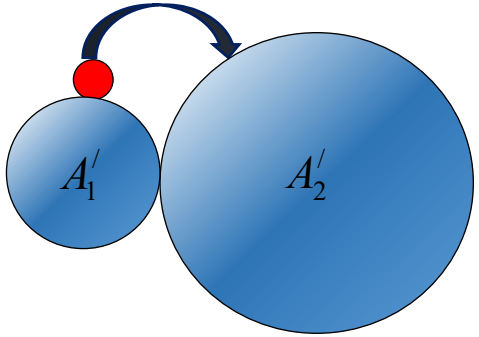
$$R_{min} \approx R_1 + R_2 + 2 \text{ fm}$$

Second stage: Transfer of nucleons



$$\eta = \frac{A_1 - A_2}{A_1 + A_2}$$

The energy of the system $U(\eta)$



$$\eta' = \frac{A_1' - A_2'}{A_1' + A_2'}$$

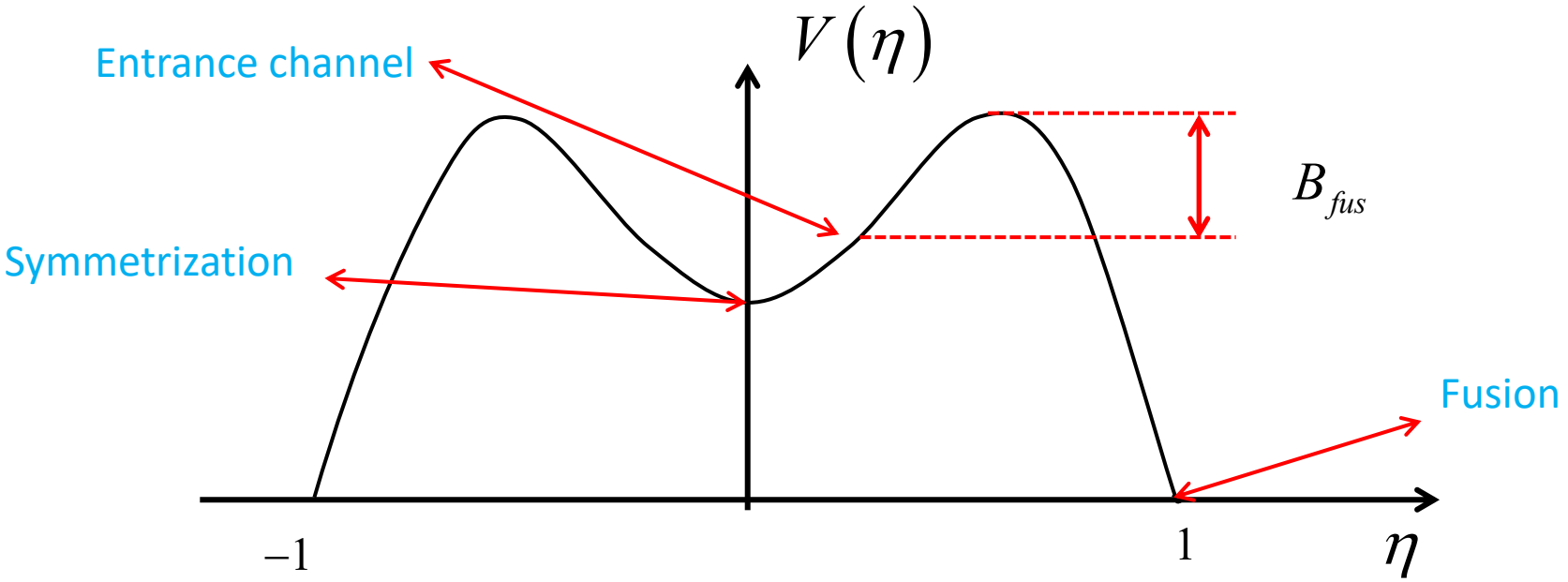
The energy of the system $U(\eta')$

$$U(\eta) = \varepsilon(A_1, Z_1) + \varepsilon(A_2, Z_2) + V_C(A_1, Z_1; A_2, Z_2)$$

$$U(\eta') = \varepsilon(A_1', Z_1') + \varepsilon(A_2', Z_2') + V_C(A_1', Z_1'; A_2', Z_2')$$

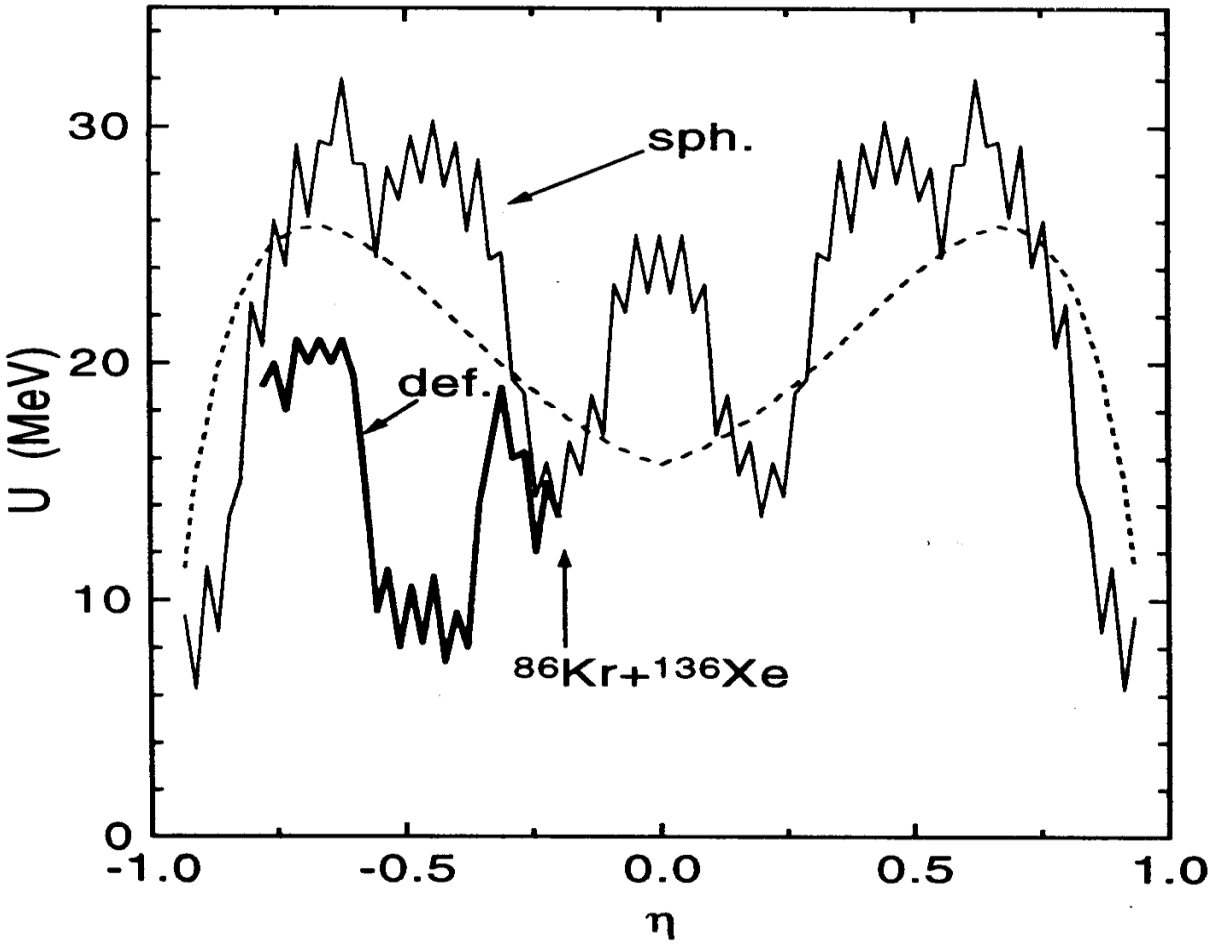
Fusion in mass asymmetry coordinate

The DNS evolves along the mass asymmetry coordinate towards fusion and along the relative distance along the quasi-fission direction.



B_{fus} -- internal fusion barrier

Real driving potential



Schematic picture

Spherical nuclei

Deformed nuclei

$^{86}Kr+^{136}Xe$ – entrance channel

Fusion probability

- **Kramers-Type Expression: The Statistical Estimate**

$$P_{CN} = \frac{\Lambda_{\eta\,\mathrm{fus}}}{\Lambda_{\eta\,\mathrm{fus}} + \Lambda_{qf}}$$

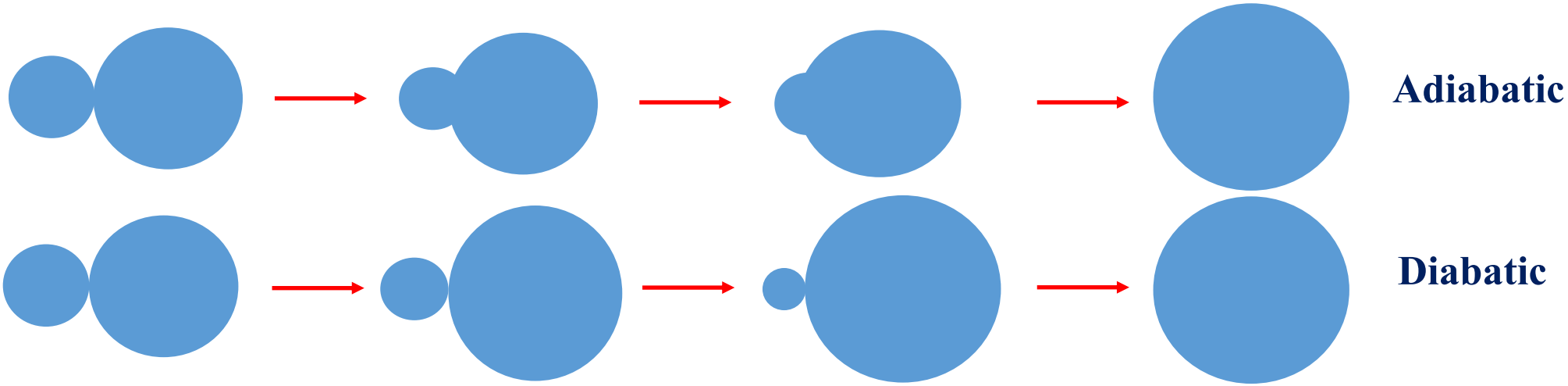
- ✓ Treats the entire dinuclear system as a ***single particle*** in a potential energy landscape with a well and a barrier.
- ✓ Uses the properties of that landscape: the ***barrier height***, the ***temperature***, and the ***curvatures*** of the well and the barrier.
- ✓ Crucially includes a ***friction coefficient*** that represents how strongly the collective motion is coupled to and slowed by the internal degrees of freedom.

- **Master Equation: Detailed description of the process**

$$\frac{dP_i(t)}{dt} = \sum \left[\lambda_{\{j\to i\}} P_j(t) - \lambda_{\{i\to j\}} P_i(t)\right]$$

- ✓ A set of coupled differential equations that describe the ***time evolution of probabilities*** for the system to be in different states.
- ✓ Discretizes the system's configuration into ***states*** (e.g., defined by mass asymmetry).
- ✓ Defines ***transition probabilities*** between these states. These probabilities are based on the potential energy difference between states and the nuclear temperature. The probability P_i of being in state i changes due to probability flowing ***into*** it from other states j and ***out*** *of* it to other states j.
- ✓ The 'fusion probability' is the ***final probability*** that ends up in the state corresponding to a fully equilibrated compound nucleus.

Diabatic (DNS) vs Adiabatic (fusion in *R*) approaches



Key Difference:

- ***Adiabatic Path:*** As two nuclei approach each other, the single-particle energy levels of the combined system shift. The adiabatic assumption is that the ***nucleons instantaneously adjust to these changing levels***. This leads to the formation of a compound nucleus where ***the energy is shared among all nucleons***. The system follows a single, smooth potential energy landscape.
- ***Diabatic Path:*** At certain energies, the ***nucleons do not have time to rearrange***. The single-particle states of the two original nuclei largely maintain their identity. This is ***due to the strong interaction and the Pauli exclusion principle*** governing nucleon.

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W. U. Schröder and J. R. Huizenga, in *Treatise on Heavy-Ion Science*, edited by D. A. Bromley (Plenum, New York, 1984), Vol. 2, p. 115.

В.В. Волков, Ядерные реакции глубоконеупругих передач (М.: Энергоиздат, 1982)

P.Fröbrich, R.Liepperhiede, Theory of nuclear reactions (Clarendon Press, Oxford, 1996)

G. G. Adamian, N. V. Antonenko, and W. Scheid, in *Lecture Notes in Physics, Clusters in Nuclei*, Vol. 2, edited by C. Beck (Springer, Berlin, 2012).

DOUBLE FOLDING POTENTIAL

$$U_N(R) = \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{R}-\mathbf{r}_2)F(\mathbf{r}_1-\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2$$

The method allows us to take into account the finite size of interacting nuclei by their densities. However, there is a question of the choice of the nucleon-nucleon interaction. The microscopic theories were developed together with the phenomenological approaches.

With the density-independent nucleon-nucleon interaction U_N is deep and does not take into account the exchange effects connected with antisymmetrization. These effects are separately treated excluding the forbidden states of the deep potential well from consideration.

The density dependence of the nucleon-nucleon interaction allows one take into account the exchange and saturation effects phenomenologically. Among that kind of interactions, the Skyrme-type interactions are often used due to their simple structure. Without momentum dependence the expression for the Skyrme interaction reduces to the expression for local interaction

$$F(\mathbf{r}_1-\mathbf{r}_2)=C_0\left(F_{in}\frac{\rho(\mathbf{r}_1)}{\rho_{00}}+F_{ex}\left(1-\frac{\rho(\mathbf{r}_1)}{\rho_{00}}\right)\right)\delta(\mathbf{r}_1-\mathbf{r}_2)$$

$$\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$$

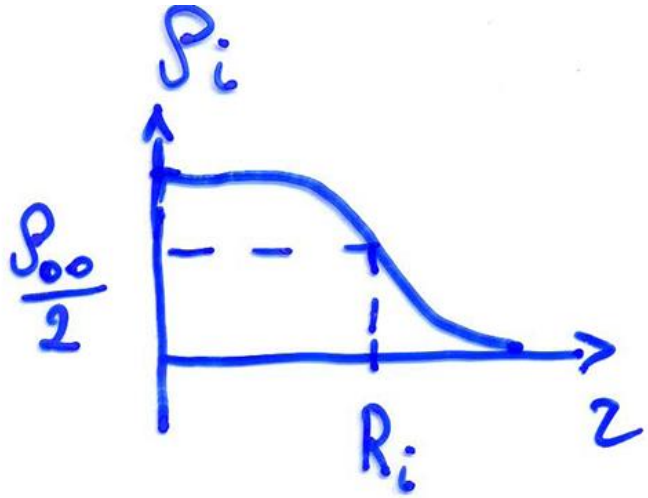
$$U_N(R)=C_0\{\frac{F_{in}-F_{ex}}{\rho_{00}}\left(\int\rho_1^2(\mathbf{r})\rho_2(\mathbf{R}-\mathbf{r})d\mathbf{r}+\int\rho_1(\mathbf{r})\rho_2^2(\mathbf{R}-\mathbf{r})d\mathbf{r}\right)$$

$$+F_{ex}\int\rho_1(\mathbf{r})\rho_2(\mathbf{R}-\mathbf{r})d\mathbf{r}\}$$

$C_0 = 300$ MeV fm^3, $\rho_{00} = 0.17$ fm^{-3}

$F_{in} \approx 0.1$ $\quad$ $F_{ex} \approx -2.6$

Two-parameter Woods-Saxon function



$$\rho_i(\mathbf{r}) = \frac{\rho_{00}}{1+\exp[(r - R_i(\theta_i,\varphi_i))/a_i]}$$

or symmetrized Woods-Saxon function

$$\rho_i(\mathbf{r}) = \frac{\rho_{00}\sinh[R_i(\theta_i,\varphi_i)/a_i]}{\cosh[R_i(\theta_i,\varphi_i)/a_i]+\cosh[r/a_i]}$$

Models of complete fusion with adiabatic and diabatic potentials

Two main collective coordinates are used for the description of the fusion process:

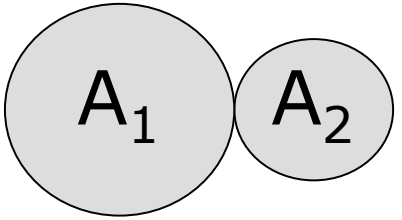
1. Relative internuclear distance R

2. Mass asymmetry coordinate η for transfer

Idea of Volkov (Dubna) to describe fusion reactions with the dinuclear system concept:

Fusion is assumed as a transfer of nucleons (or clusters) from the lighter nucleus to the heavier one in a dinuclear configuration.

This process is describable with the mass asymmetry coordinate $\eta=(A_1-A_2)/(A_1+A_2)$.



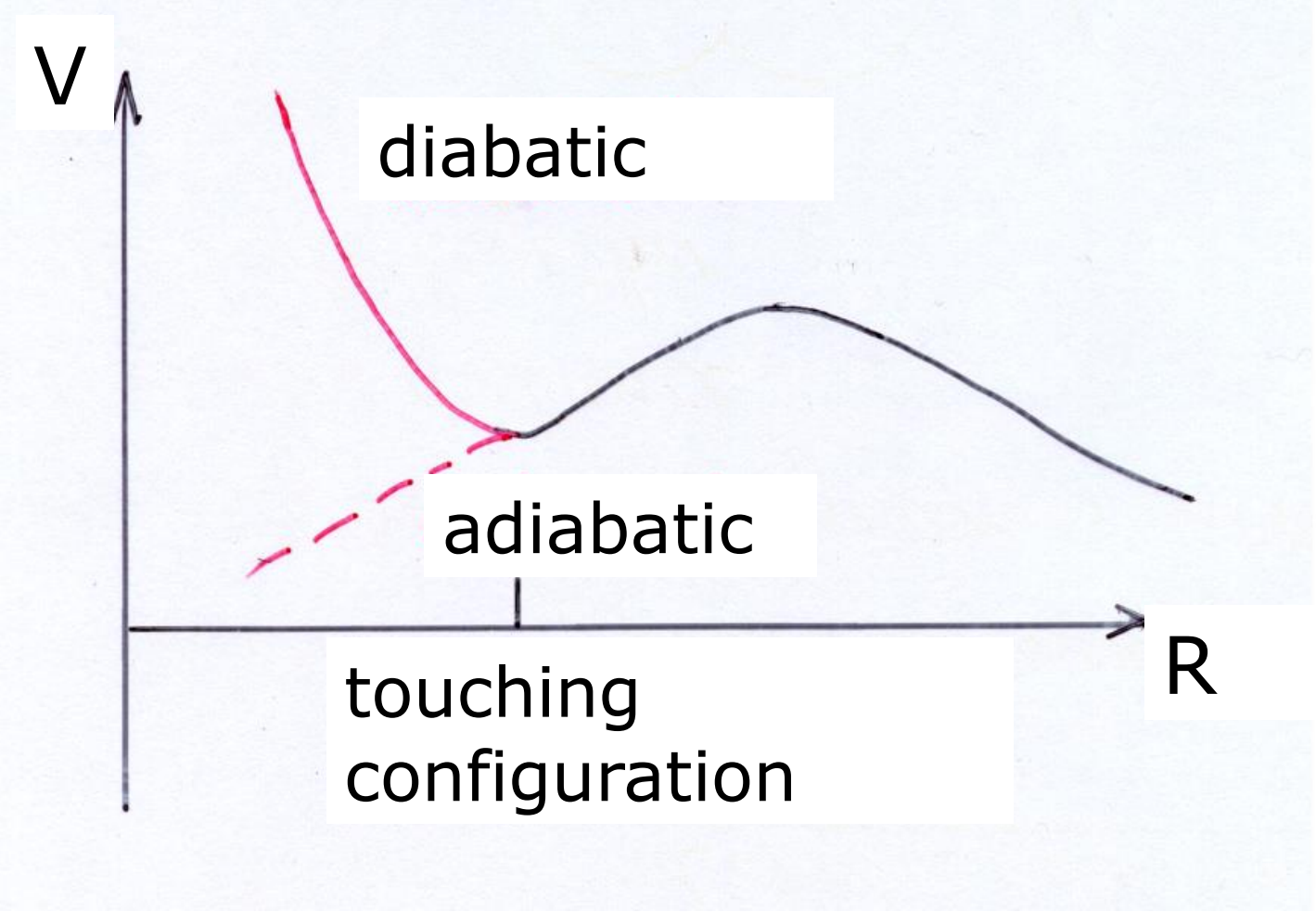
If A_1 or A_2 get small, then $|\eta| \rightarrow 1$ and the system fuses.

The dinuclear system model uses two main degrees of freedom to describe the fusion and quasifission processes:

Relative motion of nuclei, capture of target and projectile into dinuclear system,
decay of the dinuclear system: quasifission

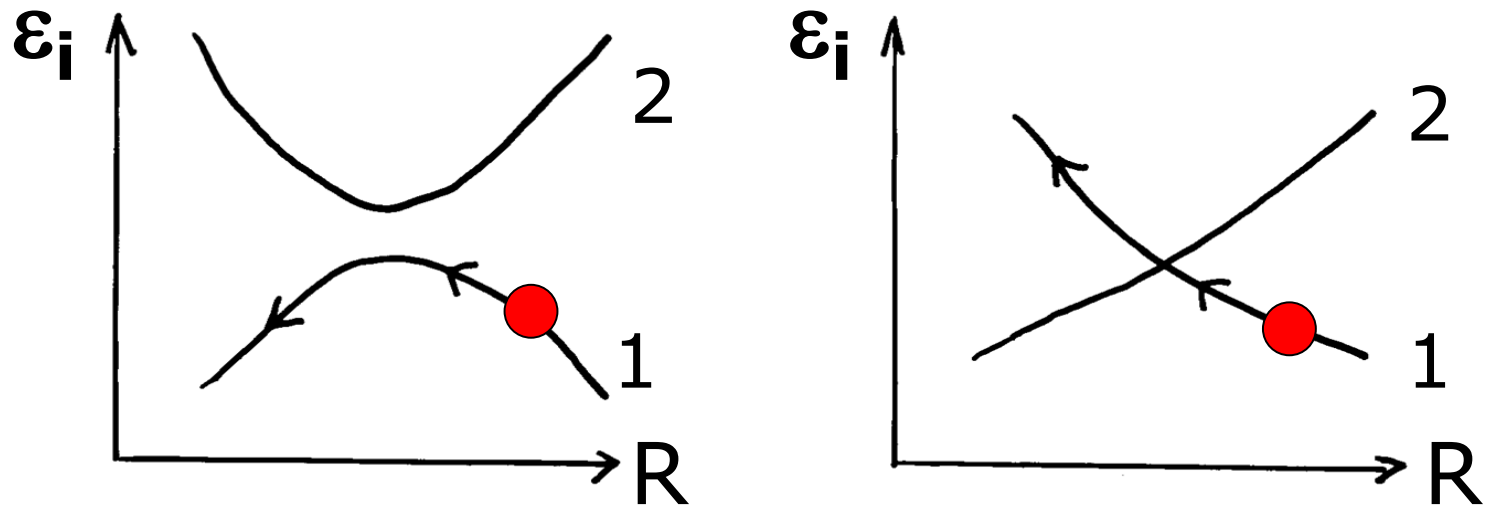
Transfer of nucleons between nuclei, change of mass and charge asymmetries leading to fusion and quasifission

Description of fusion dynamics depends strongly whether adiabatic or diabatic potential energy surfaces are assumed.



Diabatic potentials are repulsive at smaller internuclear distances $R<R_t$.

Explanation with two-center shell model:

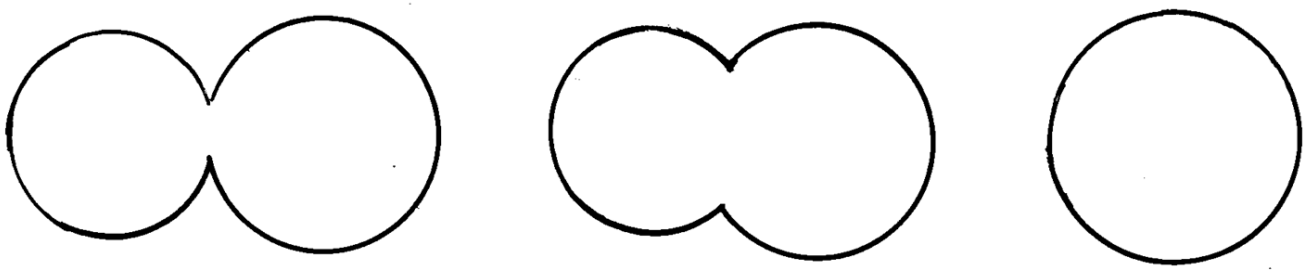


adiabatic model diabatic model

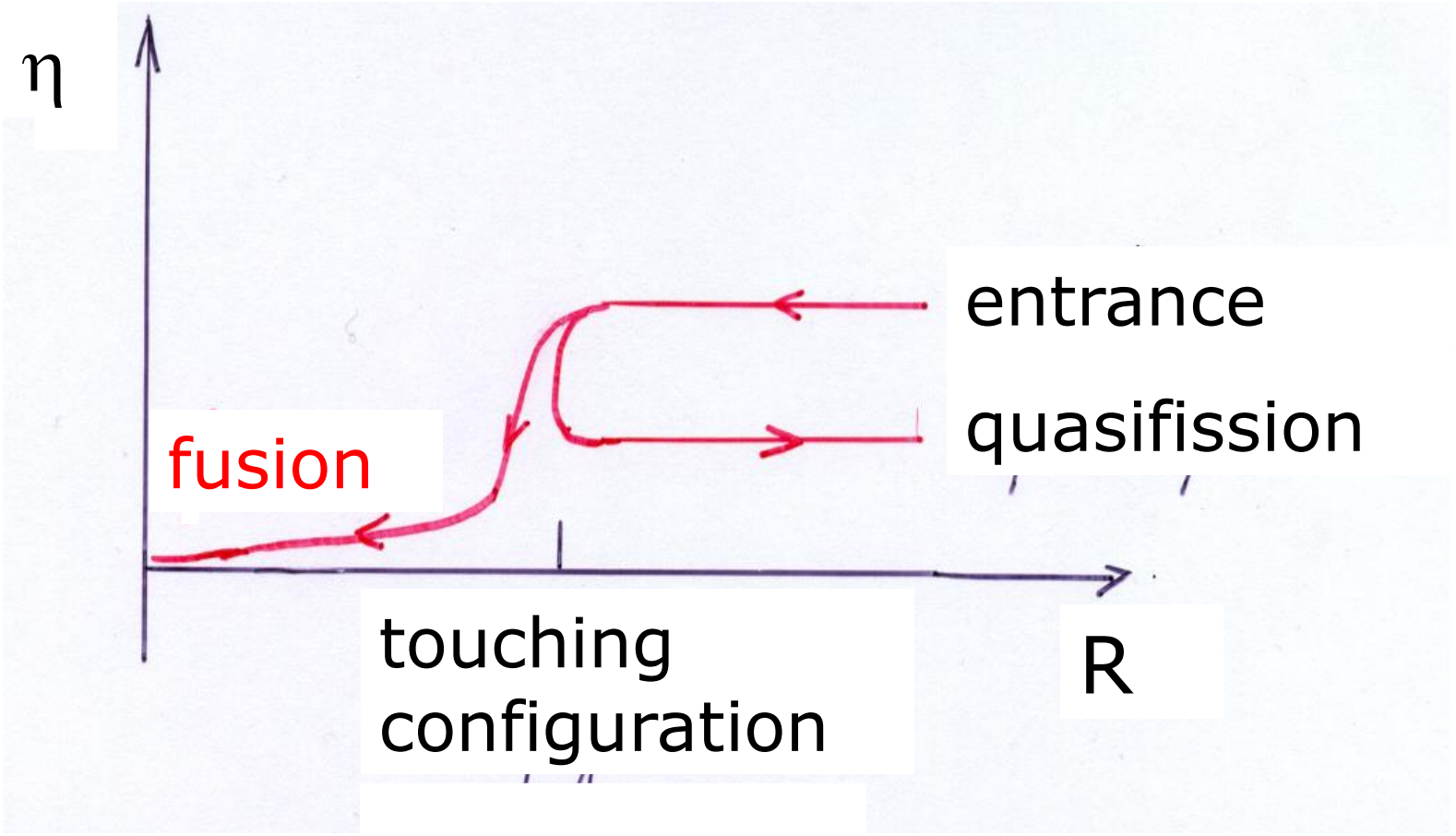
Velocity between nuclei leads to diabatic occupation of single-particle levels, Pauli principle between nuclei

Models using adiabatic potentials

Minimization of potential energy, essentially adiabatic dynamics in the internuclear distance, nuclei melt together.

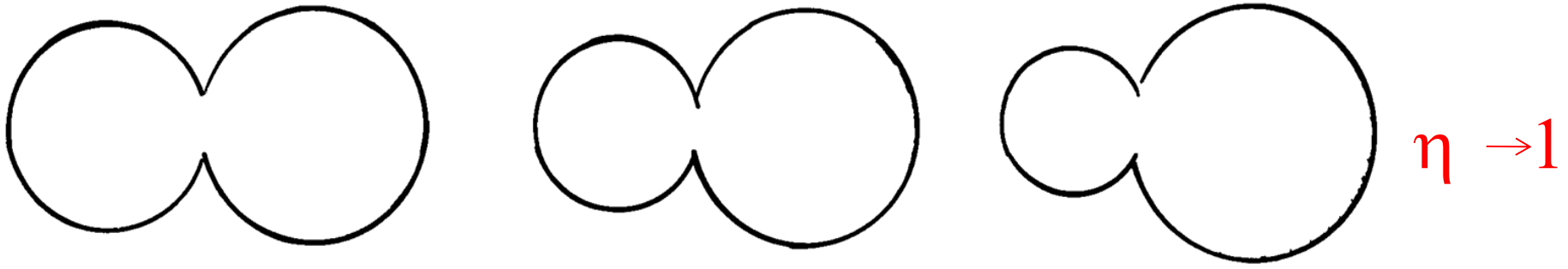


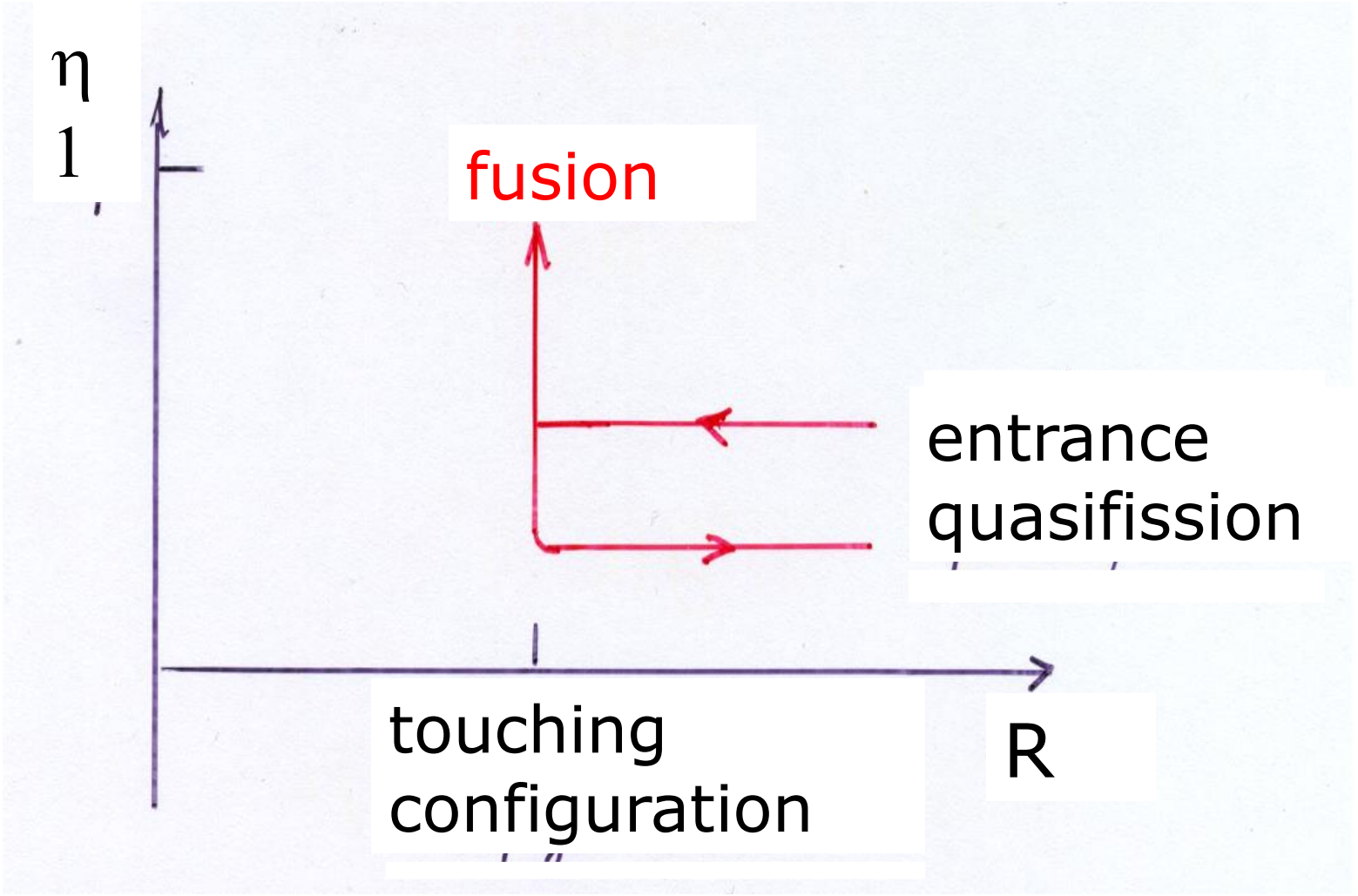
Large probabilities of fusion for producing nuclei with similar projectile and target nuclei.


η
entrance
quasifission
fusion
touching
configuration
R

b) Dinuclear system (DNS) concept

Fusion by transfer of nucleons between the nuclei (idea of V. Volkov, also von Oertzen), mainly dynamics in mass asymmetry degree of freedom, use of diabatic potentials, e.g. calculated with the diabatic two-center shell model.




η
1
fusion
entrance
quasifission
touching
configuration
R

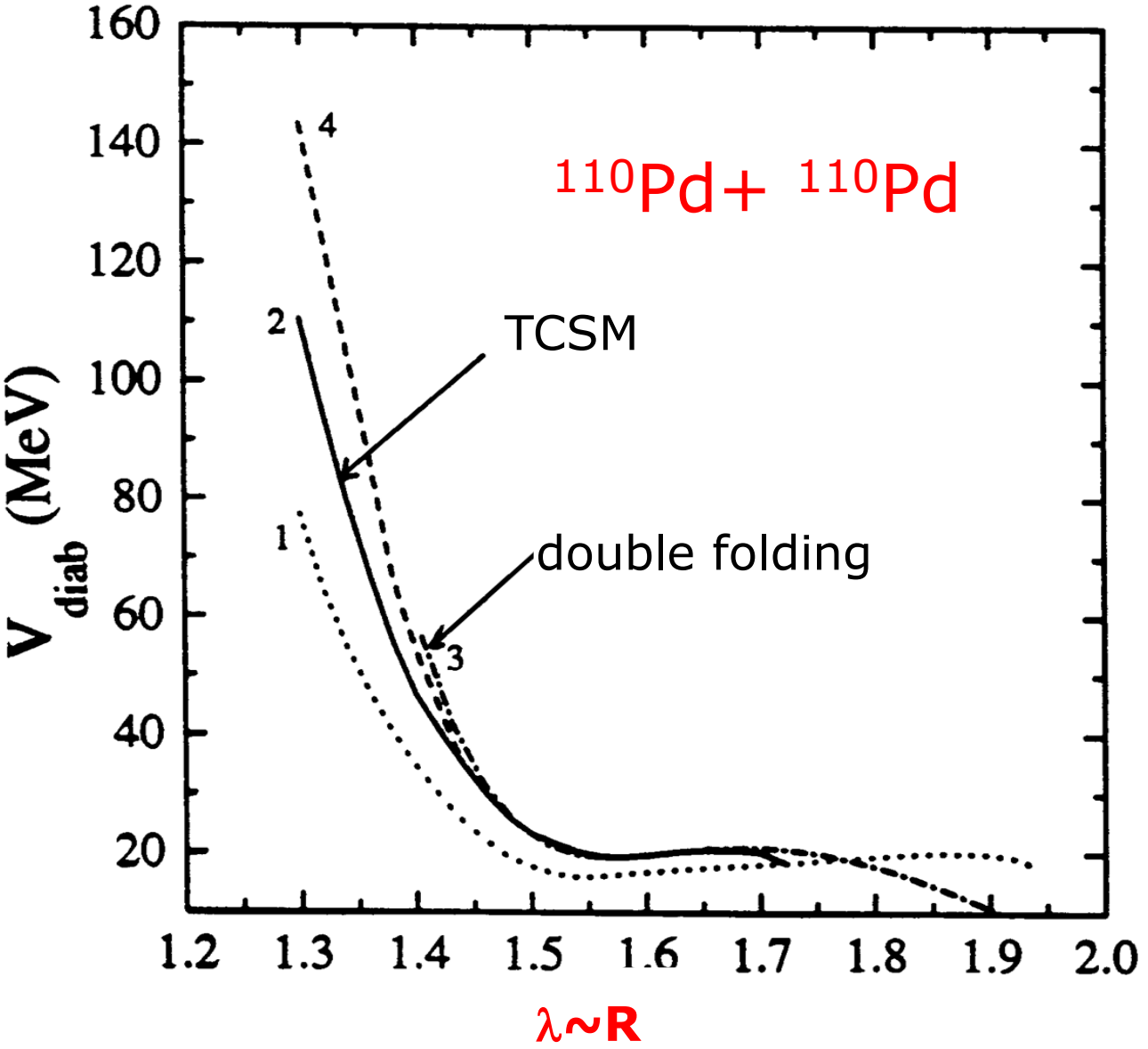
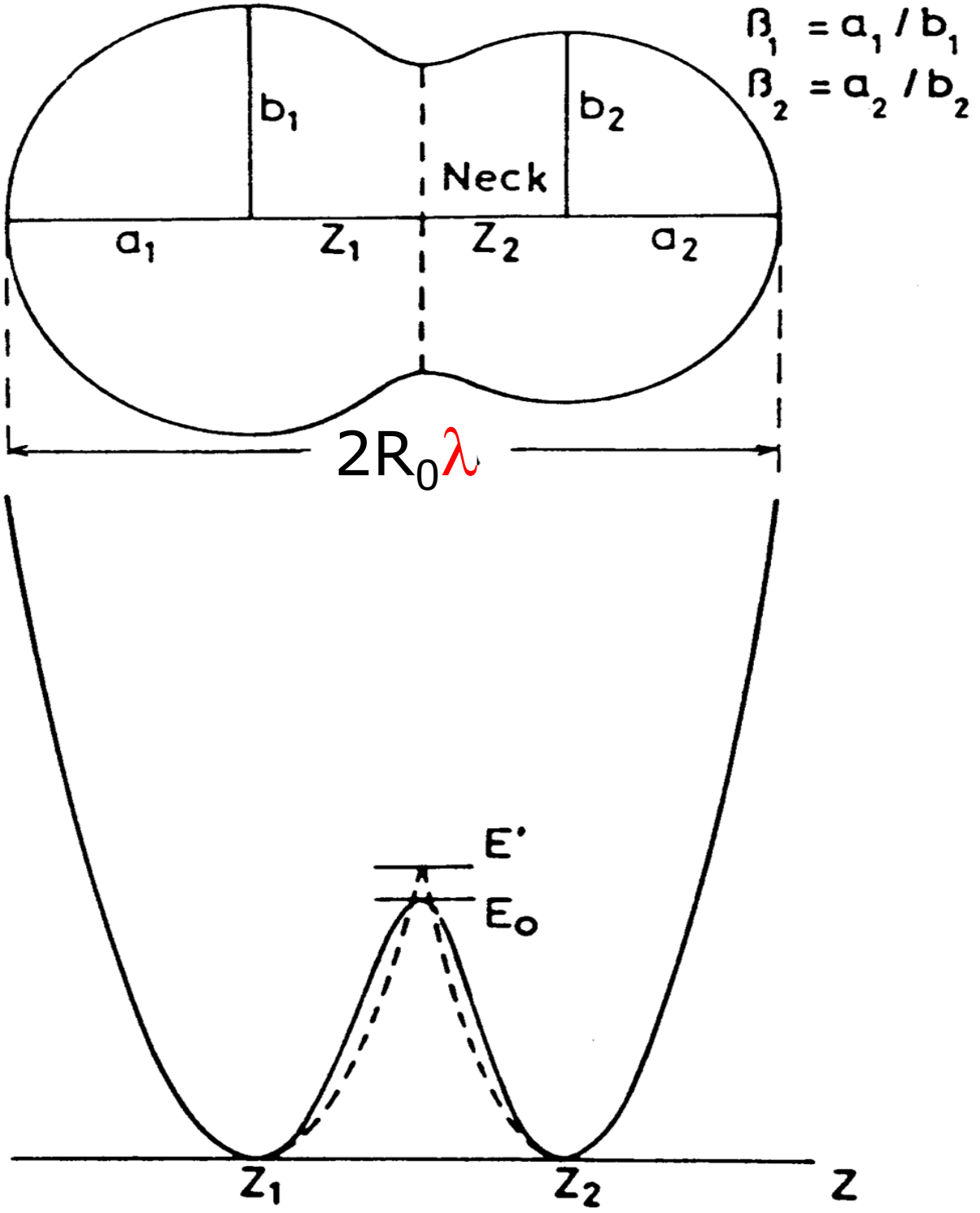


Fig.2. Diabatic potentials for the systems:
(1) ^{100}Mo + ^{100}Mo (TCSM)
(2) ^{110}Pd + ^{110}Pd (TCSM)
(3) ^{110}Pd + ^{110}Pd (double folding)
(4) Neck parameter ε is diminished with decreasing λ


β1 = a1 / b1
β2 = a2 / b2
b1
b2
Neck
a1
Z1
Z2
a2
2R0λ
E'
E0
Z1
Z2
Z

Calculation of diabatic potential:

$$U_{diab}(\lambda) = U_{adiab}(\lambda) + \sum_{\alpha} \epsilon_{\alpha}^{diab}(\lambda) n_{\alpha}^{diab}(\lambda) - \sum_{\alpha} \epsilon_{\alpha}^{adiab}(\lambda) n_{\alpha}^{adiab}(\lambda)$$

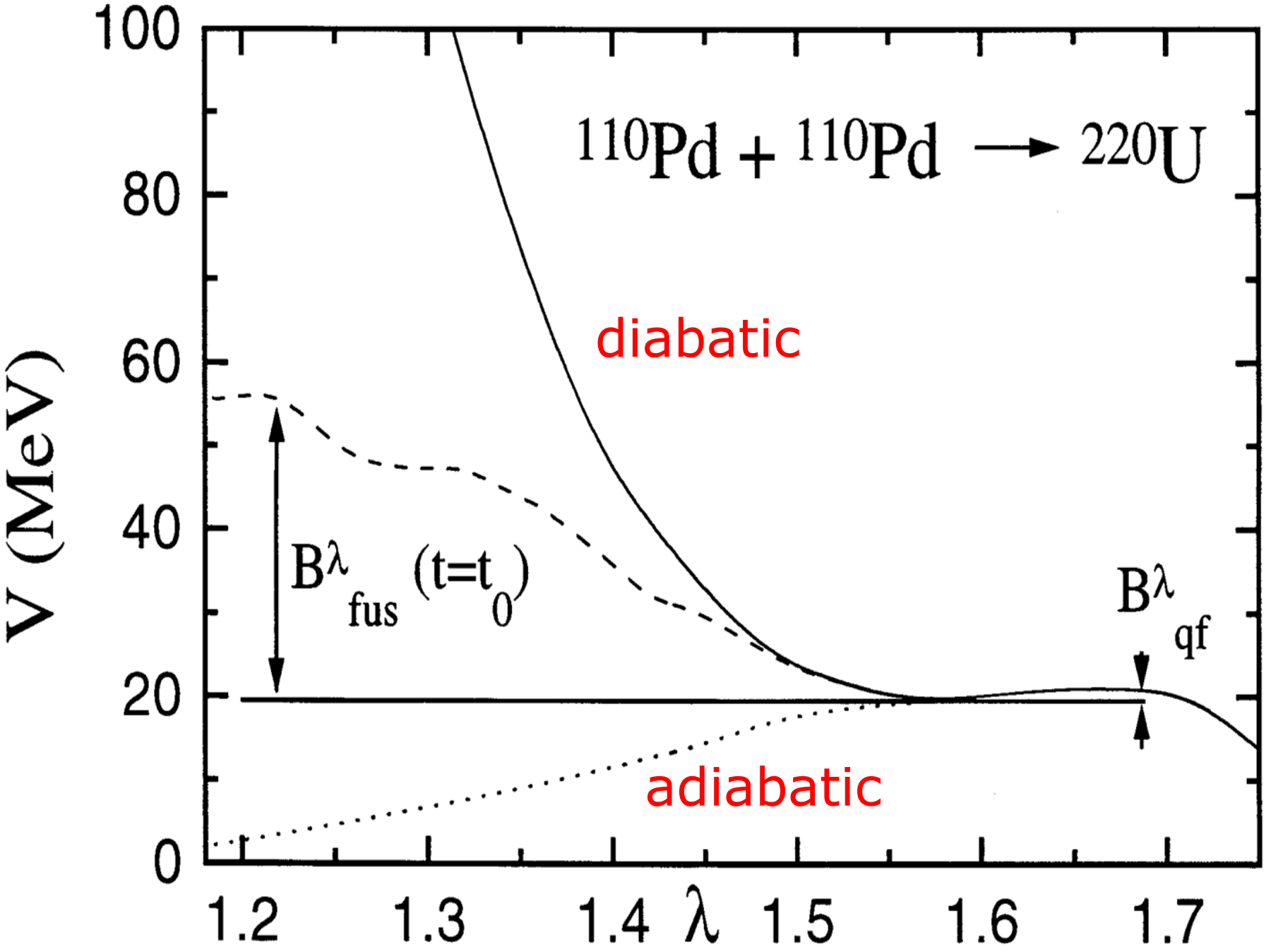
$\epsilon_{\alpha}^{diab}(\lambda), \epsilon_{\alpha}^{adiab}(\lambda)$ = single particle energies

$n_{\alpha}^{diab}(\lambda), n_{\alpha}^{adiab}(\lambda)$ = occupation numbers

Diabatic occupation numbers depend on time:

$$n_{\alpha}^{diab}(\lambda, t)$$

De-excitation of diab. levels with relaxation time, depending on single particle width.


100
80
60
40
20
0
V (MeV)
1.2
1.3
1.4
λ
1.5
1.6
1.7
$^{110}Pd + ^{110}Pd \rightarrow ^{220}U$
diabatic
adiabatic
$B^{\lambda}_{fus}(t=t_0)$
B^{λ}_{qf}

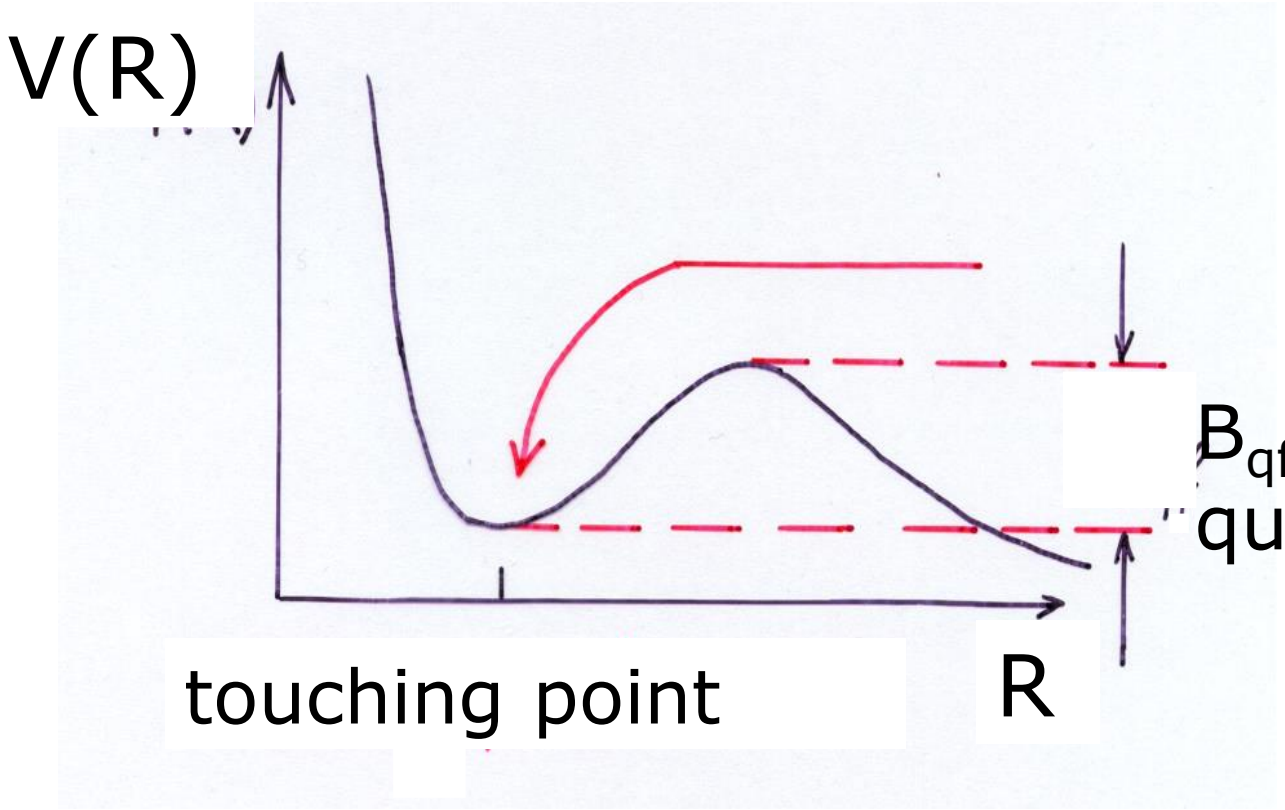
Dynamics of fusion in the dinuclear system model

Evaporation residue cross section for the production of superheavy nuclei:

$$\sigma_{ER}(E_{cm}, J) = \sum_{J=0}^{J_{max}} \sigma_{cap}(E_{cm}, J) P_{CN}(E_{cm}, J) W_{sur}(E_{cm}, J)$$

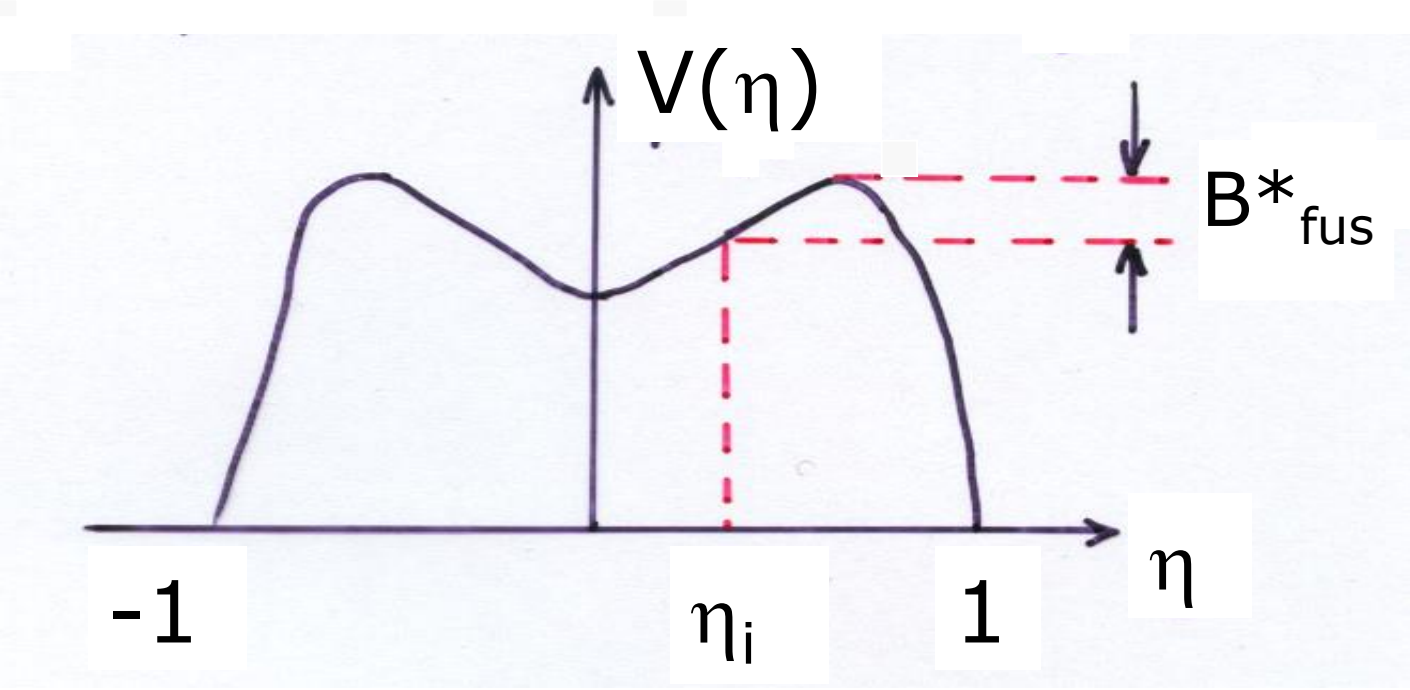
Partial capture cross section σ_{cap}

Dinuclear system is formed at the initial stage of the reaction, kinetic energy is transferred into potential and excitation energy.

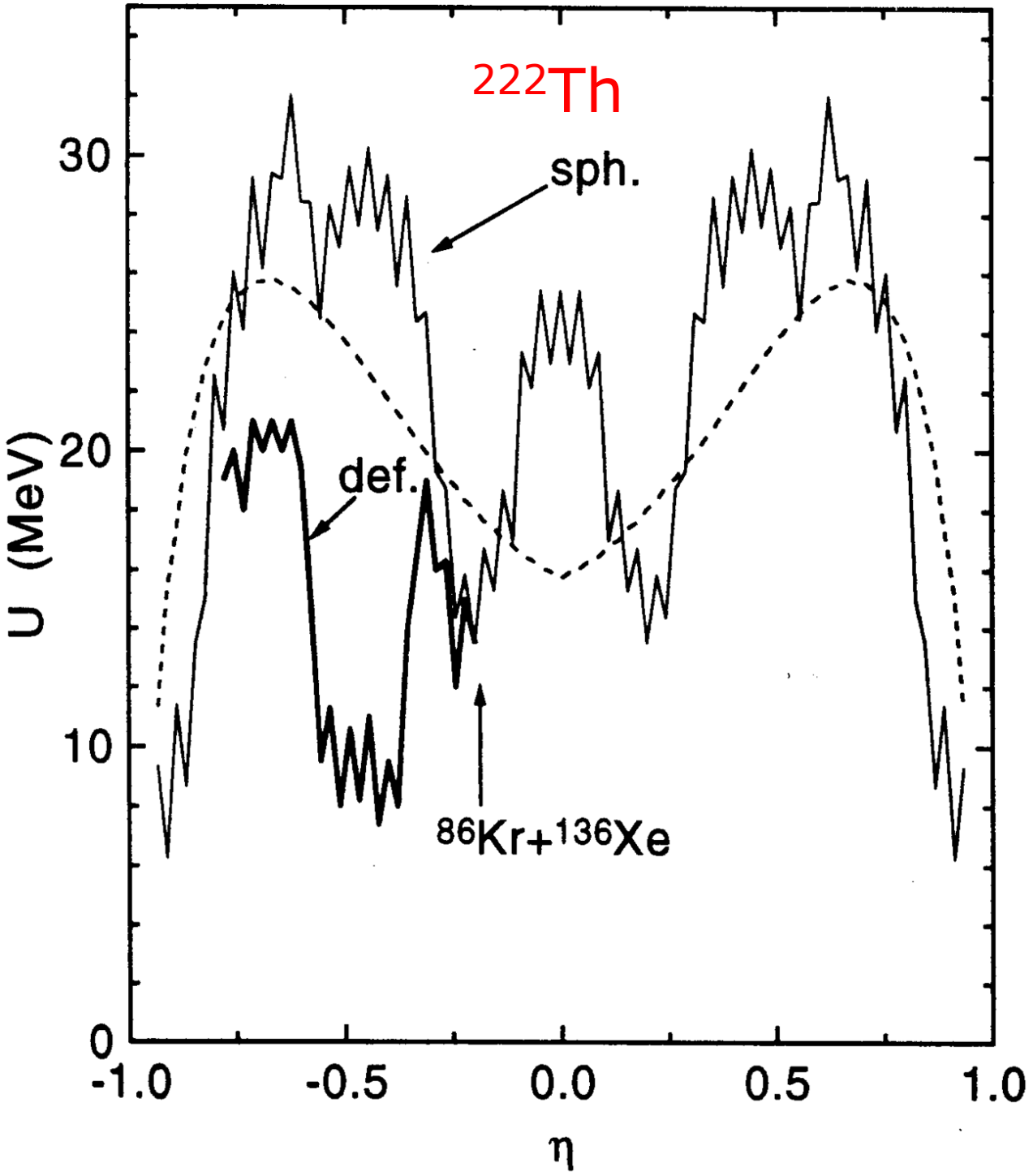


b) Probability for complete fusion P_{CN}

DNS evolves in mass asymmetry coordinate by diffusion processes toward fusion and in the relative coordinate toward the decay of the dinuclear system which is quasifission.



B^*_{fus}=inner fusion barrier


^{222}Th
sph.
def.
^{86}Kr+^{136}Xe
U (MeV)
30
20
10
0
-1.0
-0.5
0.0
0.5
1.0
η

Competition between fusion and quasifission, both processes are treated simultaneously.

Calculation of P_{CN} and mass and charge distributions in η and R:

Fokker-Planck equation, master equations, Kramers approximation

Kramers formula for P_{CN}:

Rate for fusion: $\Lambda_{\eta\ fus}$

Rate for quasifission: $\Lambda_{qf} = \Lambda_R + \Lambda_{\eta\ sym}$, i.e. decay in R and diffusion in η to more symmetric DNS.

$$P_{CN} = \frac{\Lambda_{\eta fus}}{\Lambda_{\eta fus} + \Lambda_{qf}}$$

$$P_{CN} \sim \exp(-(B^*_{fus} - B_{qf})/kT)$$

Cold fusion (Pb-based reactions): $\Lambda_R \gg \Lambda_{\eta sym}$

Hot fusion (^{48}Ca projectiles): $\Lambda_R \ll \Lambda_{\eta sym}$

$$\wedge_k^{\mathrm{Kr}} = \frac{1}{2\pi} \frac{\omega_k \omega_{\bar{k}}}{\omega_R^{B_k} \omega_\eta^{B_k}} \left(\sqrt{\left[\frac{\Gamma}{2\hbar}\right]^2 + (\omega_k^{B_k})^2} - \frac{\Gamma}{2\hbar} \right) \exp\left(-B_k/\theta\right).$$

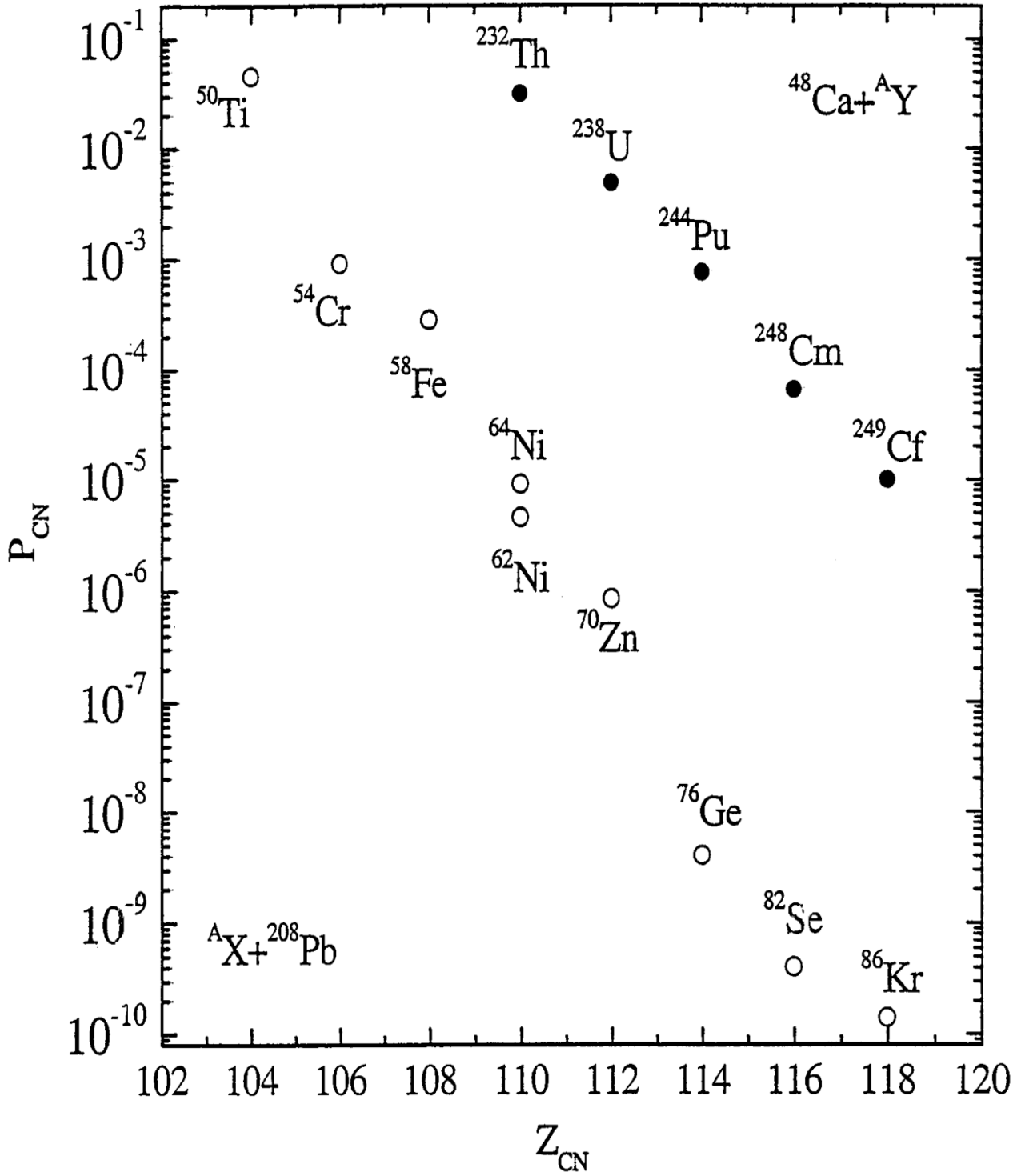
Competition between fusion and quasifission, both processes are treated simultaneously.

Calculation of P_{CN} and mass and charge distributions in η and R:

Fokker-Planck equation, master equations, Kramers approximation.

$$P_{CN} \approx \frac{1.25 \exp\left[-\left(B_{\eta}^{fus} - B_{qf}\right)/T\right]}{1 + 1.25 \exp\left[-\left(B_{\eta}^{fus} - B_{qf}\right)/T\right]}$$

$$B_{qf} = \min\left(B_{qf}^{R}, B_{qf}^{\eta}\right)$$


10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
P_{CN}
102
104
106
108
110
112
114
116
118
120
Z_{CN}
^{50}Ti
^{232}Th
^{48}Ca+^{A}Y
^{238}U
^{244}Pu
^{54}Cr
^{248}Cm
^{58}Fe
^{64}Ni
^{249}Cf
^{62}Ni
^{70}Zn
^{76}Ge
^{82}Se
^{A}X+^{208}Pb
^{86}Kr

$$
\begin{aligned}
\frac{d}{dt}P_{Z,N}(t) &= \Delta^{(-,0)}_{Z+1,N}P_{Z+1,N}(t) + \Delta^{(+,0)}_{Z-1,N}P_{Z-1,N}(t) \\
&+ \Delta^{(0,-)}_{Z,N+1}P_{Z,N+1}(t) + \Delta^{(0,+)}_{Z,N-1}P_{Z,N-1}(t) \\
&- \left(\Delta^{(-,0)}_{Z,N} + \Delta^{(+,0)}_{Z,N} + \Delta^{(0,-)}_{Z,N} + \Delta^{(0,+)}_{Z,N}\right) P_{Z,N}(t) \\
&- (\Lambda^{qf}_{Z,N} + \Lambda^{fis}_{Z,N})P_{Z,N}(t)
\end{aligned}
$$

Rates Δ depend on single-particle energies and temperature related to excitation energy.

Only one-nucleon transitions are assumed.

$\Lambda^{qf}_{Z,N}$: rate for quasifission

$\Lambda^{fis}_{Z,N}$: rate for fission of heavy nucleus

The charge and mass yields for quasifission can be expressed

$$Y_{Z,N}(t_0) = \Lambda^{qf}_{Z,N} \int_0^{t_0} P_{Z,N}(t)\, dt$$

The time t_0 of reaction is determined by solving the normalization condition

$$\sum_{Z,N} Y_{Z,N}(t_0)\, P_{CN} \approx 1$$

$$P_{CN} = \sum_{Z=Z_{BG}}^{} \sum_{N=N_{BG}}^{} P_{Z,N}(t_0)$$

Z_{BG}=8-14 in the reactions considered

Survival probability W_{sur}

De-excitation of excited compound nucleus by neutron, alpha, proton and gamma emissions in competition with fission. The survival probability under the evaporation of a certain sequence *s* of *x* particles is calculated as:

$$W_{sur}^{s}(E_{CN}) \approx P_s(E_{CN}) \prod_{i_s=1}^{x} \frac{\Gamma_{i_s}(E_{i_s})}{\Gamma_t(E_{i_s})}$$

$\boldsymbol{P_s}$ =probability of realisation of s channel The total width for the compound nucleus decay is the sum of partial widths.

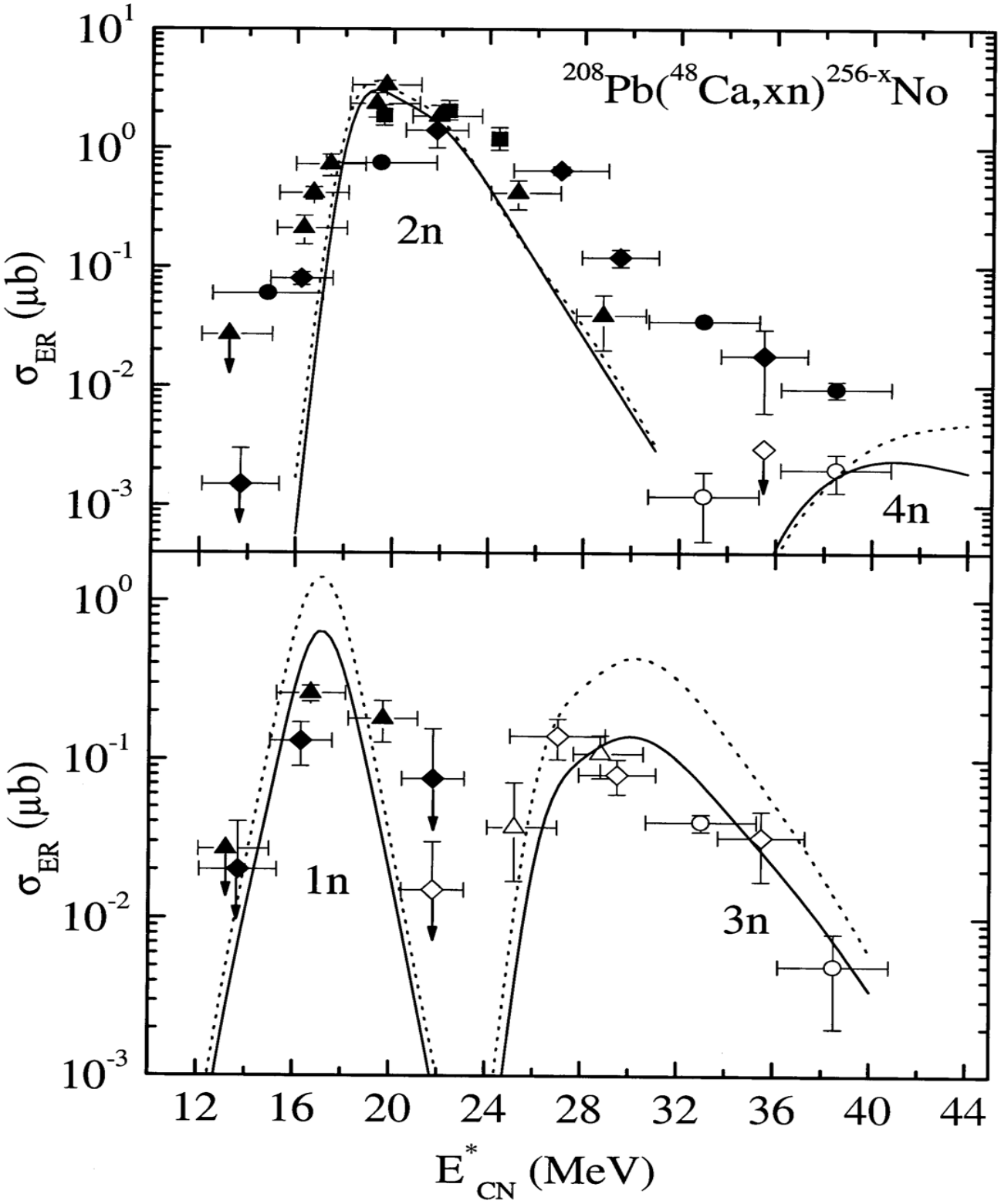
The fission barrier B_f has a liquid drop part B_f^{LD} and a microscopical part B_f^{M} .

B_f^{LD}=1.9-3.2 MeV for Pu and Cm isotopes

$$B_f^M \approx -\Delta W_{gr}^A$$

$$B_f(E_{CN}) = B_f^{LD} + B_f^M \exp[-E_{CN}/E_D]$$

$$E_D = 0.4A^{4/3}/a$$


^{208}Pb(^{48}Ca,xn)$^{256-x}$No
2n
4n
1n
3n
σ_{ER} (μb)
E^*_{CN} (MeV)

```
 OPEN (5,FILE='INPF3D.DAT',STATUS='OLD')
 OPEN(60,FILE='DEF1.DAN',STATUS='OLD',ACCESS='DIRECT',RECL=800)
 READ (5,15) STRIN
 READ (5,10) A1,A2,Z1,Z2
 READ (5,10) A01,A02
 READ (5,10) R0,RO00
 READ (5,10) TET1,TET2,AVAL1,AVAL2
 READ (5,10) TETA1,TETA2,ALFA
 READ (5,10) BET1,BET2,XL
 READ (5,10) C0,FIN,FINP,FEX,FEXP
 READ (5,16) PROMPT

F+Al
18.00      26.00       9.00    13.00
 0.550111    0.550111
 1.15      0.17        -0.10
 0.000       3.140      0.500    0.500
 0.000       0.000      0.000
 0.00       0.00          0.
 300.00     0.09      0.42       -2.59     0.54
 OUTPUT.DAT
```

F+Al

A1 = 18.000000000000000 A2 = 26.000000000000000

R0 = 1.150000000000000 RO00 = 1.700000000000000E-001

R01 = 3.013849574527699 R02 = 3.406866619300192

A01 = 5.501110000000000E-001 A02 = 5.501110000000000E-001

TET1 = 0.000000000000000E+000 TET2 = 3.140000000000000

TETA1 = 0.000000000000000E+000 TETA2 = 0.000000000000000E+000ALFA = 0.000000000000000E+000

BET1 = 0.000000000000000E+000 BET2 = 0.000000000000000E+000

5.34	31.4993	10.5260	42.0253	.0000
5.54	30.3969	-4.9729	25.4240	.0000
5.74	29.3553	-16.2482	13.1070	.0000
5.94	28.3739	-23.8396	4.5343	.0000
6.14	27.4515	-28.3295	-.8780	.0000
6.34	26.5854	-30.3107	-3.7253	.0000
6.54	25.7721	-30.3580	-4.5859	.0000
6.74	25.0070	-29.0034	-3.9964	.0000
6.94	24.2861	-26.7166	-2.4305	.0000
7.14	23.6055	-23.8913	-.2858	.0000
7.34	22.9621	-20.8404	2.1217	.0000

DRIV.DAT

DM1,DM2,QV,DM12
8.73000000000000E-001 -12.210000000000000 -11.337000000000000
-37.548000000000000
VMIN,VDR
-4.585915389331262 21.625084610668740

Driving potential

```
 OPEN(100,FILE='DEF1.DAN',STATUS='OLD',ACCESS='DIRECT',
 *FORM='UNFORMATTED',RECL=800)

  OPEN (3,FILE='RIDIN.DAT',STATUS='OLD')

  READ (3,12) Z1,A1
  READ (3,12) Z2,A2
  READ (3,11) KL
  READ (3,11) L


   20.       40.
   20.       48.
1
0
```

```
FUNCTION VPR(X)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON/CPR/CPROX,SO,R1,R2,R12,G,ETA,RRED
ADEN=0.56
PI=3.1415927
R=X-R12
ADEN1=ADEN+DABS(ETA)*(0.50-ADEN)
AL=11.47-17.32*ADEN*ADEN1+2.069*RRED
D=1.03*(10.96-0.8*RRED)*2.*PI*ADEN*ADEN1*RRED
VPR= D*(DEXP(-2.*AL*R/R12)-2.*DEXP(-AL*R/R12))
RETURN
END
```

$$W_{sur}(E_{\mathrm{c.m.}}, J) = W_{sur}(E_{\mathrm{c.m.}}, J = 0)\exp\left[-J(J+1)/(J_{max}(J_{max}+1))\right]$$

$$\sum_J \sigma_c(E_{\mathrm{c.m.}}, J) W_{sur}(E_{\mathrm{c.m.}}, J) \approx \sigma_c(E_{\mathrm{c.m.}}) W_{sur}(E_{\mathrm{c.m.}}),$$

$$W_{sur}(E_{\mathrm{c.m.}}) = W_{sur}(E_{\mathrm{c.m.}}, J = 0)$$

$$\sigma_c(E_{\mathrm{c.m.}}) = \frac{\pi\hbar^2}{2\mu E_{\mathrm{c.m.}}}(J_{max}+1)^2 T(E_{\mathrm{c.m.}})$$

$$\begin{aligned}\sigma_{ER}(E_{\mathrm{c.m.}}) &\approx P_{CN}(E_{\mathrm{c.m.}}) \sum_J \sigma_c(E_{\mathrm{c.m.}}, J) W_{sur}(E_{\mathrm{c.m.}}, J) \approx \\ &\approx \sigma_c(E_{\mathrm{c.m.}}) P_{CN}(E_{\mathrm{c.m.}}) W_{sur}(E_{\mathrm{c.m.}}).\end{aligned}$$

$$W^{s}_{sur}(E^{*}_{CN},J) \approx P_{s}(E^{*}_{CN},J)\prod_{i_s=1}^{x}\frac{\Gamma_i(E^{*}_{i_s},J_{i_s})}{\Gamma_t(E^{*}_{i_s},J_{i_s})}$$

$$\Gamma_t = \sum\nolimits_i \Gamma_i + \Gamma_\gamma + \Gamma_f$$

$$\Gamma_t \approx \Gamma_n + \Gamma_f$$

$$W_{sur}(E^{*}_{CN}) \approx P_{xn}(E^{*}_{CN})\prod_{i=1}^{x}\frac{\Gamma_n(E^{*}_{i})}{\Gamma_n(E^{*}_{i})+\Gamma_f(E^{*}_{i})} \approx P_{xn}(E^{*}_{CN})\prod_{i=1}^{x}\frac{\Gamma_n(E^{*}_{i})}{\Gamma_f(E^{*}_{i})},$$

$$W_{sur}(E^{*}_{CN}) \approx P_{1n}(E^{*}_{CN})\frac{\Gamma_n(E^{*}_{CN})}{\Gamma_f(E^{*}_{CN})}.$$

$$P_s = P_{xn} = P(x) - P(x+1),$$

$$P(x) = 1 - \exp[-\Delta_x/\overline{T}](1 + \sum_{i=1}^{2x-3} \frac{(\Delta_x/\overline{T})^i}{i!})$$

$$\Delta_x = E^*_{CN} - \sum_{k=1}^{x} B_k$$

$$\overline{T} = \sqrt{E^*_{CN}/1.5a_{CN}}$$

if $B_f < B_{x+1}$, $B_{x+1} \dashrightarrow B_f$ in the expression for Δ_x

$$P_{1n}(E^*_{CN}) = \exp[-(E^*_{CN} - B_n - 2T)^2/(2\sigma^2)]$$

$$T = \sqrt{E^*_{CN}/a_{CN}} \qquad \sigma = 2.5 \text{ МэВ}$$

$$B_f(E^*) = B_f^{LD} + B_f^M(E^* = 0)\exp[-E^*/E_D]$$

$$E_D = 0.4A^{4/3}/a$$

$$B_f^M = B_f^M(E^* = 0) = |\delta W_{gr}^A(E^* = 0)|$$

or

$$a(A, E^* - E_c) = \tilde{a}(A)\left[1 + \frac{1 - \exp\{-(E^* - E_c)/E'_D\}}{E^* - E_c}\delta W\right]$$

$$\frac{\Gamma_n}{\Gamma_f} = \frac{0.41A^{2/3}a_f U_n}{2[a_n U_n]^{1/2} - 1}\exp[2a_n^{1/2}U_n^{1/2} - 2a_f^{1/2}U_f^{1/2}]$$

$$U_n = E^*_{CN} - B_n - \delta, \quad U_f = E^*_{CN} - (B_f - \delta)\exp[-E^*_{CN}/E_d] - \delta$$

$E_d = 25$ MeV $\quad \delta = 22/A_0^{1/2}$ and $11/A_0^{1/2}$ $\quad a_f/a_n = 1.07$ 1.045