Cluster features of nuclear structure

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Содержание:

- ✓ **Collective excitations of atomic nuclei**
- ✓ Clusterization and its role in nuclear structure
- ✓ **Cluster degrees of freedom in Dinuclear System Model**
- ✓ Collective spectra of heavy nuclei in cluster model

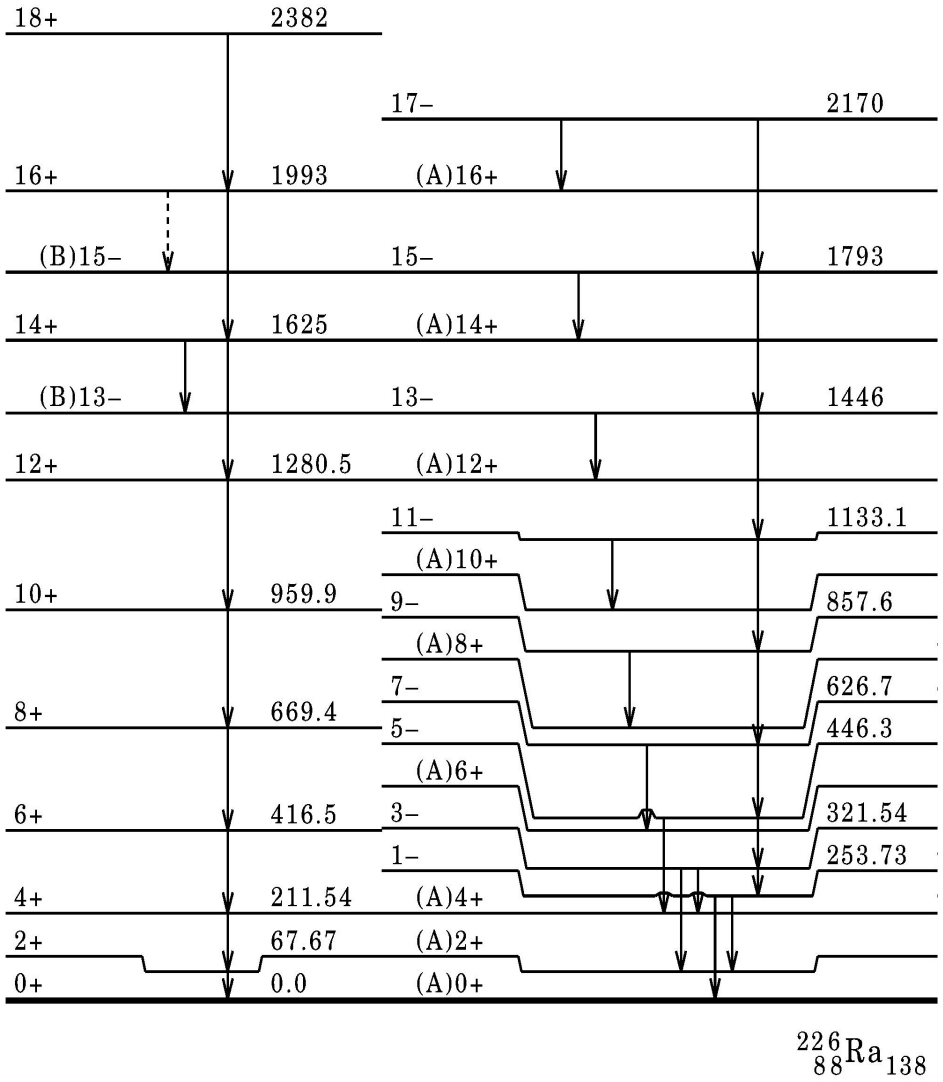
Excitation Spectrum of Deformed Nucleus

States are marked as

$$L^p$$

L- angular momentum,

$p = \pm 1$ -parity



Mean Field Models

$$\sum_{i=1}^{A}\left(-\frac{\hbar^2\Delta_i}{2m_i}\right)\psi_{nucleus} + \frac{1}{2}\sum_{i=1}^{A}\sum_{j=1}^{A}V(\vec{r}_i-\vec{r}_j)\psi_{nucleus} = E_{nucleus}\psi_{nucleus}$$

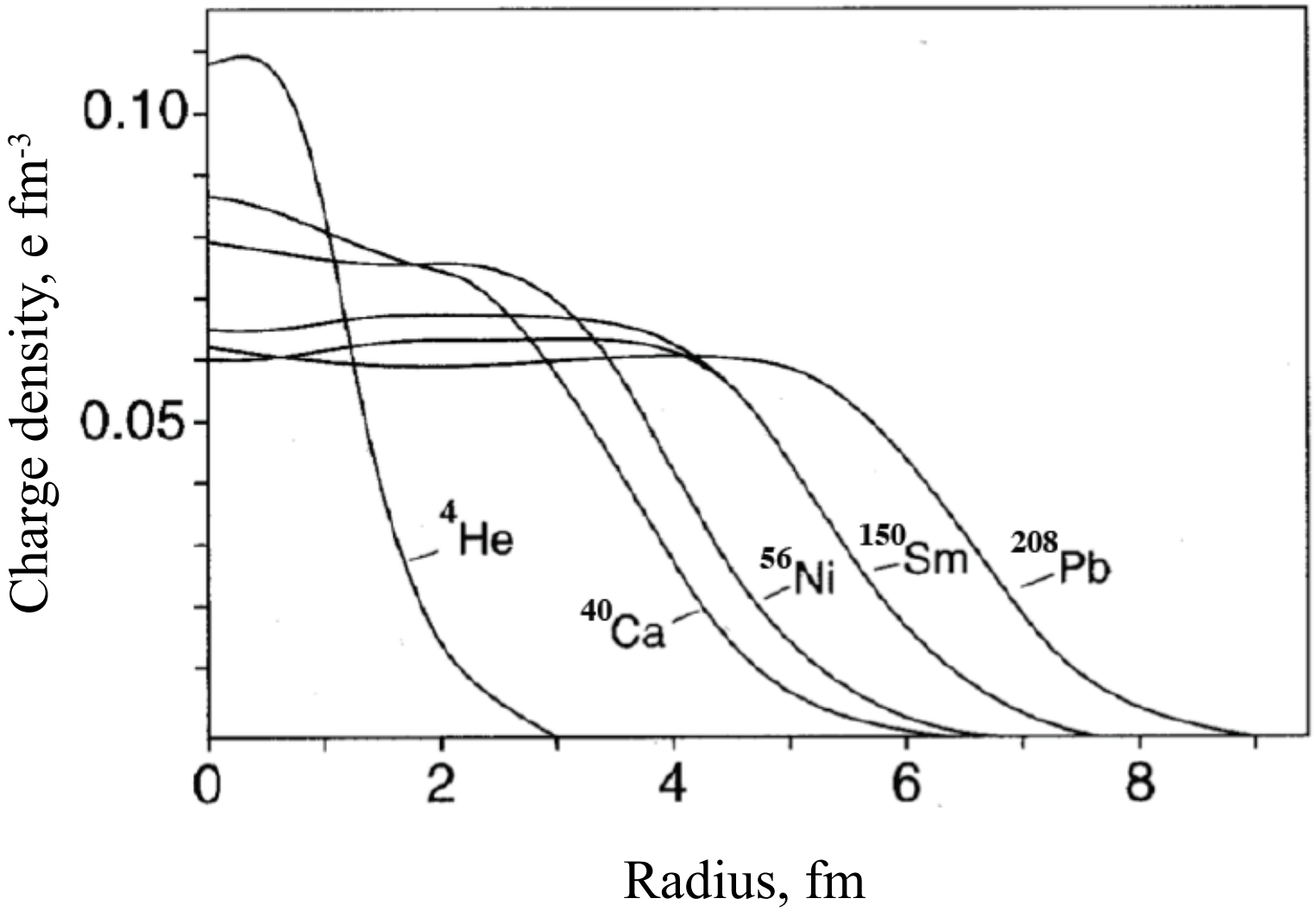
In the mean field model the interaction between nucleons are approximated by the average field (mean field).

Nucleons are moving independently in the mean field.

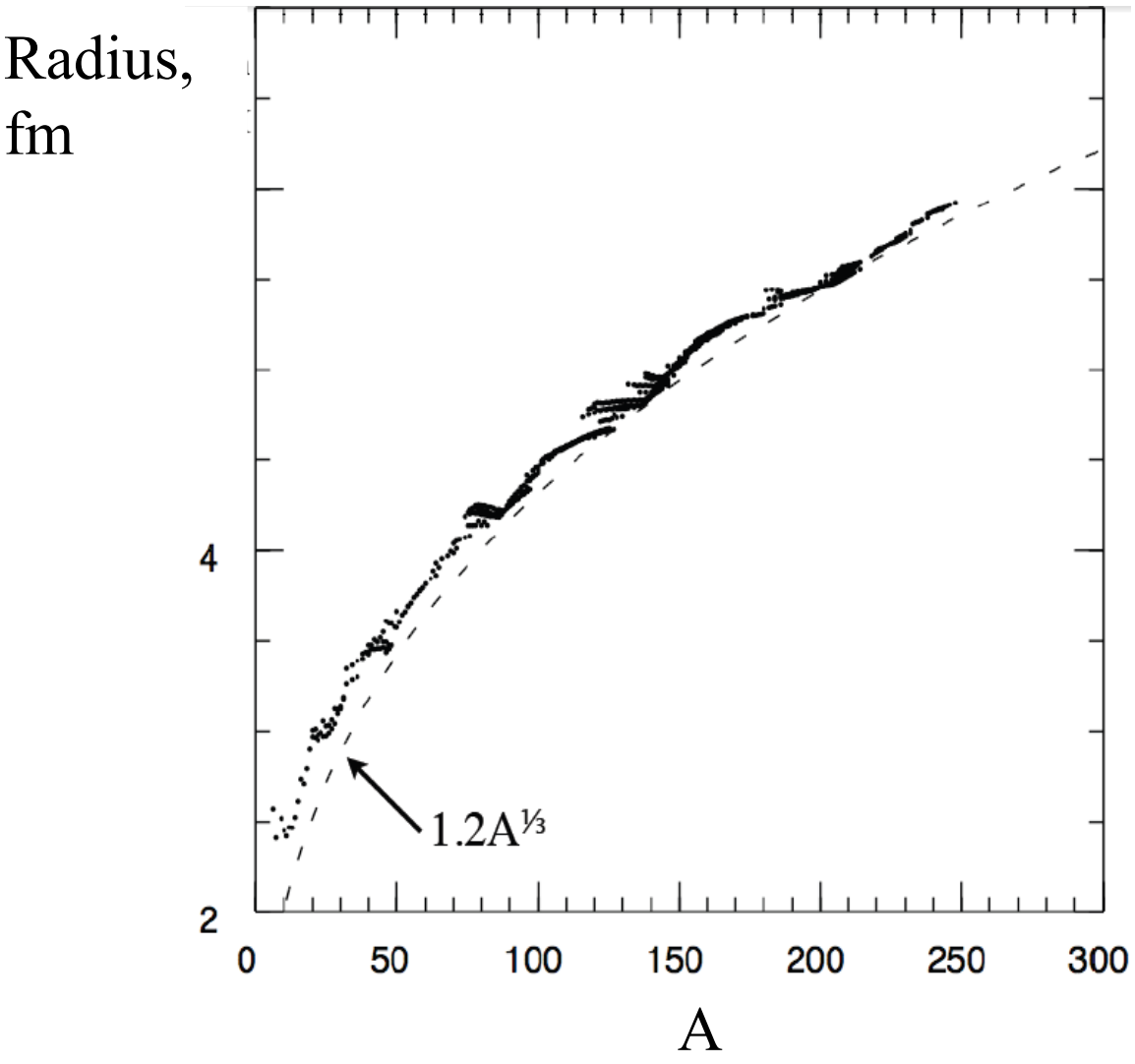
$$\frac{1}{2}\sum_{i=1}^{A}\sum_{j=1}^{A}V(\vec{r}_i-\vec{r}_j) \Rightarrow \sum_{j=1}^{A}U_{av}(\vec{r}_i)$$

Nuclear Schrödinger equation is reduced to the A independent equations for each nucleon.

Density is almost constant inside

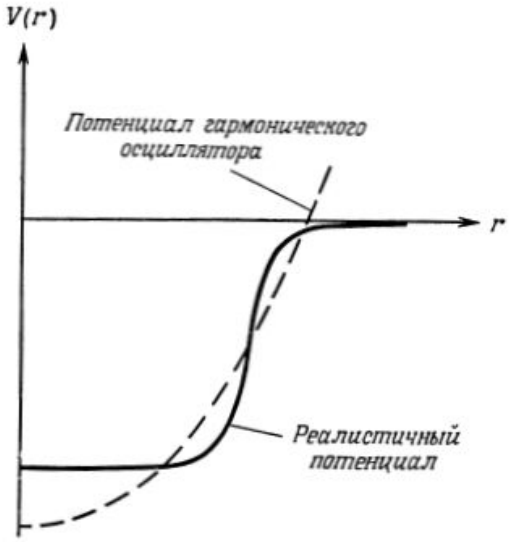


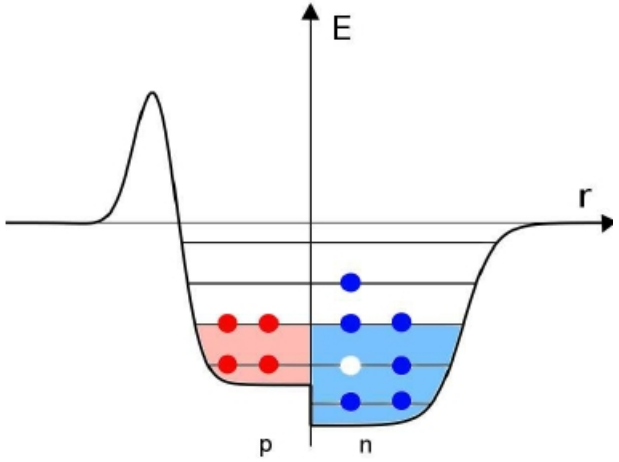
Volume grows linearly with mass

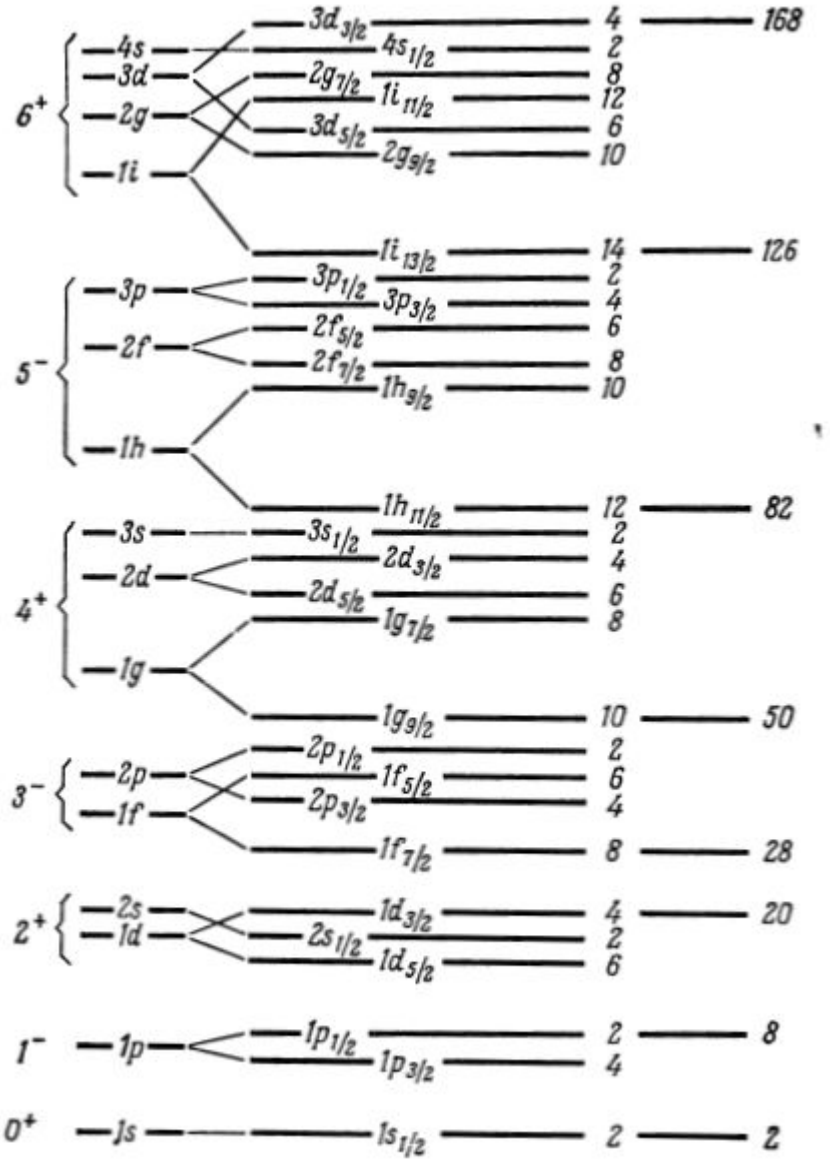


For nuclei with $A>20$ $\quad R \approx 1.2A^{1/3}$

Nuclear excitations

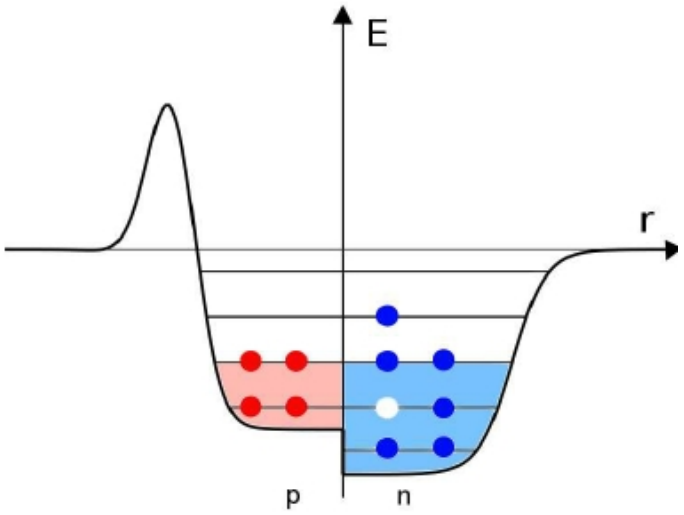




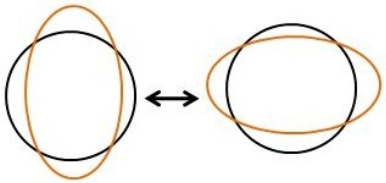


Nuclear excitations

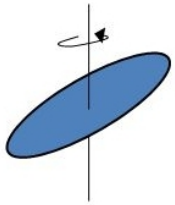
Single-particle excitations



Collective excitations

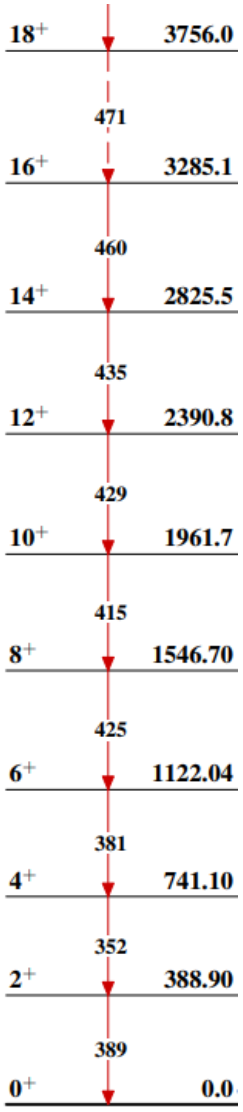


Vibrations

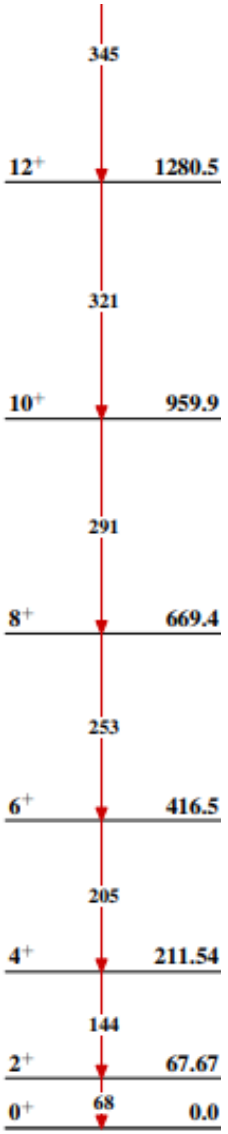


Rotations

The excited nucleus is unstable and can be de-excited by emission of γ

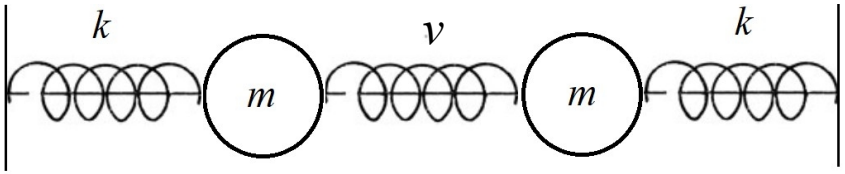


$^{218}Ra_{88}$



$^{226}Ra_{88}$

Lowering of collective excitations



$$L = T - U$$

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

$$U = \frac{1}{2} kx_1^2 + \frac{1}{2} kx_2^2 - \frac{1}{2} v(x_1 - x_2)^2$$

$$\begin{cases} m\ddot{x}_1 = -kx_1 + v(x_1 - x_2) \\ m\ddot{x}_2 = -kx_2 - v(x_1 - x_2) \end{cases}$$

Interaction between the oscillatiors
Leads to a lowering of the vibrational mode

$$\begin{cases} X = x_1 + x_2 \\ Y = x_1 - x_2 \end{cases}$$

$$\omega_X = \sqrt{k/m}$$

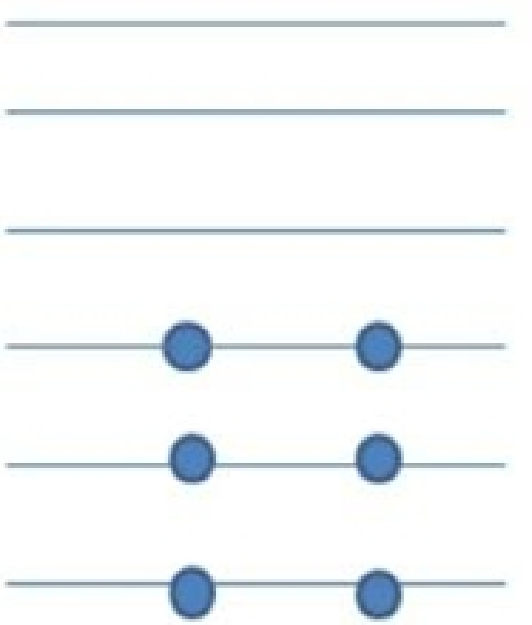
$$\begin{cases} m\ddot{X} = -kX \\ m\ddot{Y} = -(k - 2v)Y \end{cases}$$

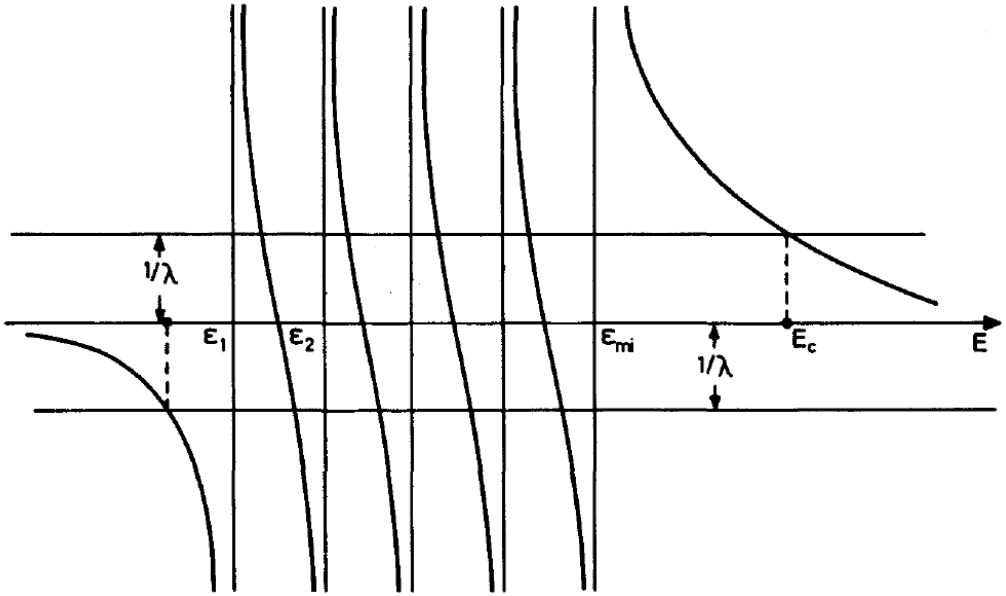
$$\omega_Y = \sqrt{(k - 2v)/m}$$

Lowering of collective excitations (Tamm-Dancoff method)

$$V(1,2) = \lambda(Q(1) \cdot Q(2))$$

$$\frac{1}{\lambda} = \sum_{mi} \frac{|\langle m | Q | i \rangle|^2}{E - \varepsilon_m + \varepsilon_i}$$

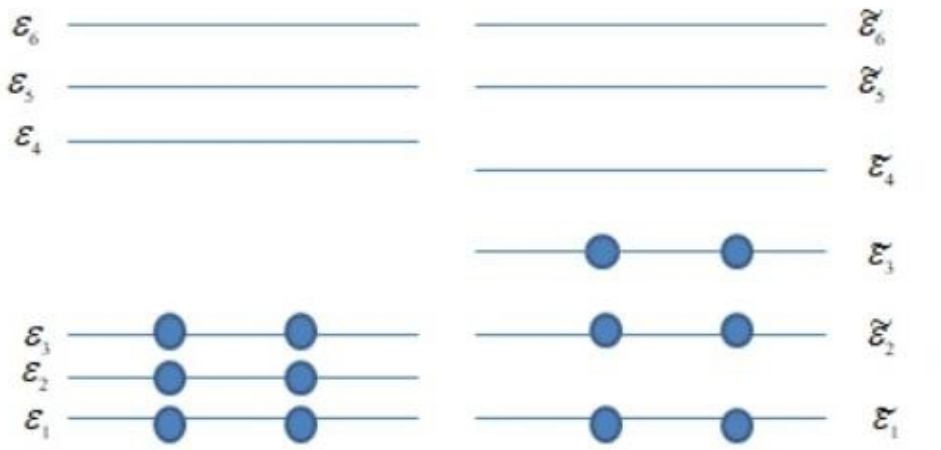




$$E_c = \varepsilon + \lambda \sum_{mi} |\langle m | Q | i \rangle|^2$$

Residual interaction

$$V_{res} = \frac{1}{2}\sum_{i=1}^{A}\sum_{j=1}^{A} V(\vec{r}_i - \vec{r}_j) - \sum_{j=1}^{A} U_{av}(\vec{r}_i)$$



$$\begin{aligned}
\langle \Psi_{fin}|V|\Psi_{in}\rangle &= \int e^{-i(\mathbf{p}'_1\mathbf{r}_1+\mathbf{p}'_2\mathbf{r}_2)} V(|\mathbf{r}_1-\mathbf{r}_2|) e^{i(\mathbf{p}_1\mathbf{r}_1+\mathbf{p}_2\mathbf{r}_2)} d\mathbf{r}_1 d\mathbf{r}_2 \\
&\sim -V_0 \int d\mathbf{r} e^{-i\mathbf{p}\mathbf{r}} e^{-r^2/\mu^2} = -V_0 \int d\mathbf{r} e^{-\left(r+\frac{i}{2}\mathbf{p}\mu^2\right)^2/\mu^2} e^{-p^2\mu^2/4} \\
&= -\pi^3/2\mu^3 V_0 e^{-p^2\mu^2/4}.
\end{aligned}$$

Residual interaction

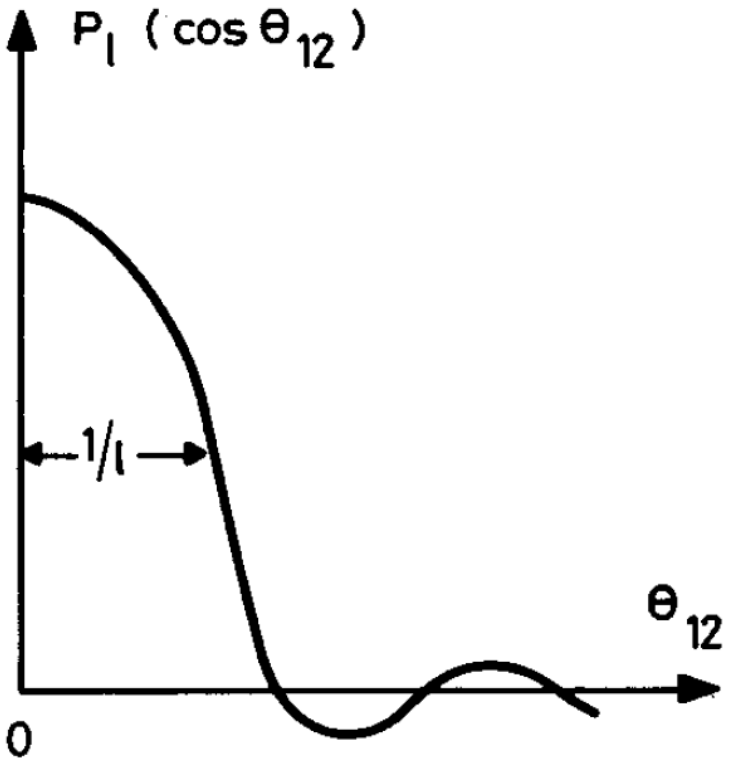
$$V(|\vec{r}_1-\vec{r}_2|)=\sum_l V_l(r_1,r_2)\sum_m Y^*_{lm}(\theta_1,\phi_1)Y_{lm}(\theta_2,\phi_2)$$

$$=\sum_l \frac{4\pi}{2l+1}V_l(r_1,r_2)P_l(\cos\theta_{12})$$

For δ-force:

$$V(|\vec{r}_1-\vec{r}_2|)=-V_0\delta(\vec{r}_1-\vec{r}_2)$$

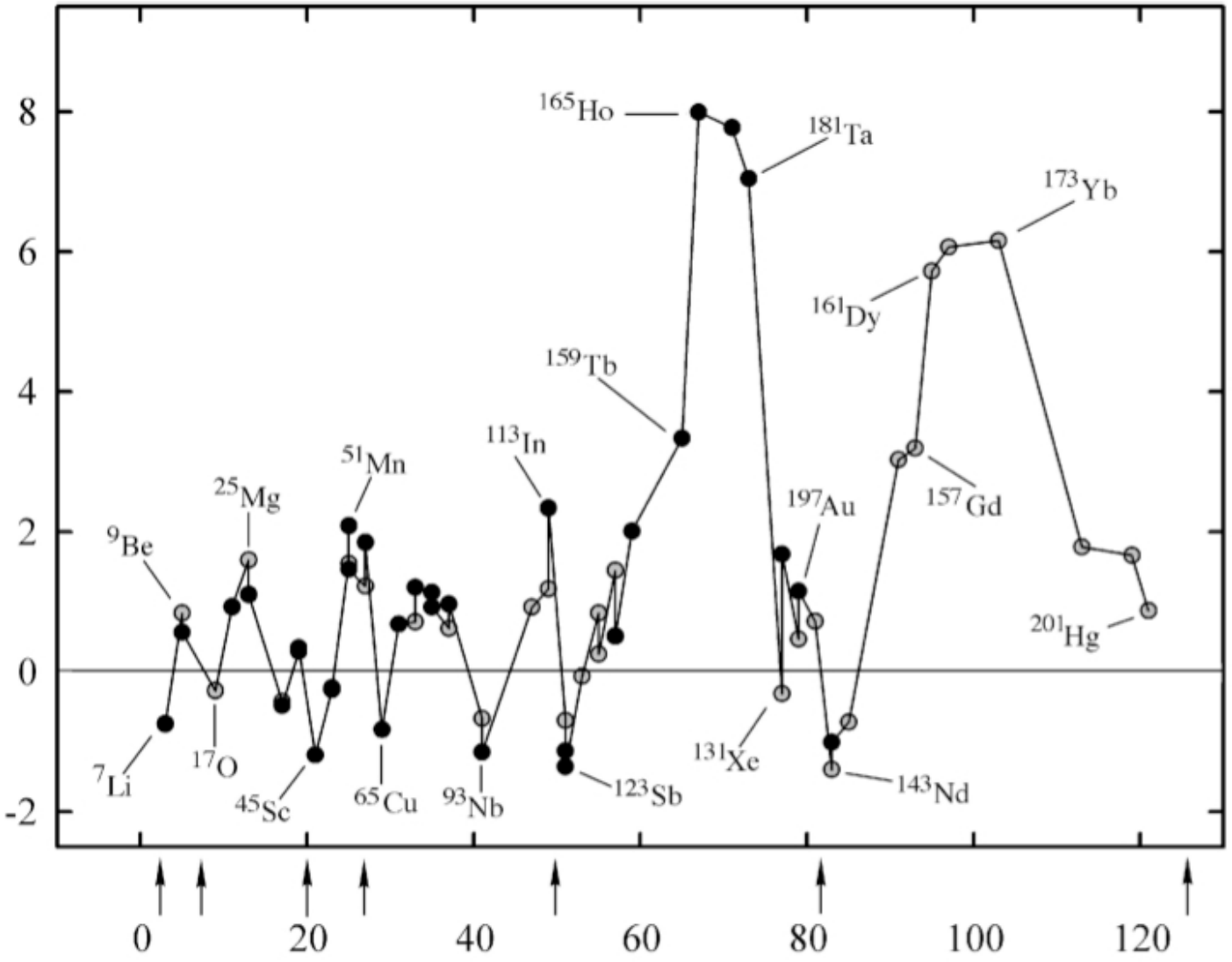
$$=\sum_l \frac{4\pi}{2l+1}\frac{\delta(r_1-r_2)}{r_1r_2}P_l(\cos\theta_{12})$$

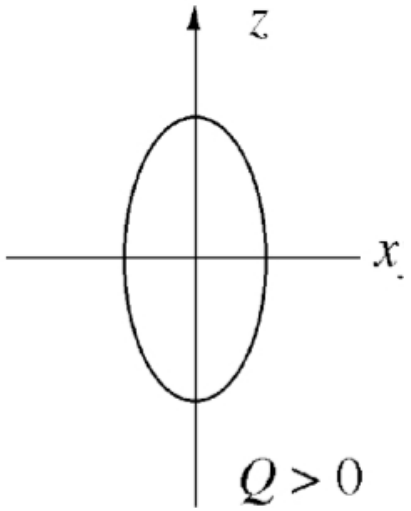


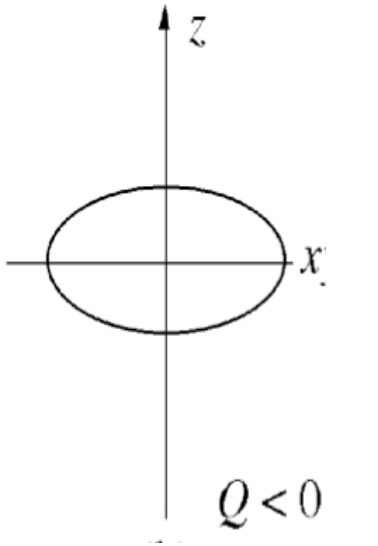
l=2,3
long-range quadrupole and octupole interactions.

All the rest is replaced by the short-range pairing interaction.

Nuclear Deformation

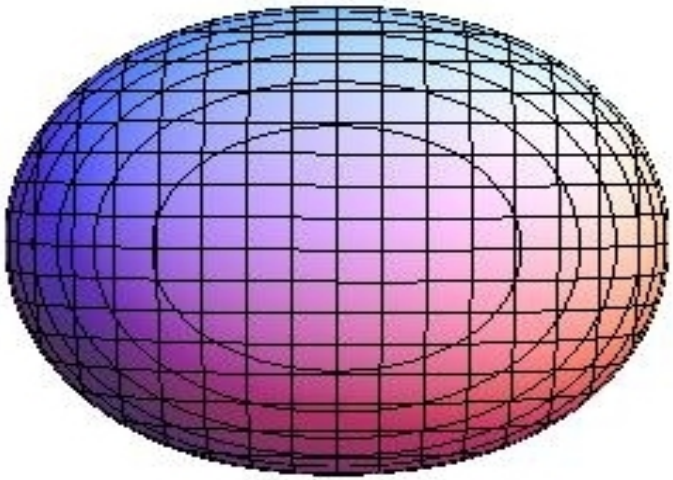






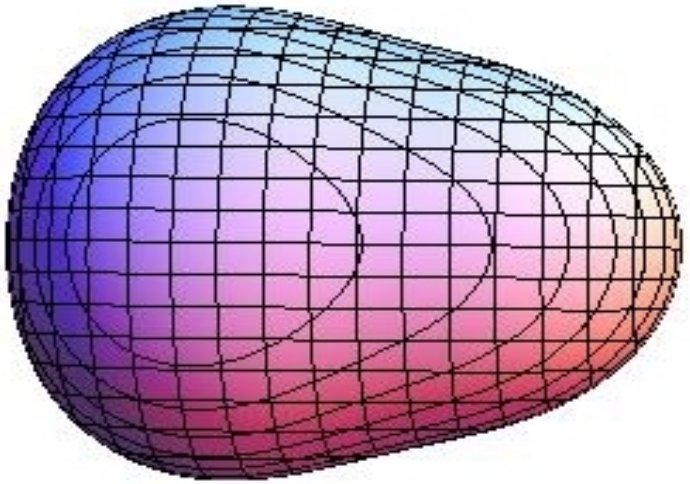
Nuclear Deformation

$$R(\Omega) = c(\beta)R_0\left[1 + \sum_{\mu=-2}^{2}\beta_{2\mu}Y^*_{2\mu}(\Omega) + \sum_{\mu=-3}^{3}\beta_{3\mu}Y^*_{3\mu}(\Omega)\right]$$



Quadrupole deformation

β_{20}=0.6, β_{30}=0.0



Quadrupole +Octupole deformation

β_{20}=0.6, β_{30}=0.5

Collective quadrupole vibrations

For small oscillations about spherical equilibrium:

$$H = \frac{1}{2} D \sum_m |\dot{\beta}_{2m}|^2 + \frac{1}{2} C \sum_m |\beta_{2m}|^2$$

Pauli quantization procedure:

$$T = \frac{1}{2} \sum_{\mu\nu} g_{\mu\nu}(\{q\}) \dot{q}_\mu \dot{q}_\nu \leftrightarrow \hat{T} = -\frac{\hbar^2}{2} \sum_{\mu\nu} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial q_\mu} \sqrt{\det g}\, g^{-1}_{\mu\nu} \frac{\partial}{\partial q_\nu}$$

and the volume element is $d\tau = \left|\sqrt{\det g}\right| \prod_\mu q_\mu$.

$$\hat{H} = \frac{\hbar^2}{2} D \sum_m (-1)^m \frac{\partial}{\partial \beta_{2m}} \frac{\partial}{\partial \beta_{2-m}} + \frac{1}{2} C \sum_m |\beta_{2m}|^2$$

$$|0^+)$$

$$|2^+ M, E = \sqrt{C/D}) = \beta^+_{2M} |0^+)$$

$$|L^+ M, E = 2\sqrt{C/D}) = \sum_{\mu\upsilon} C^{LM}_{2\mu 2\upsilon} \beta^+_{2\mu} \beta^+_{2\upsilon} |0^+), L = 0,2,4$$

Collective quadrupole rotation

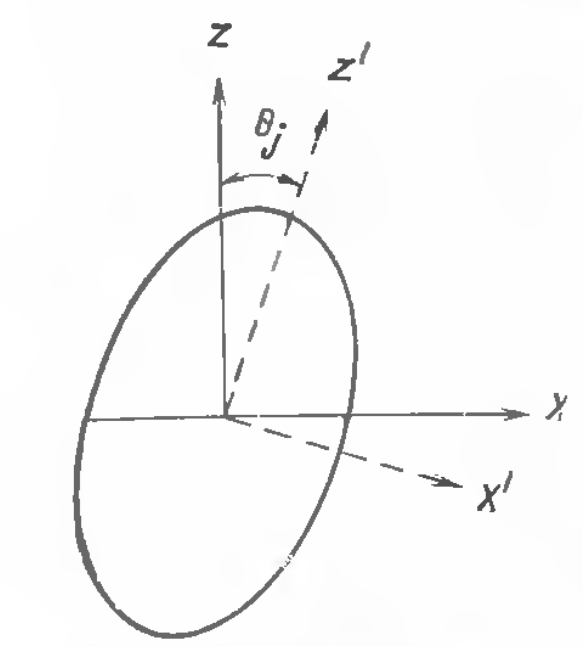
If nucleus is deformed, the Hamiltonian should be transformed into the intrinsic coordinate system:

$$\beta_{2m} = \sum_{\mu} D^{2}_{m\mu}(\theta_1, \theta_2, \theta_3)\overline{\beta}_{2\mu},$$

$$\overline{\beta}_{20} = \beta\cos(\gamma),$$

$$\overline{\beta}_{21} = \overline{\beta}_{2-1} = 0,$$

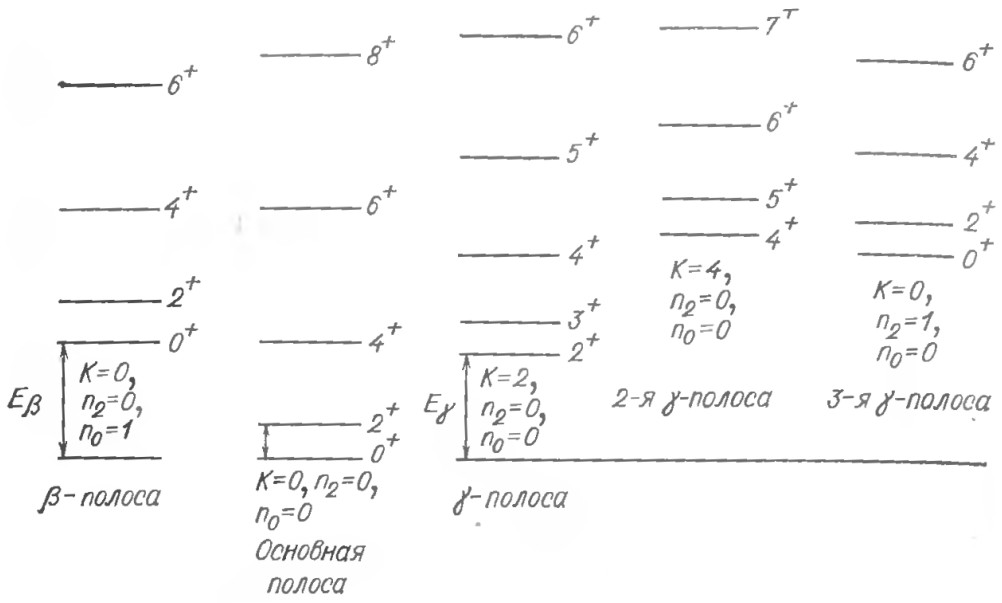
$$\overline{\beta}_{22} = \overline{\beta}_{2-2} = \frac{1}{\sqrt{2}}\beta\sin(\gamma)$$

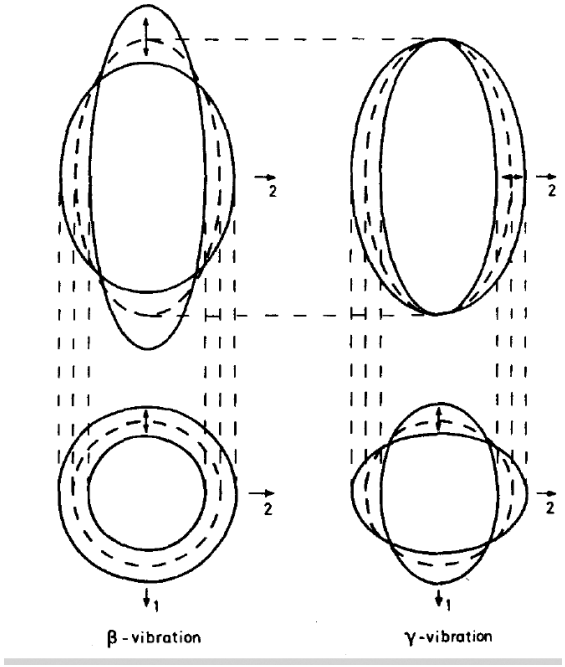


$$H = \frac{1}{2}\sum_{k=1}^{3} I_k\omega_k + \frac{1}{2}D(\dot{\beta}^2 + \beta^2\dot{\gamma}^2) + \frac{1}{2}C_0(\beta - \beta_0)^2 + \frac{1}{2}C_2\gamma^2$$

Collective quadrupole rotation (Bohr Hamiltonian)

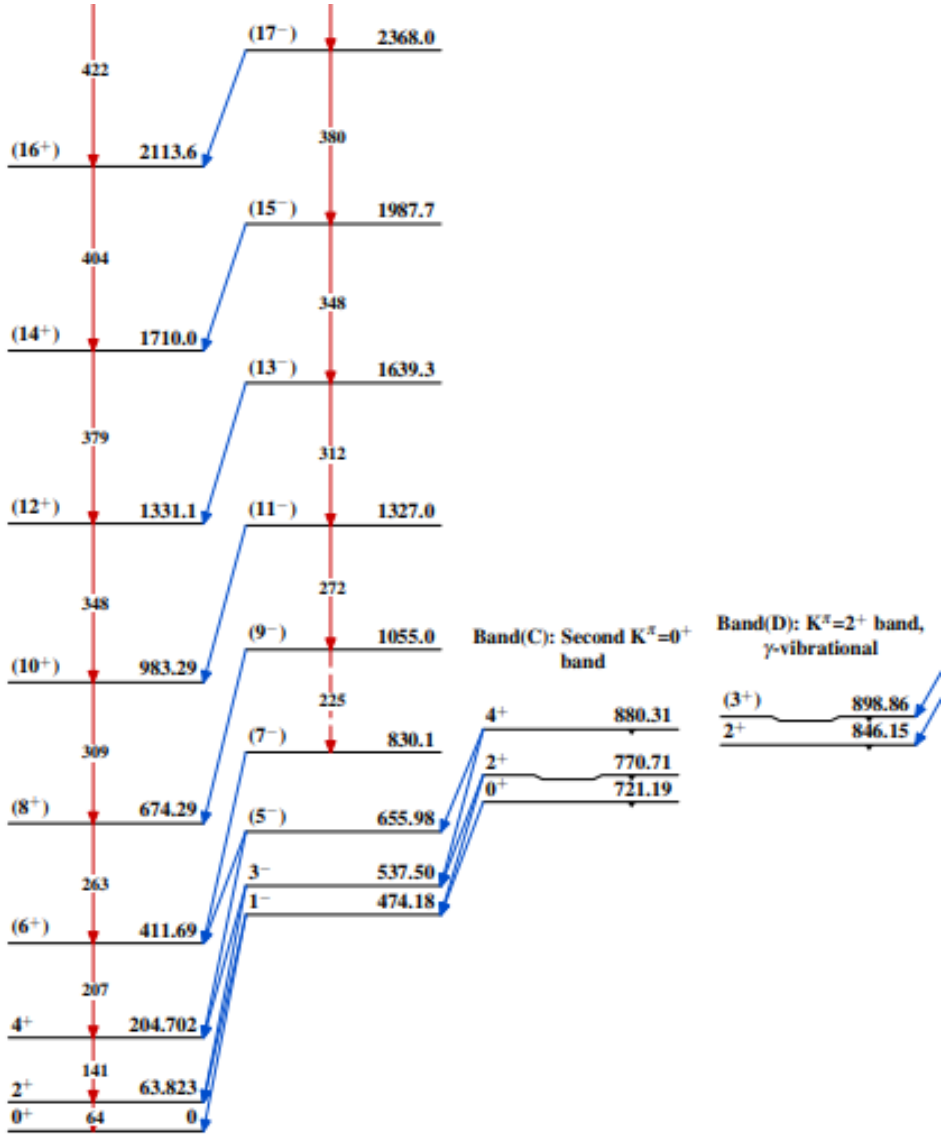
$$\hat{H} = \frac{\hbar^2}{2}\sum_{k=1}^{3}\frac{\hat{l}_k^2}{I_k(\beta,\gamma)} - \frac{\hbar^2}{2D}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right) + V(\beta,\gamma)$$





$$E_{Ln_\beta n_\gamma} = \frac{\hbar^2}{2I_0}\left(L(L+1)-K^2\right) + \hbar\sqrt{\frac{C_0}{D}}\left(n_\beta+\frac{1}{2}\right) + \hbar\sqrt{\frac{C_2}{D}}\left(n_\gamma+|\frac{|K|}{2}+1\right)$$

Excitation spectrum of deformed nucleus



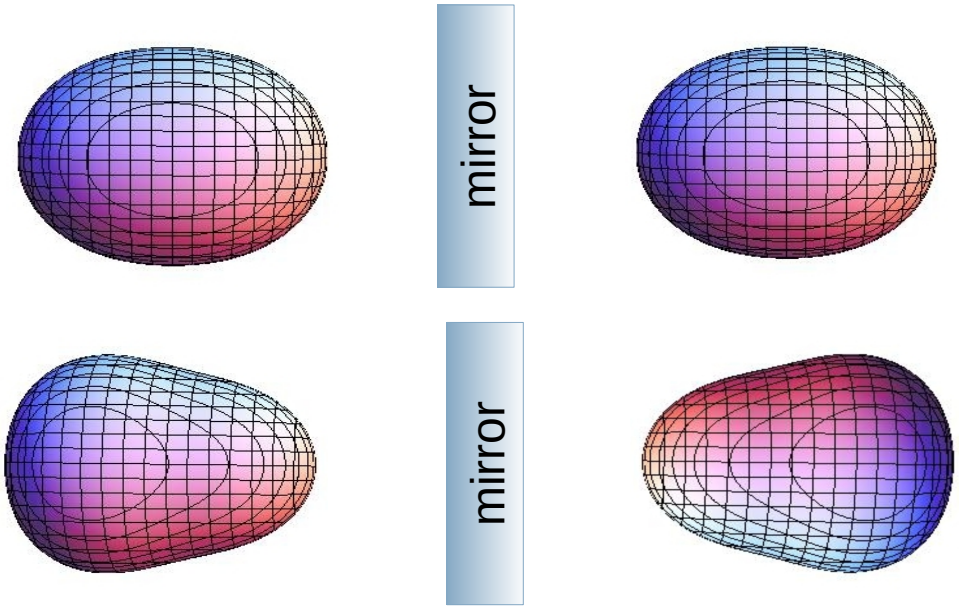
$^{228}_{88}Ra_{140}$

Reflection-Asymmetric Shapes

Parity:

$$\hat{P}\Psi(\mathbf{r}) = \Psi(-\mathbf{r}) = p\Psi(\mathbf{r})$$

$$\hat{P}\hat{P}\Psi(\mathbf{r}) = p^2\Psi(\mathbf{r}) = \Psi(\mathbf{r}); \quad p = \pm 1$$



In strong and e.-m. interactions parity is a good quanum number.

Quadrupole deformation (lowest excitations): 0^+, 2^+, 4^+,...
Octupole deformation (lowest excitations): 0^+, 1^-,2^+,3^- ,4^+

Deformations violating reflection symmetry

$$\Psi_{p,IMK} = \sqrt{\frac{2I+1}{16\pi^2}}\left(\Phi_{n,K}(\xi)D^I_{MK} + p(-1)^{I+K}\Phi_{n,\overline{K}}(\xi)D^I_{M,-K}\right)$$

Wave function in ξ defined by the equation:

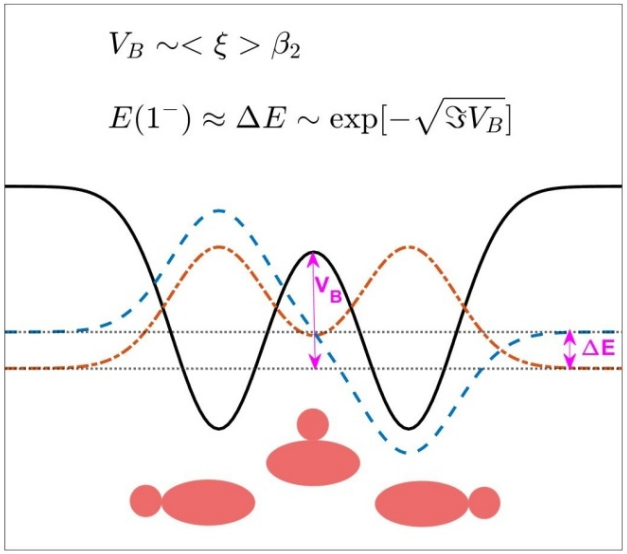
$$\left(-\frac{\hbar^2}{2B_\xi}\frac{d^2}{d\xi^2} + U(\xi) + \frac{\hbar^2}{2\Im(\xi)}I(I+1)\right)\Psi_{n,K}(\xi) = E_{n,K}\Psi_{n,K}(\xi),$$

where

$$\Im(\xi) = 0.85(\Im_1^r + \Im_2^r + m_0\frac{A_1A_2}{A}R^2)$$

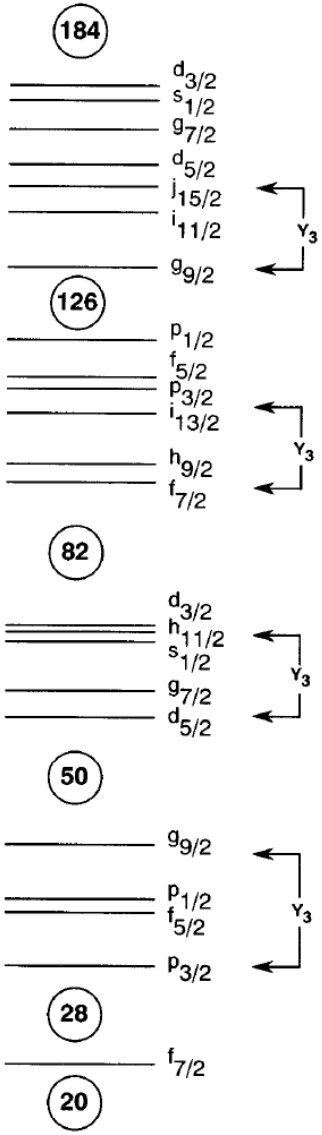
Exitation spectra:

$$I^p(\text{ for } K=0) = 0^+, 1^-, 2^+ \ldots$$
$$I^p(\text{ for } K\neq 0) = K^\pm, (K+1)^\pm \ldots$$



$$\hat{P}\cdot\hat{R}_y(\pi) \rightarrow p(-1)^{I+K} = 1$$

Low-energy collective states of negative parity

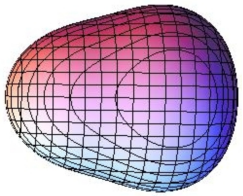


(fig. from: P.A.Butler, W. Nazarewicz, Rev. Mod. Phys. 68, 349 (1996)

N(Z) ≈34, 56,88,134

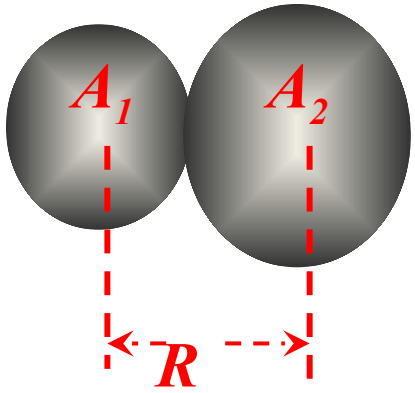
Octupole deformation

$$R(\Omega) = c(\beta)R_0\left[1 + \sum_{\mu=-2}^{2}\beta_{2\mu}Y^*_{2\mu}(\Omega) + \sum_{\mu=-3}^{3}\beta_{3\mu}Y^*_{3\mu}(\Omega)\right]$$



β_{20}=0.6, β_{30}=0.5

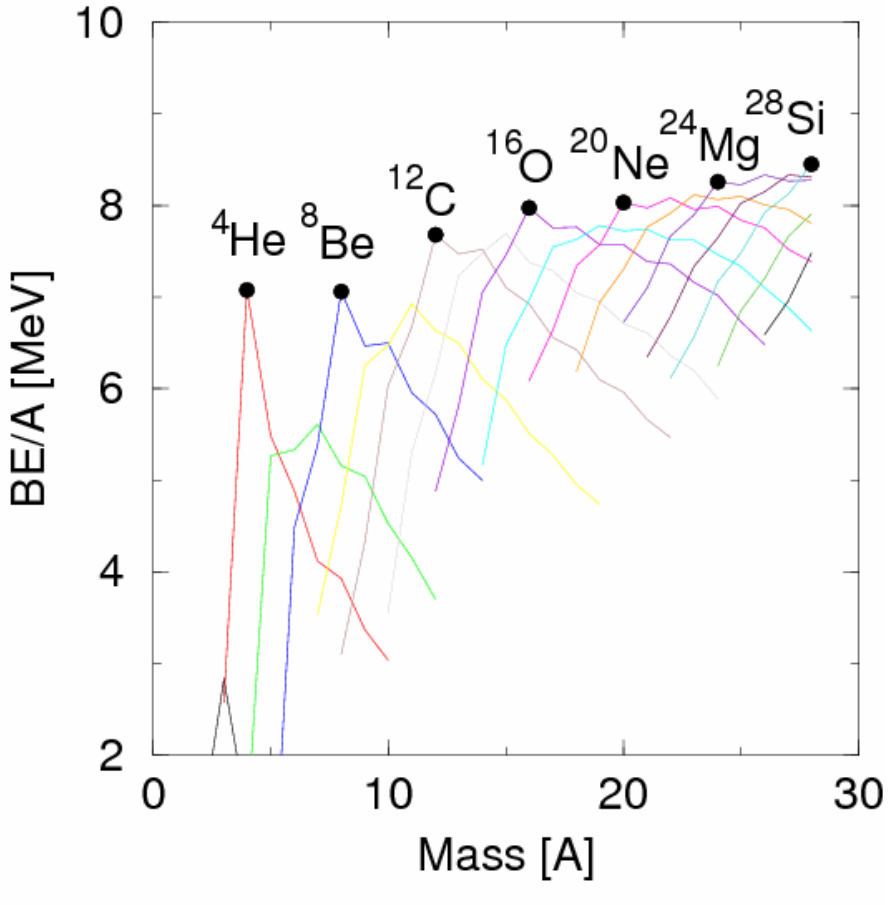
Mass asymmetry

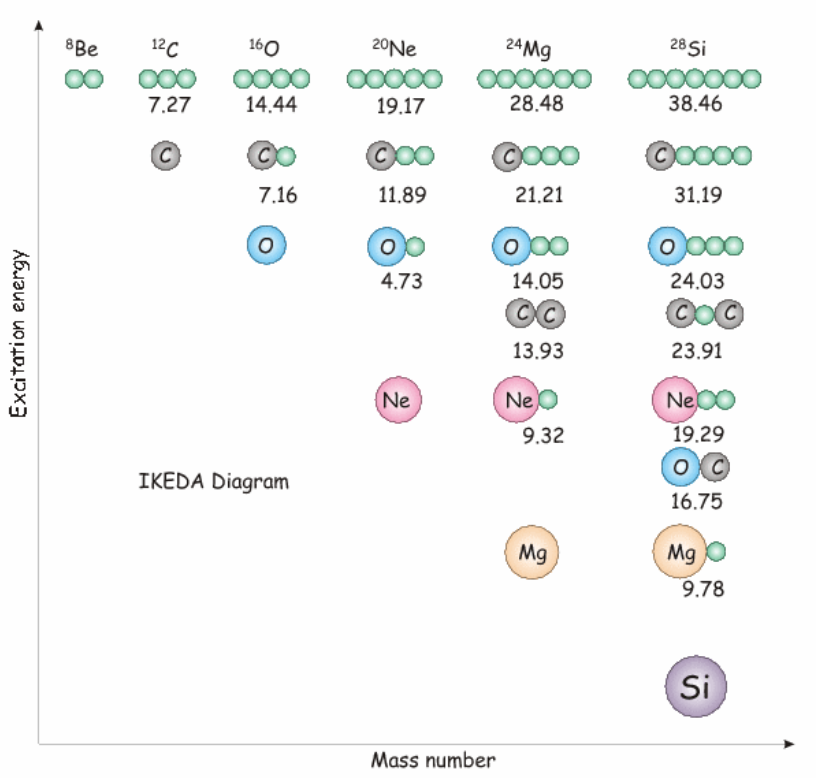


$$\xi = \frac{2A_1}{A_1 + A_2}$$

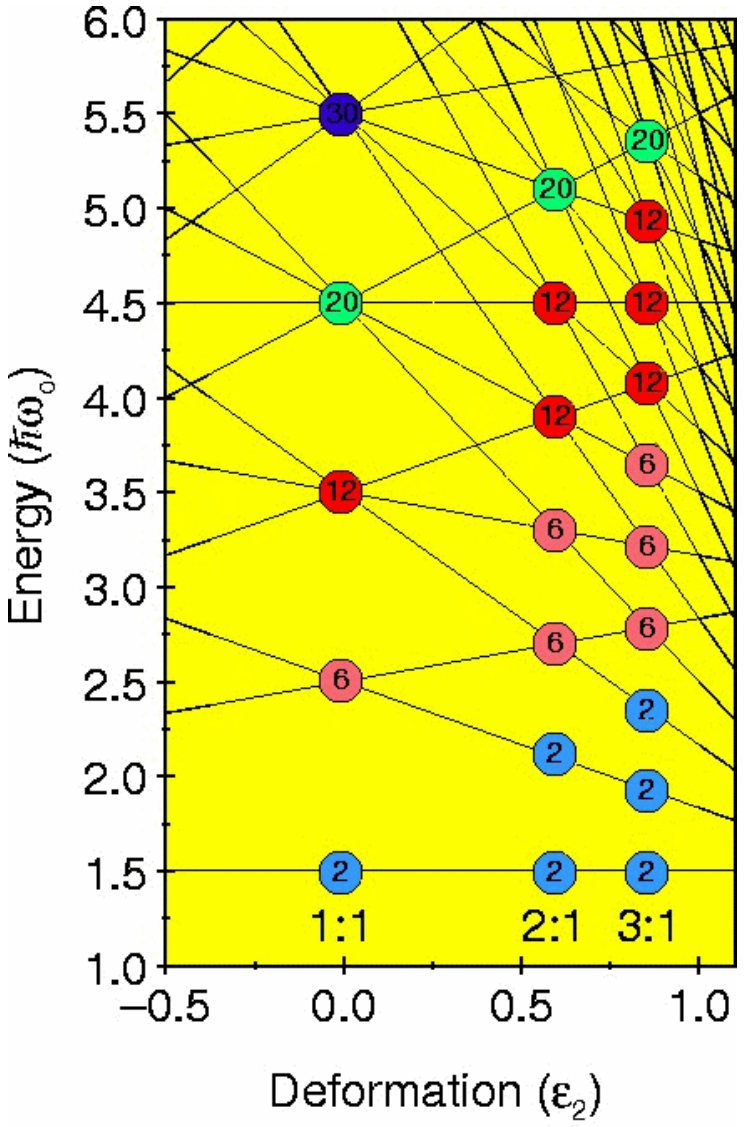
Clusterization in light nuclei

The ***p-n*** attraction is much stronger than the ***n-n*** and ***p-p*** attractions. The proton and neutron have a $^3S_1$ bound state with a binding energy of 1.1 MeV per nucleon (deuteron). This attraction leads to strong spatial correlations between valence neutrons and protons, i.e., to the formation of clusters (**nuclear molecules**).





Clusterization and deformation



Deformed magic numbers	Spherical Magic Numbers	Spherical Constituents
N	**Superdeformation - dimers**	
4	2 + 2	$\alpha+\alpha$
10	8 + 2	$^{16}O+\alpha$
16	8 + 8	$^{16}O+^{16}O$
28	8 + 20	$^{16}O+^{40}Ca$
N	**Hyperdeformation, chains**	
6	2 + 2 + 2	$\alpha+\alpha+\alpha$
12	2 + 8 + 2	$\alpha+^{16}O+\alpha$
24	8 + 8 + 8	$^{16}O+^{16}O+^{16}O$
36	8 + 20 + 8	$^{16}O+^{40}Ca+^{16}O$
48	20 + 8 + 20	$^{40}Ca+^{16}O+^{40}Ca$
60	20 + 20 + 20	$^{40}Ca+^{40}Ca+^{40}Ca$

Mean field models and clusterization

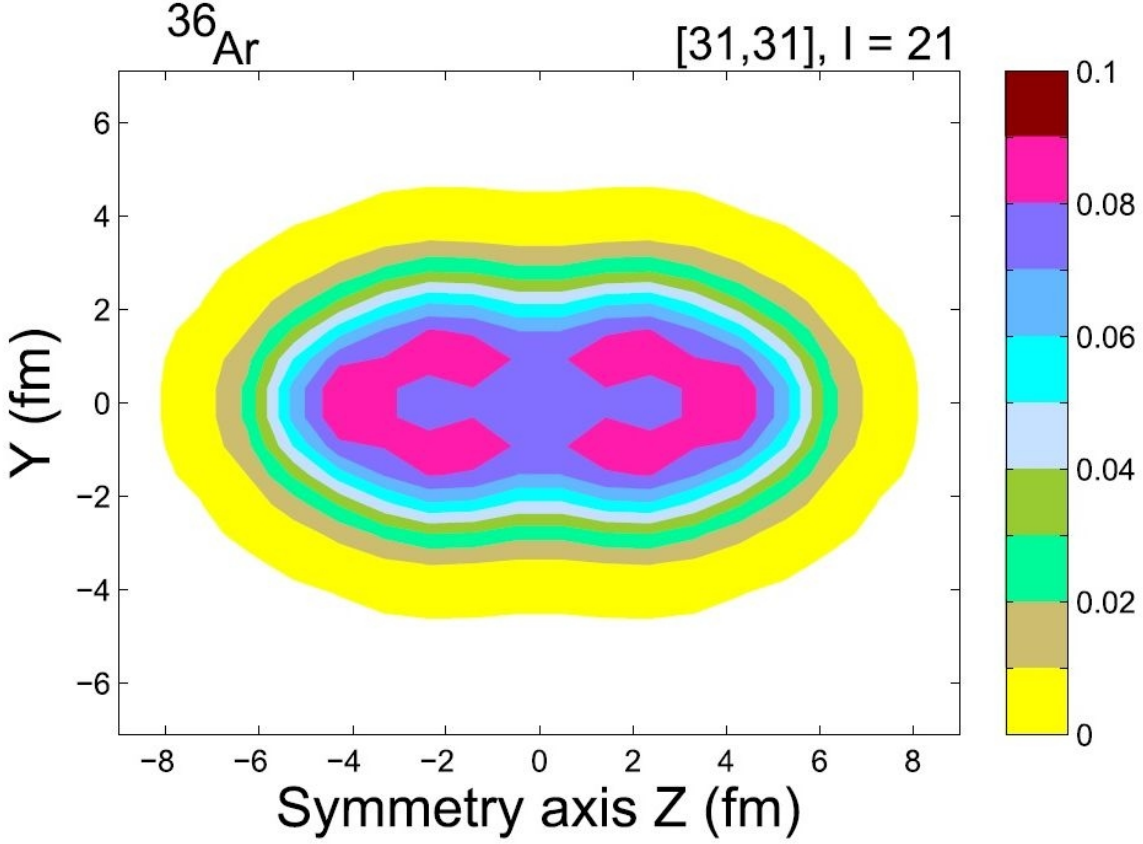
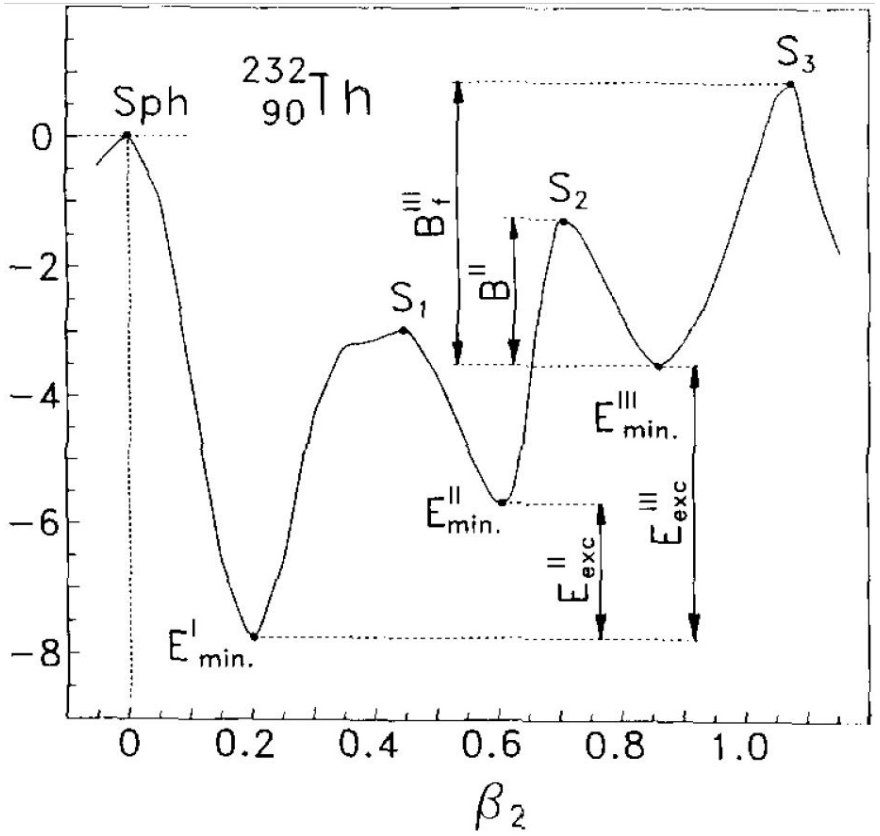


Figure 2. Total neutron density (in fm^{-3}) of the megadeformed [31, 31] configuration in ^{36}Ar obtained in the CRMF calculations with the NL3* functional.

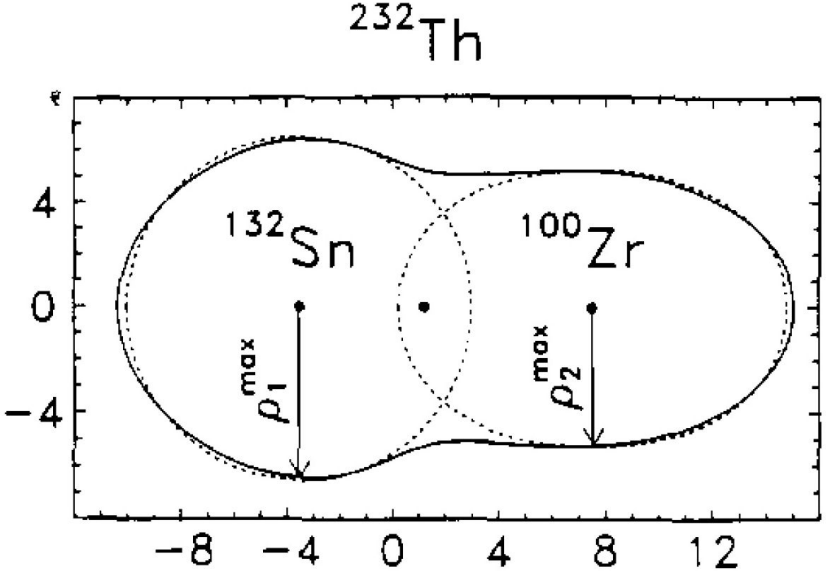
A V Afanasjev et al., Phys. Scr. ***93*** *034002 (2018)*

Mean field models and clusterization

- Hyperdeformed nuclear states are similar to the nuclear molecules



Potential energy as a function β_2 along fission path (fission barrier)

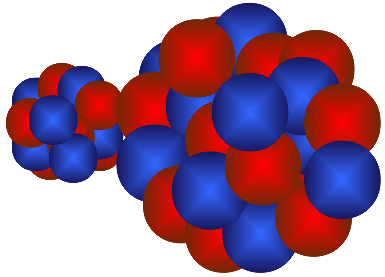


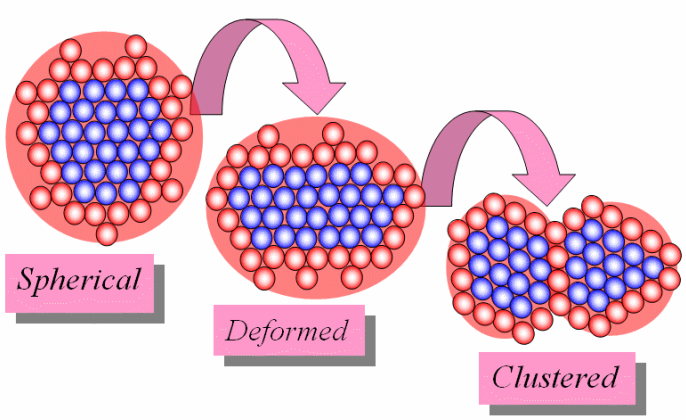
Nuclear shape (HD) in the third minimum

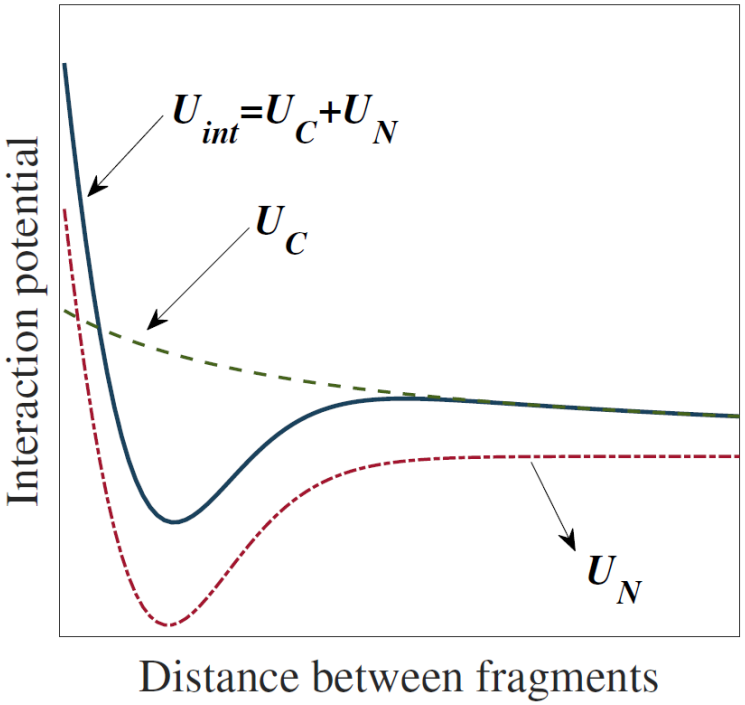
S. Cwiok ,W . Nazarewicz et al., Phys. Lett. B ***322 (1994) 304-310***

Dinuclear Sytem

Dinuclear System (Nuclear Molecule) - two clusters in touching

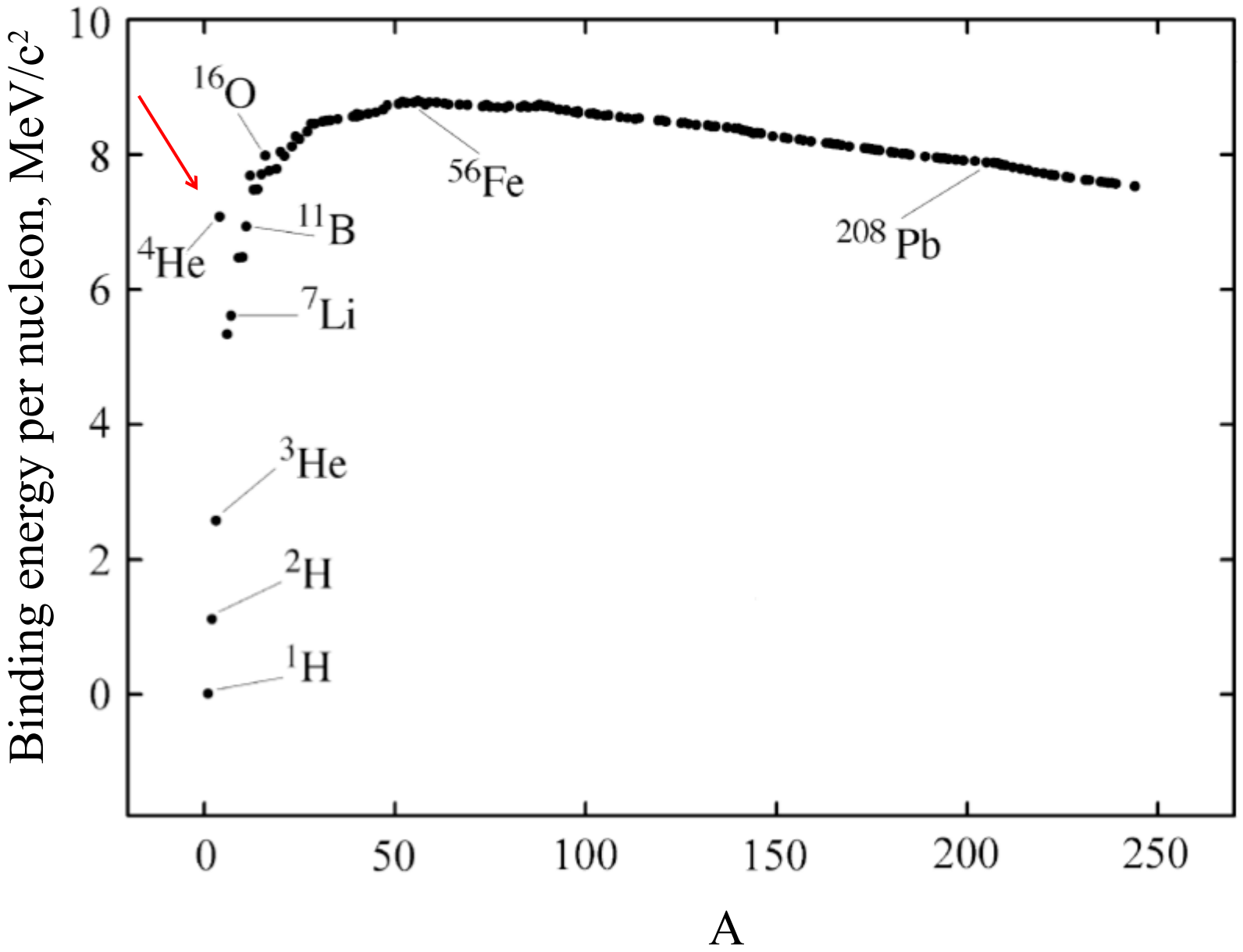






$$V = -B(A_1, Z_1) - B(A_2, Z_2) + V_N(R, \{A_i, Z_i\}) + V_C(R, \{A_i, Z_i\})$$

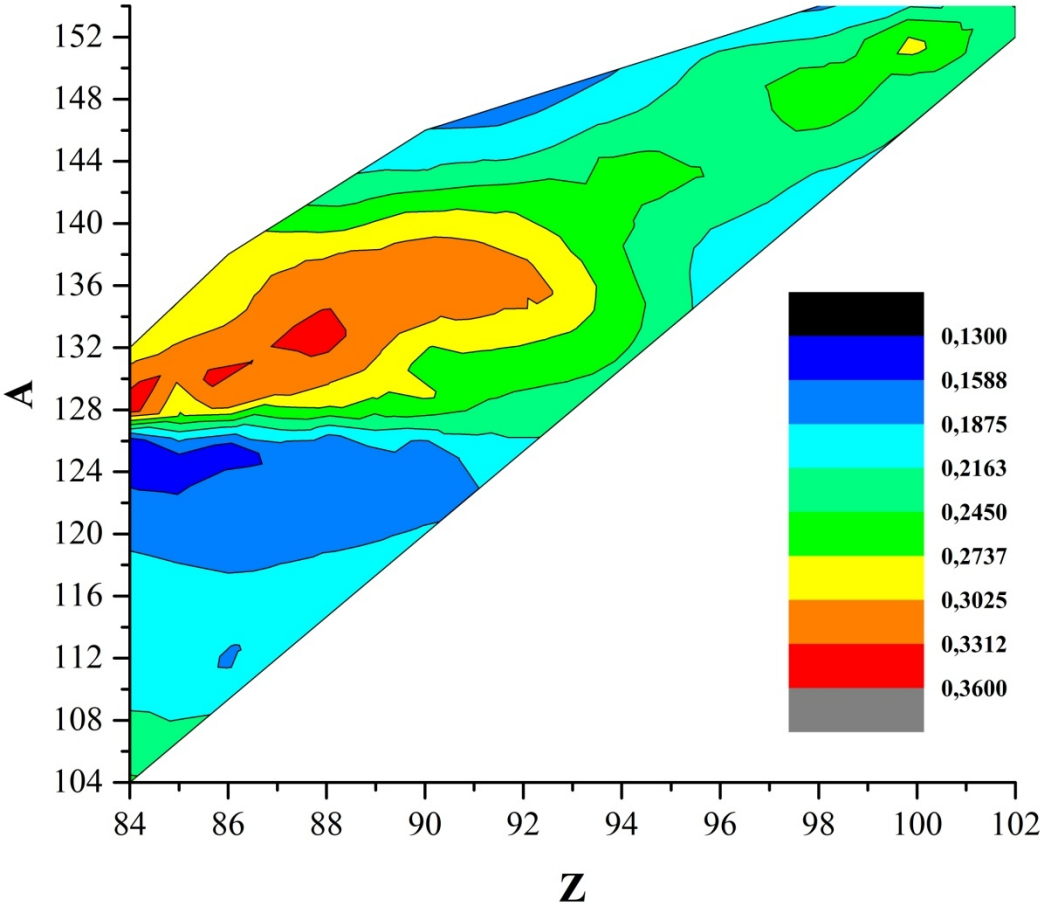
Binding Energy of Nucleus

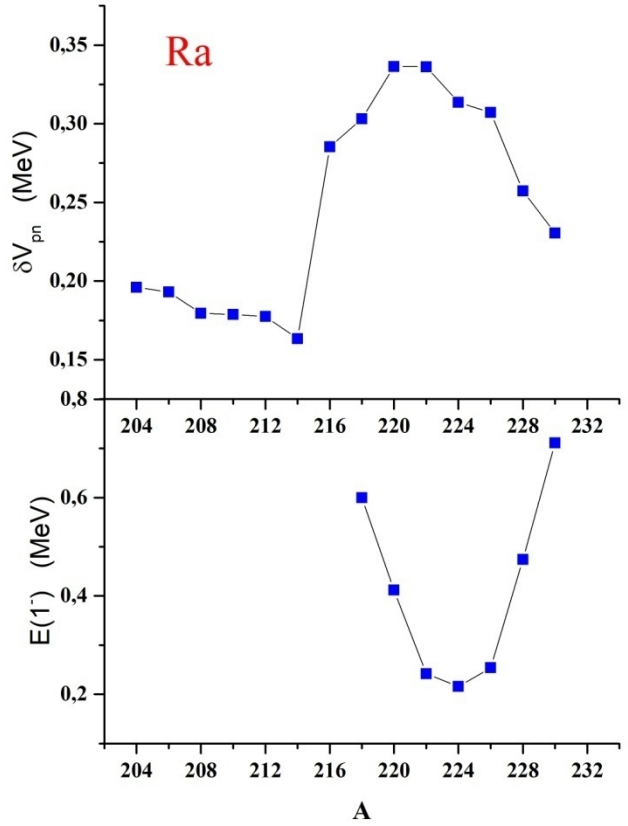


Role of alpha-clusterization

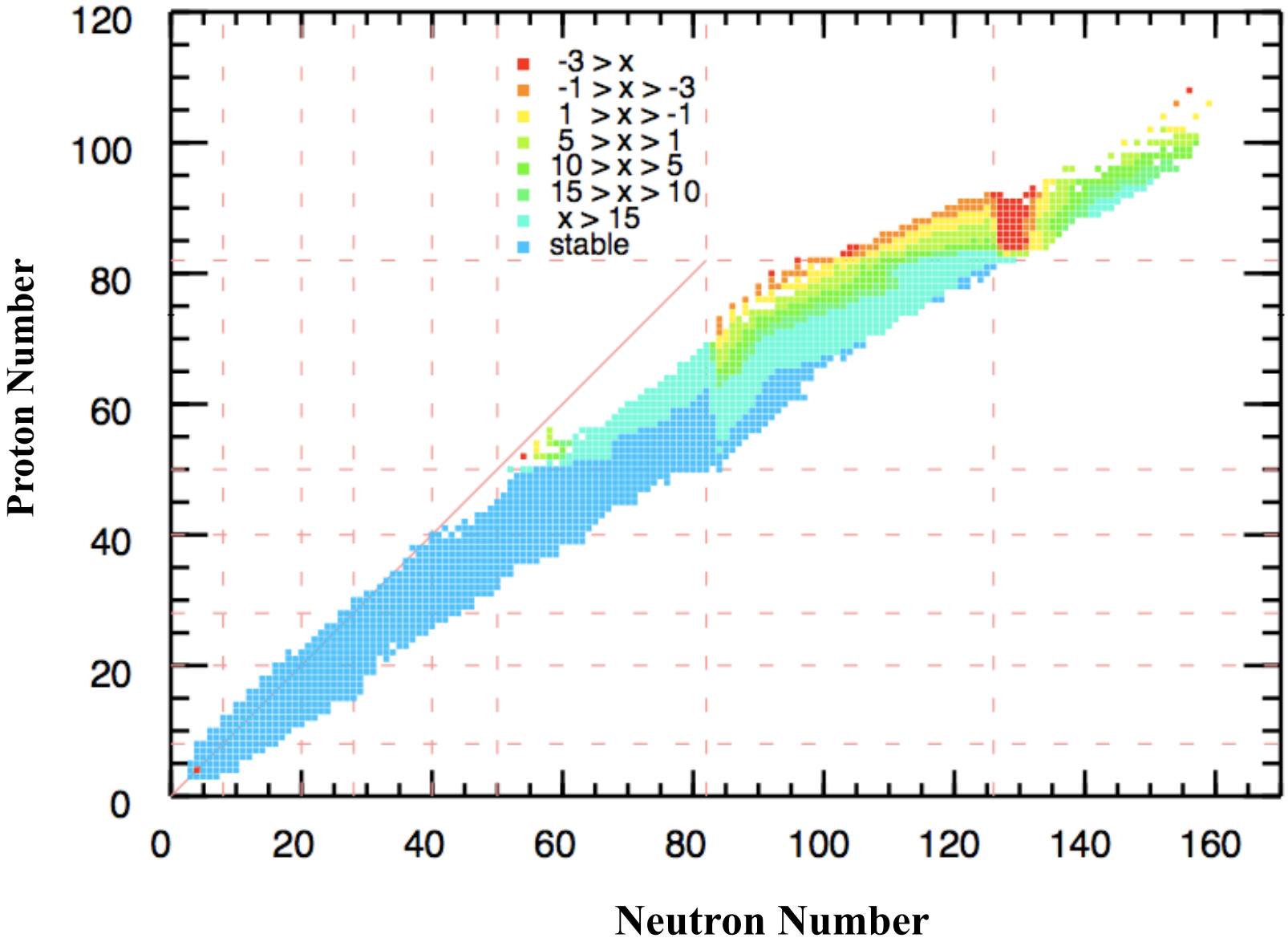
$$\delta V_{pn} = -0.25\left[B(Z,N) - B(Z,N-2) - B(Z-2,N) + B(Z-2,N-2)\right]$$
$$= -0.25\left[S_{2n} + S_{2p} - S_{\alpha}\right]$$

R.F. Casten & R.B. Cakirli, Phys. Scri 91, 033004 (2016)





Alpha-decay


-3 > x
-1 > x > -3
1 > x > -1
5 > x > 1
10 > x > 5
15 > x > 10
x > 15
stable
Proton Number
Neutron Number
0
20
40
60
80
100
120
140
160

α-распад

Одним из наиболее прямых указаний на кластеризацию в тяжелых ядрах является существование α- и кластерных распадов.

Период полураспада

$$T_{1/2} = \frac{\hbar \ln 2}{\Gamma}$$

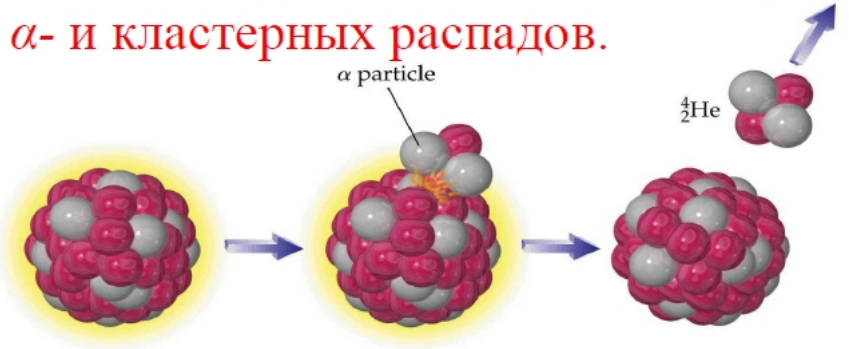
Ширина распада

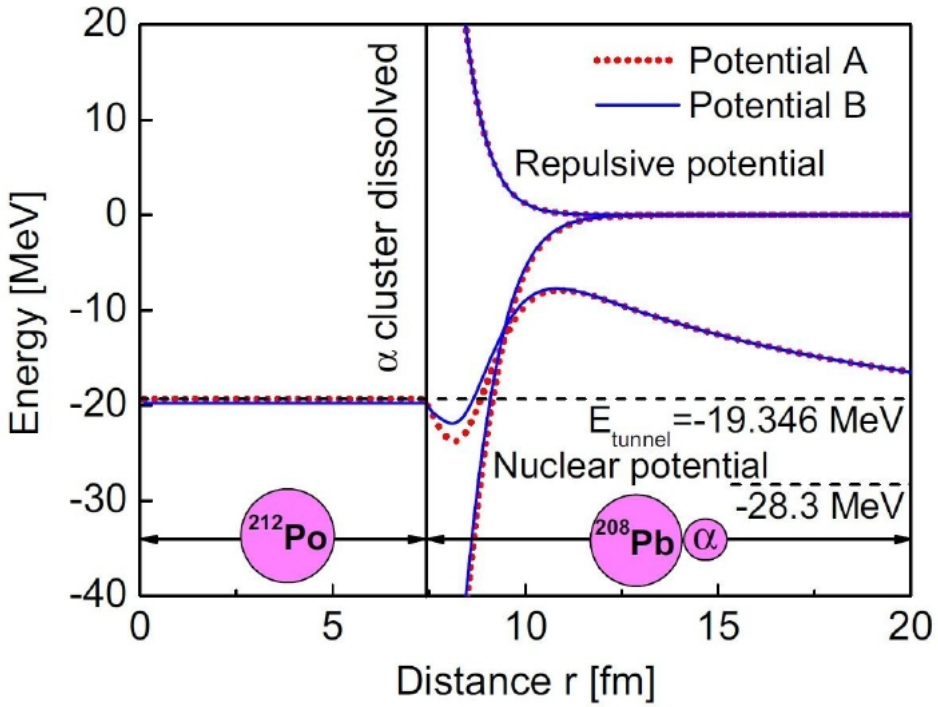
$$\Gamma = \frac{\hbar \omega_\alpha}{\pi} S_\alpha P_\alpha$$

ω_α - частота соударений

S_α - спектроскопический фактор

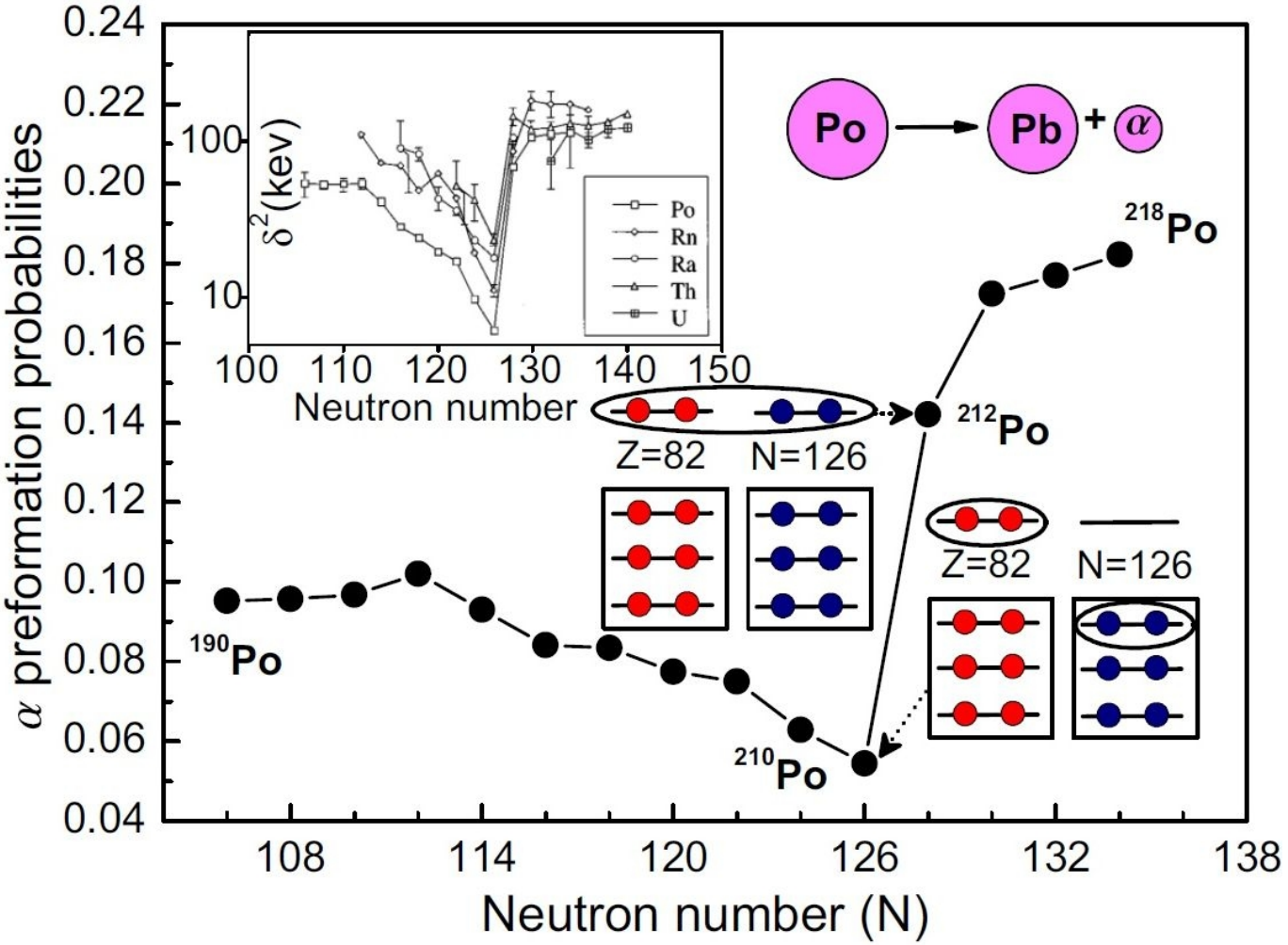
P_α - вероятность туннелирования





A. Tohsaki, H. Horiuchi, P. Schuck, and G. Röpke, PRL, **87,** 192501 **(2001).**

Spectroscopic factors



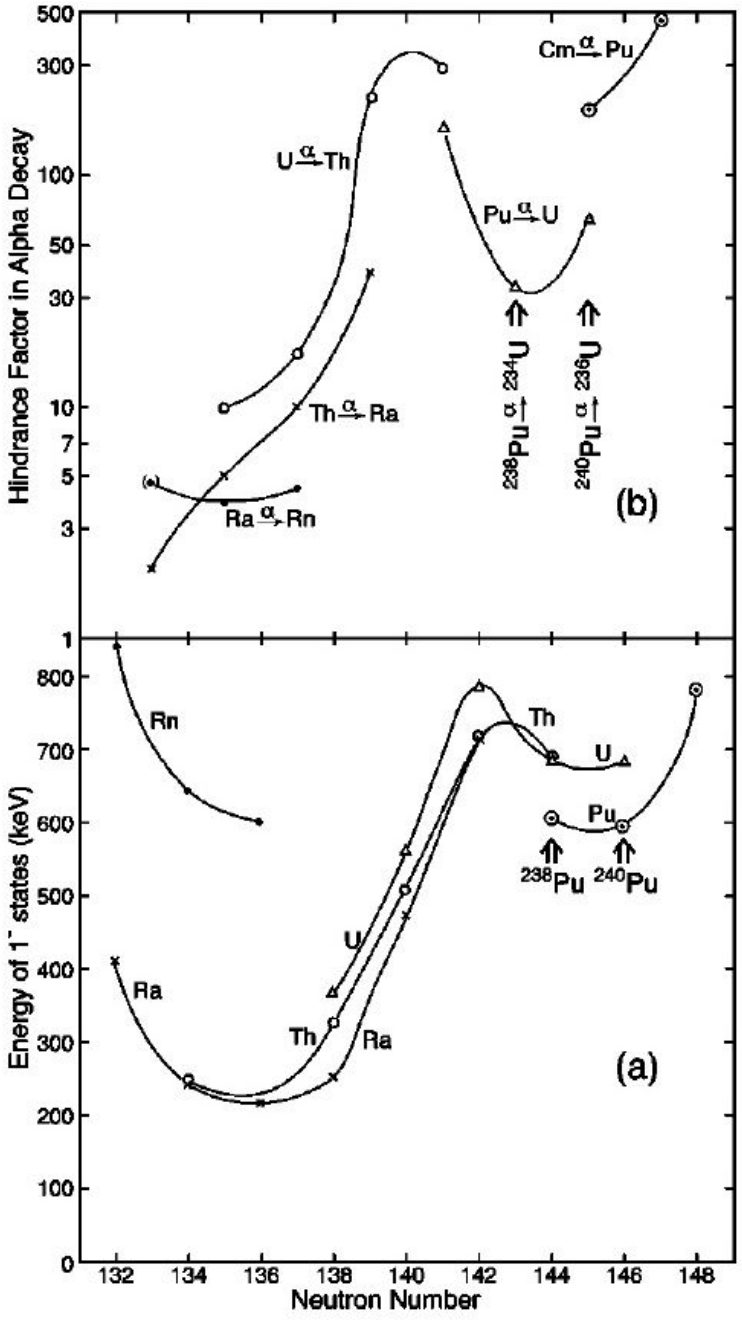
Chang Xu et al., PRC ***95, 061306(R) (2017)***

Alpha-decay and octupole correlations in actinides

A very low hindrance factor is observed in actinides for alpha decay from the nuclear ground state to low-lying 1^- states of the daughter nucleus.
*(R.K. Sheline, M. A. Riley, PRC **61, 057301 (2000)**)*

The nature of these states is related to the excitation of a collective mode that breaks the reflection (mirror) symmetry of the nucleus (octupole vibrations).
(*K. Alder, A. Bohr, T. Huus, B. Mottelson, A. Winther, Rev. Mod. Phys.* ***28, 432 (1956).***

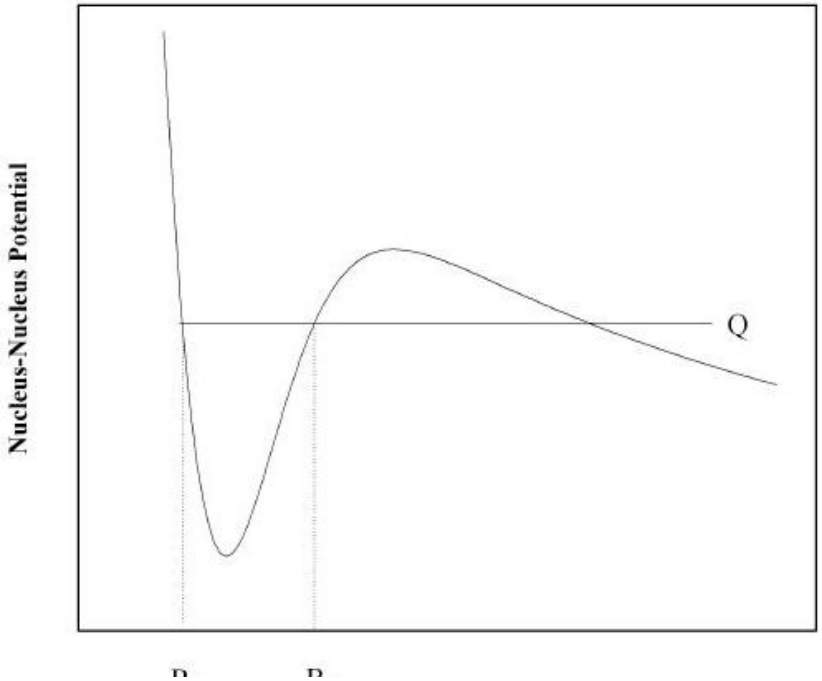
Formation of cluster at the nuclear surface→ mirror asymmetry
(*T.M. Shneidman, G.G. Adamian, N.V. Antonenko, R.V. Jolos, W. Scheid, PLB* ***526****, 322 (2002).*



WKB Approximation

According to the WKB approximation the probability

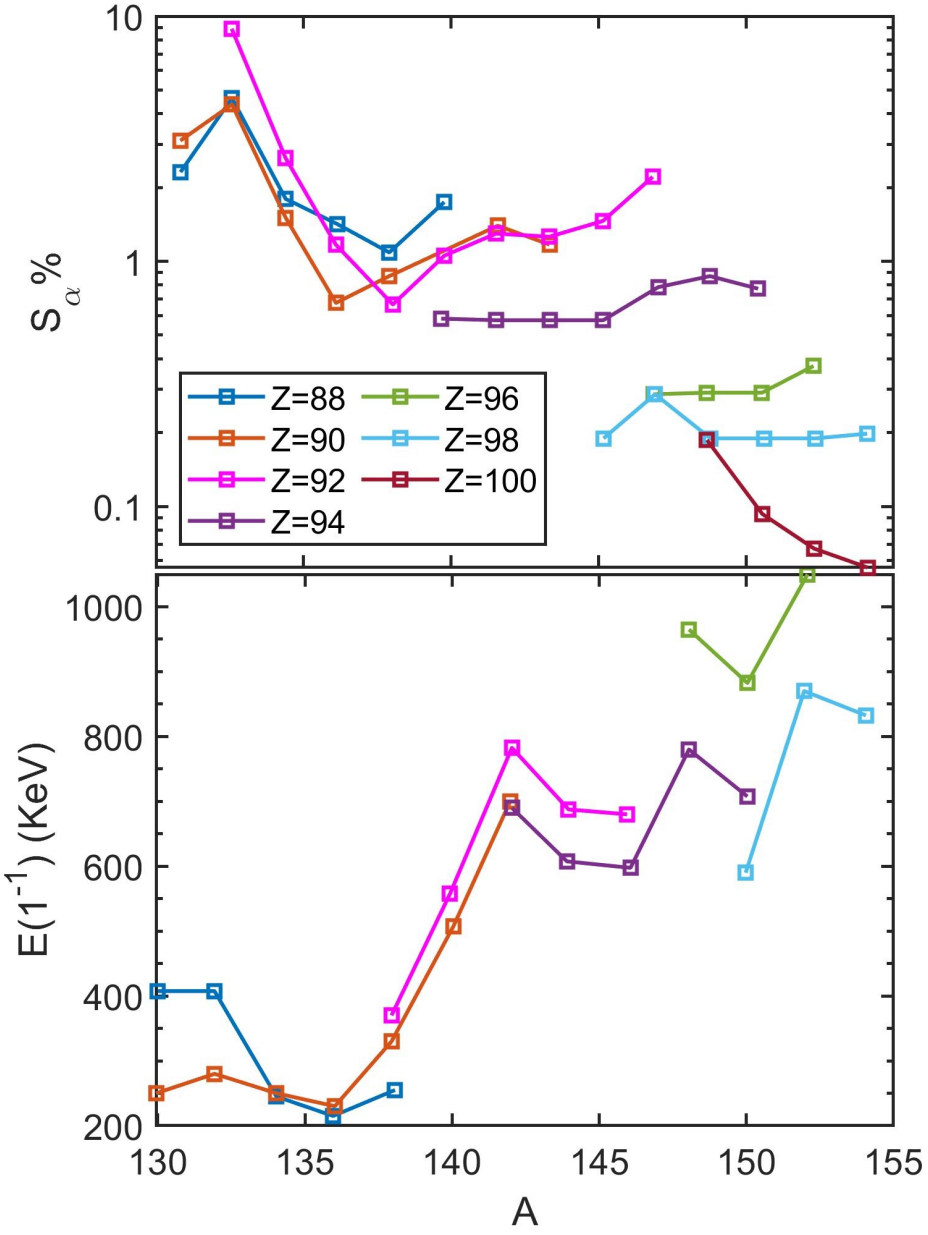
$$P_\alpha = \exp\left[-\frac{2}{\hbar}\int_{R_1}^{R_2}\sqrt{2\mu(U(R,\eta_\alpha)-Q)}\right]$$



R_1, R_2 – are the turning points of the WKB action intergral determined by the equation

$$U(R_{1,2}, \eta_\alpha) = Q$$

Alpha-decay and octupole correlations in actinides



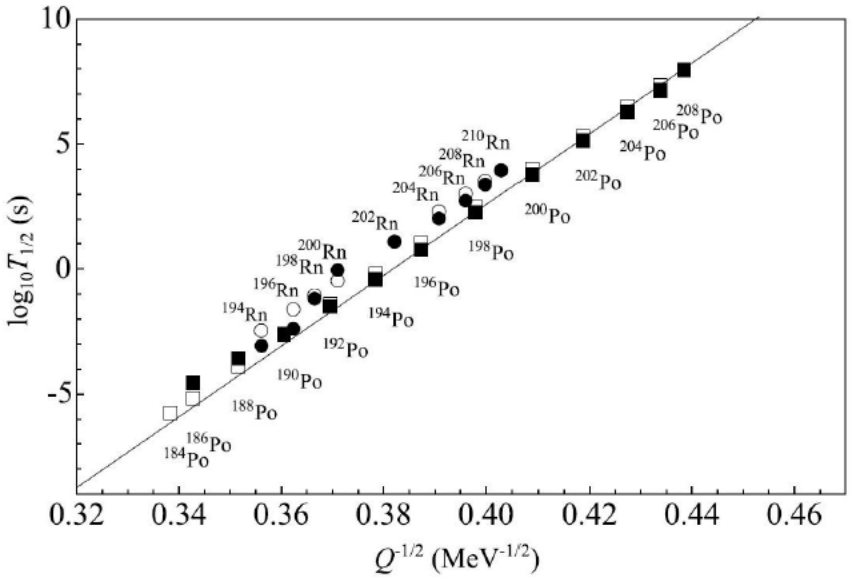
The alpha-particle spectroscopic factor can be estimated from experimental data on nuclear half-lives with respect to alpha decay.

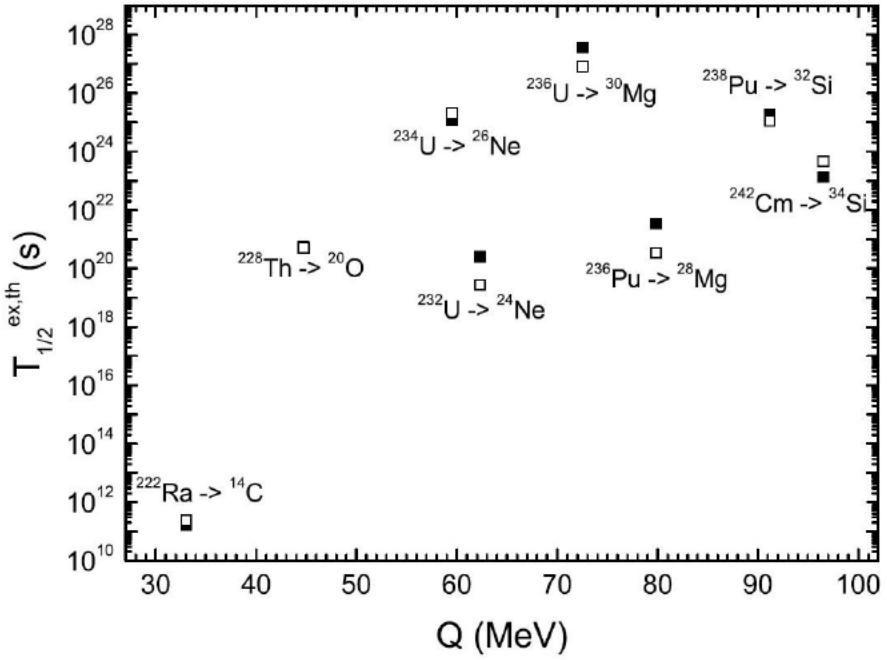
$$S_\alpha = T_{1/2} / T_{1/2}^{\exp}$$

α- и кластерный распады

Спектроскопический фактор можно связать с весом кластерной системы в полной волновой функции ядра

$$\Psi(r_1, r_2, \ldots.) = \Psi_m(r_1, r_2, \ldots.) + S_\alpha{}^{1/2}\Psi_\alpha(r_1, r_2, \ldots.) + \ldots + S_C{}^{1/2}\Psi_C(r_1, r_2, \ldots.) + \ldots$$





S.N. Kuklin, T.M. Shneidman, G.G. Adamian, N.V. Antonenko, Eur. Phys. J. A (2012) ***48: 112***

Cluster model-1

--H. Schultheis, R. Schultheis and K. Wildermuth, PLB. 53 ***325*** *(1974).*
--Wildermuth and Tang, A unified theory of the nucleus, Vieweg, Braunschweig, 1977

$$\Psi\left(\vec{r}_1,...,\vec{r}_{A_0},t\right)=\sum_h\sum_{ijk}a^h_{ijk}(t)\psi^h_{ijk}\left(\vec{r}_1,...,\vec{r}_{A_0}\right),\ \text{with}$$

$$\psi^h_{ijk}\left(\vec{r}_1,...,\vec{r}_{A_0}\right)=\hat{A}\left[\phi_i(A_h)\phi_j(A_l)\chi_k\left(\vec{R}\right)\right],\ \text{where}$$

$\phi_i(A_h)$: wave function of the i^{th} internal state of the heavy cluster $(A_h=Z_h+N_h)$

$\phi_j(A_l)$: wave function of the j^{th} internal state of the light cluster $(A_l=Z_0-Z_h+N_0-N_h)$

$\chi_k\left(\vec{R}\right)$: wave function of the k^{th} state of relative collective motion ($\vec{R}=\vec{R}_h-\vec{R}_l$, vector distance of the centres of mass of the two clusters)

$\hat{A}$: antisymmetrization operator

Cluster model-2

The two-cluster expansion given above allows for any overlap of clusters. If the overlap is strong, the antisymmetrization operator washes out the effects of cluster decomposition.

There are two regions where antisymmetrization effects play a minor role:

1) clusters well separated in momentum space → strong overlap in coordinate space ($\mathbf{R \approx 0}$) → no connection with nuclear shape → peculiar role of $Z = 82$ and $N = 126$ shell closures in actinide ground states;

2) clusters well separated in coordinate space → higly excited state of relative motion → clusters in low-lying internal states of excitation → directly connected with nuclear shape (reflection asymmetry).

Cluster model-3

On the basis of the above considerations, the two-cluster expansion can be written as the sum of two terms

$$\Psi\left(\vec{r}_1,\ldots,\vec{r}_{A_0};s_\alpha\right)=\Phi_I\left(\vec{r}_1,\ldots,\vec{r}_{A_0};s_\alpha\right)+\Phi_{II}\left(\vec{r}_1,\ldots,\vec{r}_{A_0};s_\alpha\right),\ \text{where}$$

$s_\alpha=(\alpha_1,\alpha_2,\ldots)$ is a set of collective coordinates describing the nuclear shape,

$\Phi_I\left(\vec{r}_1,\ldots,\vec{r}_{A_0};s_\alpha\right)$: linear combination of two-cluster states with overlapping clusters → interior of the intermediate nucleus,

$\Phi_{II}\left(\vec{r}_1,\ldots,\vec{r}_{A_0};s_\alpha\right)$: linear combination of two-cluster states with clusters in touching configuration → surface region of the intermediate nucleus.

Cluster model-4

To begin with, one defines the **mass asymmetry** coordinate

$\eta = (A_1 - A_2)/(A_1 + A_2)$ (mononucleus: $\eta = \pm 1$; symmetric configuration: $\eta = 0$)

or, more commonly

$\xi = 1 - \eta = 2A_2/(A_1 + A_2)$ (mononucleus: $\xi = 0,2$; symmetric configuration: $\xi = 1$)

and, correspondingly, the **charge asymmetry** coordinate

$\eta_Z = (Z_1 - Z_2)/(Z_1 + Z_2) \rightarrow \xi_Z = 1 - \eta_Z = 2Z_2/(Z_1 + Z_2)$

Cluster effects are all included in the Φ_{II} function ; neglecting antisymmetrization,

$$\Phi_{II}(\vec{r}_1, \vec{r}_2, ...) = \sum_{\xi} a_{\xi} \Phi_{\xi}(\vec{r}_1, \vec{r}_2, ...), \text{ with}$$

$$\Phi_{\xi}^{(LM)}(\vec{r}_1, \vec{r}_2, ...) = \sum_{ijk} b_{ijk} \left[\phi_i^{(l_1)}(A_1) \times \phi_j^{(l_2)}(A_2) \times g_k^{(l_0)}(\vec{R}) \right]^{(LM)}, \text{ where}$$

l_1 (l_2) : angular momentum of cluster of mass A_1 (A_2),

l_0 : angular momentum of relative motion ,

L : total angular momentum with 3rd component M .

Cluster model-5

In order to compute the collective excitations of the nucleus we need the wave function of the DNS at given **elongation** (touching configuration).

The simplifying assumptions are made:

The quadrupole deformations of the clusters are chosen so as to minimize the energy of the DNS.

Intrinsic excitations of the clusters are not allowed.

The relative distance, **R**, is not an independent variable and is fixed, for a given mass asymmetry, at the touching pole-to-pole configuration of the clusters.

Cluster model-6

Considering the mass asymmetry ξ as a continuous variable, the sum over ξ is replaced with an integral and the wave function of the intrinsic state can be written in the Hill-Wheeler form

$$\left|\Phi_{LM}\left(q_2;\vec{r}_1,...,\vec{r}_{A_0}\right)\right\rangle = \int b(\xi,\Omega)\left|\phi_{\mathrm{int}\,r}\left(\xi;\vec{r}_1,...,\vec{r}_{A_0}\right)\right\rangle \gamma(\xi)d\xi\,,$$

with $\Omega = (\Omega_1,\Omega_2,\Omega_0)$ and $\gamma(\xi)d\xi$ an infinitesimal volume element;

The weight function is

$$b(\xi,\Omega) = \sum_{l_0 l_1 l_2} a_{l_0 l_1 l_2}(\xi)\left[Y_{l_1}(\Omega_1)\times Y_{l_2}(\Omega_2)\times Y_{l_0}(\Omega_0)\right]_{LM}$$

and the intrinsic wave function is

$$\phi_{\mathrm{int}\,r}\left(\xi;\vec{r}_1,...,\vec{r}_{A_0}\right) = \phi_1\left(A_1,\beta_1,R_m(\xi);\{\vec{r}_i\}\right)\phi_2\left(A_2,\beta_2,R_m(\xi);\{\vec{r}_j\}\right).$$

The weight function can be derived from the Hill - Wheeler equation

$$\int\left\langle\phi_{\mathrm{int}\,r}\left(\xi;\vec{r}_1,...,\vec{r}_{A_0}\right)\right|\hat{H}-E\left|\phi_{\mathrm{int}\,r}\left(\xi;\vec{r}_1,...,\vec{r}_{A_0}\right)\right\rangle b(\xi,\Omega)\gamma(\xi)d\xi = 0\,.$$

Cluster model-7

On the assumption of a sharply peaked overlap integral the Hill-Wheeler equation can be rewritten in the form of a Schrödinger equation obeyed by the weight function ***b(ξ,Ω)***, depending on the collective coordinate ξ

$$-\frac{\hbar^2}{2M(\xi,E)}\frac{1}{\sqrt{\gamma(\xi)}}\frac{d}{d\xi}\left(\sqrt{\gamma(\xi)}\frac{db(\xi,\Omega)}{d\xi}\right)+V(\xi)b(\xi,\Omega)=Eb(\xi,\Omega),$$

where $M(\xi,E)$ is an effective mass, $\gamma(\xi)\equiv\mu^3(\xi)$, with $\mu(\xi)$ the reduced mass and the potential energy is

$$V(\xi)=\left\langle\phi_{\text{int}\,r}(\xi,\vec{r}_1,...,\vec{r}_{A_0})\middle|\hat{H}\middle|\phi_{\text{int}\,r}(\xi,\vec{r}_1,...,\vec{r}_{A_0})\right\rangle.$$

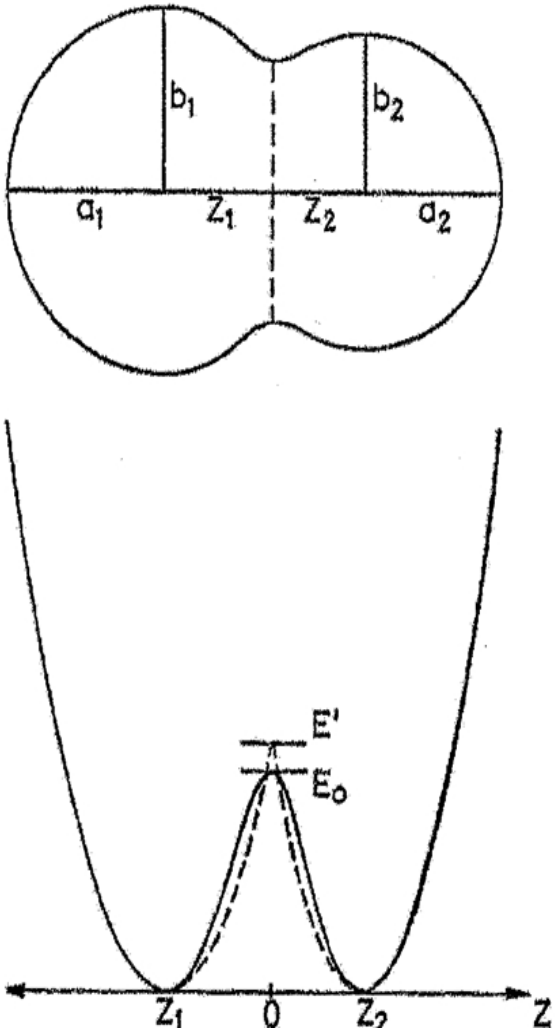
$\hat{H}$ is the sum of the Hamiltonians of the two clusters and of their interaction.

One transforms from the laboratory frame to the intrinsic coordinate systems of the two clusters:

$$\vec{r}_i=\vec{R}_1+D(\Omega_1)\tilde{r}_i\ ,\ (i=1,...,A_1)\qquad \vec{r}_j=\vec{R}_2+D(\Omega_2)\tilde{r}_j\ ,\ (i=A_1+1,,...,A_0)$$

where D is a Wigner rotation matrix.

Two-center shell model-1



Two-center shell model-2

In the original version with harmonic oscillator potentials the Hamiltonian in cylindrical coordinates z,ρ,φ is

$$H=-\frac{\hbar^2\nabla^2}{2m_N}+V(\rho,z)+V_{LS}(\vec{r},\vec{p},\vec{s})+V_{L^2}(\vec{r},\vec{l}),\ \text{where}$$

$$V(\rho,z)=\begin{cases}\dfrac{1}{2}m_N\omega_{z1}^2z'^2+\dfrac{1}{2}m_N\omega_{\rho1}^2\rho^2,\ z<z_1\\ \dfrac{f_0}{2}m_N\omega_{z1}^2z'^2\left(1+c_1z'+d_1z'^2\right)+\dfrac{1}{2}m_N\omega_{\rho1}^2\rho^2\left(1+g_1z'^2\right)\rho^2,\ z_1<z<0\\ \dfrac{f_0}{2}m_N\omega_{z2}^2z'^2\left(1+c_2z'+d_2z'^2\right)+\dfrac{1}{2}m_N\omega_{\rho2}^2\rho^2\left(1+g_2z'^2\right)\rho^2,\ 0<z<z_2\\ \dfrac{1}{2}m_N\omega_{z2}^2z'^2+\dfrac{1}{2}m_N\omega_{\rho2}^2\rho^2,\ z>z_2\end{cases}$$

with

$$z'=\begin{cases}z-z_1,\ z<0\\ z-z_2,\ z>0\end{cases}$$

Two-center shell model-3

The model has five parameters ($\rightarrow$ collective coordinates)

$\frac{\omega_{\rho 2}}{\omega_{\rho 1}}$: mass asymmetry

$\frac{\omega_{zi}}{\omega_{\rho i}}$ $(i = 1,2)$: ellipsoidal deformations of the prefragments

$z_2 - z_1$: separation of the prefragments (elongation)

$\frac{E_0}{E'}$: neck parameter, permitting a variable barrier height

$$\left(E' = \frac{1}{2} m_N \omega_{z1} z_1^2 = \frac{1}{2} m_N \omega_{z2} z_2^2 \right)$$

Potential Energy of the Dinuclear System

$$U(R,\xi,\beta_{2\mu}^{(1)},\beta_{2\mu}^{(2)}) = B_1(\beta^{(1)}) + B_2(\beta^{(2)}) - B_{12} + V(R,\xi,\beta_{2\mu})$$

where, B_1, B_2 and B_{12} are the binding energies of the fragments and the compound nucleus, respectively.

The nucleus-nucleus potential

$$V(R,\xi,\beta_{2\mu}^{(1)},\beta_{2\mu}^{(2)}) = V_{Coul}(R,\xi,\beta_{2\mu}^{(1)},\beta_{2\mu}^{(2)}) + V_{nucl}(R,\xi,\beta_{2\mu}^{(1)},\beta_{2\mu}^{(2)})$$

is the sum of the nuclear interaction potential $V_{nucl}(R,\xi,\beta_{2\mu}^{(1)},\beta_{2\mu}^{(2)})$ and of the Coulomb potential

$$V_{Coul}(R,\xi,\beta_{2\mu}) = \frac{e^2 Z_1 Z_2}{R} + \frac{3}{5}\frac{e^2 Z_1 Z_2}{R^3} R_{01}^2 \sum_{i,\mu} \beta_{2\mu}^{(i)*} Y_{2\mu}(\theta_i,\phi_i) + \ldots$$

Nuclear Interaction in Dinuclear System

$$V_{nucl}(R,\xi,\beta_{2\mu}) = \int \rho_1(\mathbf{r}_1)\rho_2(\mathbf{R}-\mathbf{r}_2)F(\mathbf{r}_1-\mathbf{r}_2)\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2$$

$$\rho_i(\mathbf{r}) = \frac{\rho_{00}}{1+\exp\left(\frac{s(\mathrm{r})}{a_{0i}}\right)}, \quad \rho_{00} = 0.17 \text{ fm}^{-3}$$

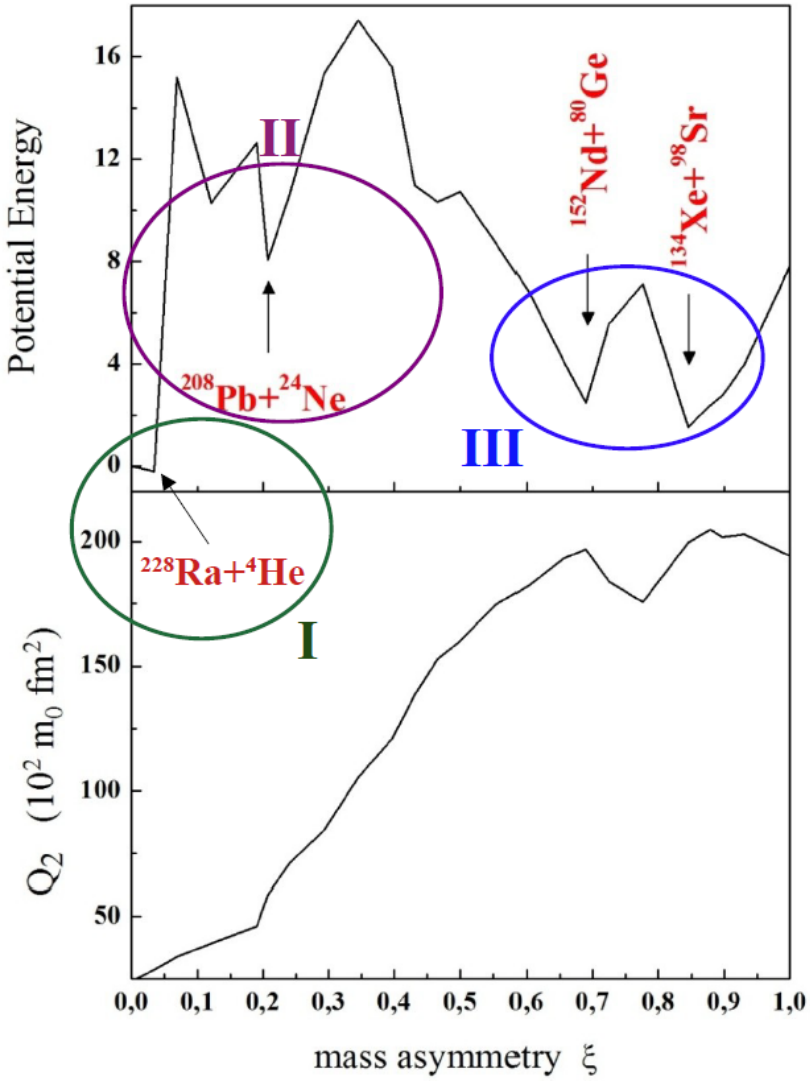
$$F(\mathbf{r}_1-\mathbf{r}_2) = C_0\left(F_{in}\frac{\rho_0(\mathbf{r}_1)}{\rho_{00}} + F_{ex}\left(1-\frac{\rho_0(\mathbf{r}_1)}{\rho_{00}}\right)\right)\delta(\mathbf{r}_1-\mathbf{r}_2)$$

$$\rho_0(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$$

$$F_{in,ex} = f_{in,ex} + f'_{in,ex}\frac{N_1-Z_1}{A_1}\frac{N_2-Z_2}{A_2}$$

$$C_0 = 300 \text{ MeV fm}^3, \; f_{in} = 0.09, \; f_{ex} = -2.59, \; f'_{in} = 0.42, \; f'_{ex} = 0.54$$

Потенциальная энергия для ^{232}U как функция массовой асимметрии.



$$V(\xi) = B_1(\xi) + B_2(\xi) + V_N(R,\xi) + V_C(R,\xi)$$

I: свойства основного состояния, α-распад

II: СД-состояния, Кластерный распад

III: ГД-состояния, деление

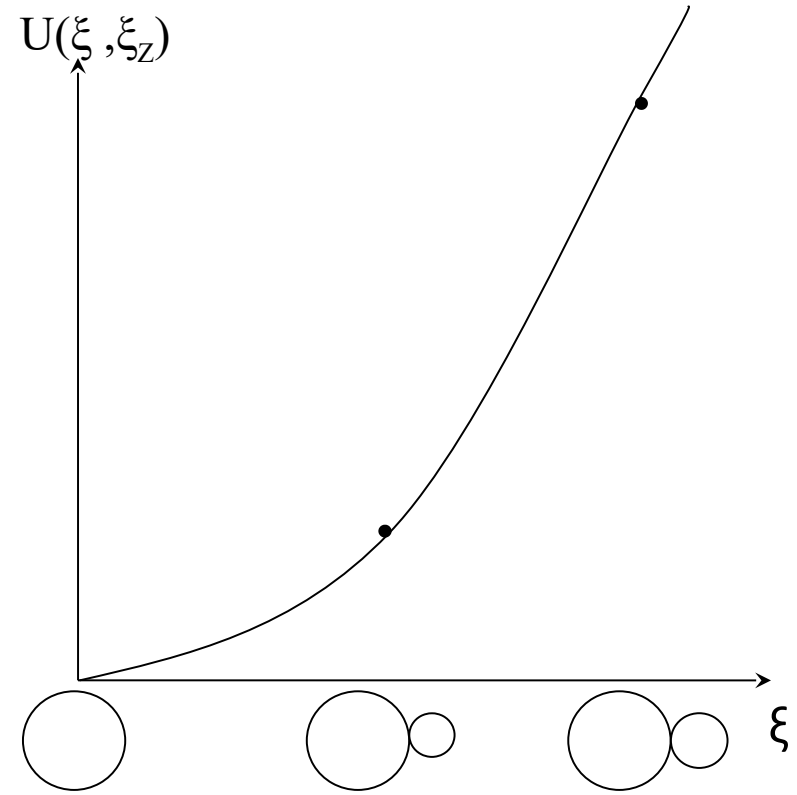
деформации ядер взяты из ***S.Raman et al., At. Data and Nuclear Data tables, Vol. 78, 2001***

Potential Energy of the Dinuclear System

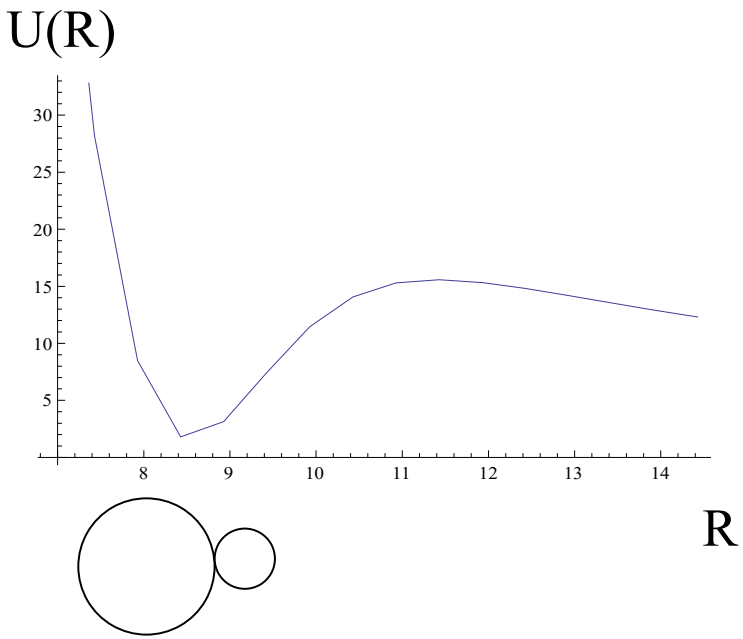
$$U(\xi,\xi_Z) = B_1(\xi,\xi_Z) + B_2(\xi,\xi_Z) - B + V(\mathbf{R}=\mathbf{R_m},\xi,\xi_\mathbf{Z})$$

$$V(\mathbf{R},\xi,\xi_\mathbf{Z}) = \mathbf{V_{coul}}(\mathbf{R},\xi,\xi_\mathbf{Z}) + \mathbf{V_N}(\mathbf{R},\xi,\xi_\mathbf{Z})$$

$$V_N(\mathbf{R},\xi,\xi_Z) = \int \rho_1(\mathbf{r_1})\rho_2(\mathbf{r_2})F(\mathbf{r_1}-\mathbf{r_2}+\mathbf{R})d\mathbf{r_1}d\mathbf{r_2}$$



mononucl. *$^{216}Ra+^{4}He$* *$^{213}Fr+^{7}Li$*



Touching configuration
$R=R_m$

Quantization procedure

The method outlined above is applicable to only one generator coordinate (mass asymmetry ξ), but, since more collective coordinates are necessary to describe collective excitations and fission, it would become too cumbersome for practical use.

It is more convenient to write down the classical Hamilton function appropriate to the model and then quantize it by standard procedures. If the classical kinetic energy is of the form

$$T = \frac{1}{2}\sum_{\mu\nu} g_{\mu\nu}(\{q\})\dot{q}_\mu \dot{q}_\nu$$

the quantum - mechanical kinetic operator is

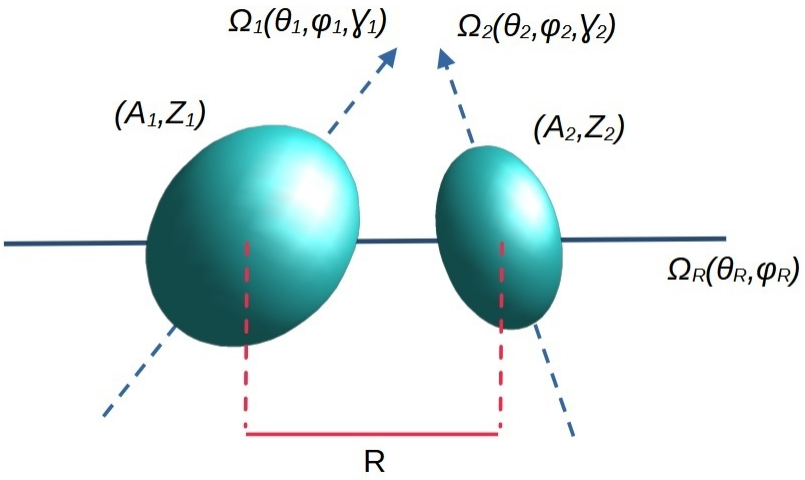
(Pauli prescription)

$$\hat{T} = -\frac{\hbar^2}{2}\sum_{\mu\nu}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial q_\mu}\sqrt{\det g}\, g^{-1}_{\mu\nu}\frac{\partial}{\partial q_\nu}$$

and the volume element is $d\tau = \left|\sqrt{\det g}\right|\prod_\mu q_\mu$.

DNS model

The wave function is a superposition of binary components $(A_1,Z_1)+(A_2,Z_2)$, including mononucleus $(A_1=A, Z_1=Z)+(A_2=0, Z_2=0)$.



Mass asymmetry $\xi = \dfrac{2A_2}{A_1 + A_2}$

Charge asymmetry $\xi_Z = \dfrac{2Z_2}{Z_1 + Z_2}$

To describe the dynamics of motion in such system we shall treat:

- *relative motion of the fragments;*
- *rotation of fragments;*
- *intrinsic excitations of fragments (collective vibrations);*
- *transfer of nucleons.*

Hamiltonian of the DNS model

$$\hat{T} = -\frac{\hbar^2}{2B(\xi_0)}\frac{1}{\mu^{3/2}(\xi)}\frac{\partial}{\partial\xi}\mu^{3/2}(\xi)\frac{\partial}{\partial\xi} - \frac{\hbar^2}{2\mu(\xi)}\frac{1}{R^2}\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R}$$

$$+\frac{\hbar^2}{2\mu(\xi)R^2}\hat{l}_0^2 + \frac{\hbar^2}{2}\sum_{n=1}^{2}\sum_{k=1}^{3}\frac{\hat{l}_{(n)k}^2}{I_k^{(n)}(\beta_n,\gamma_n)} \qquad \left(\equiv \hat{T}_{rot}\right)$$

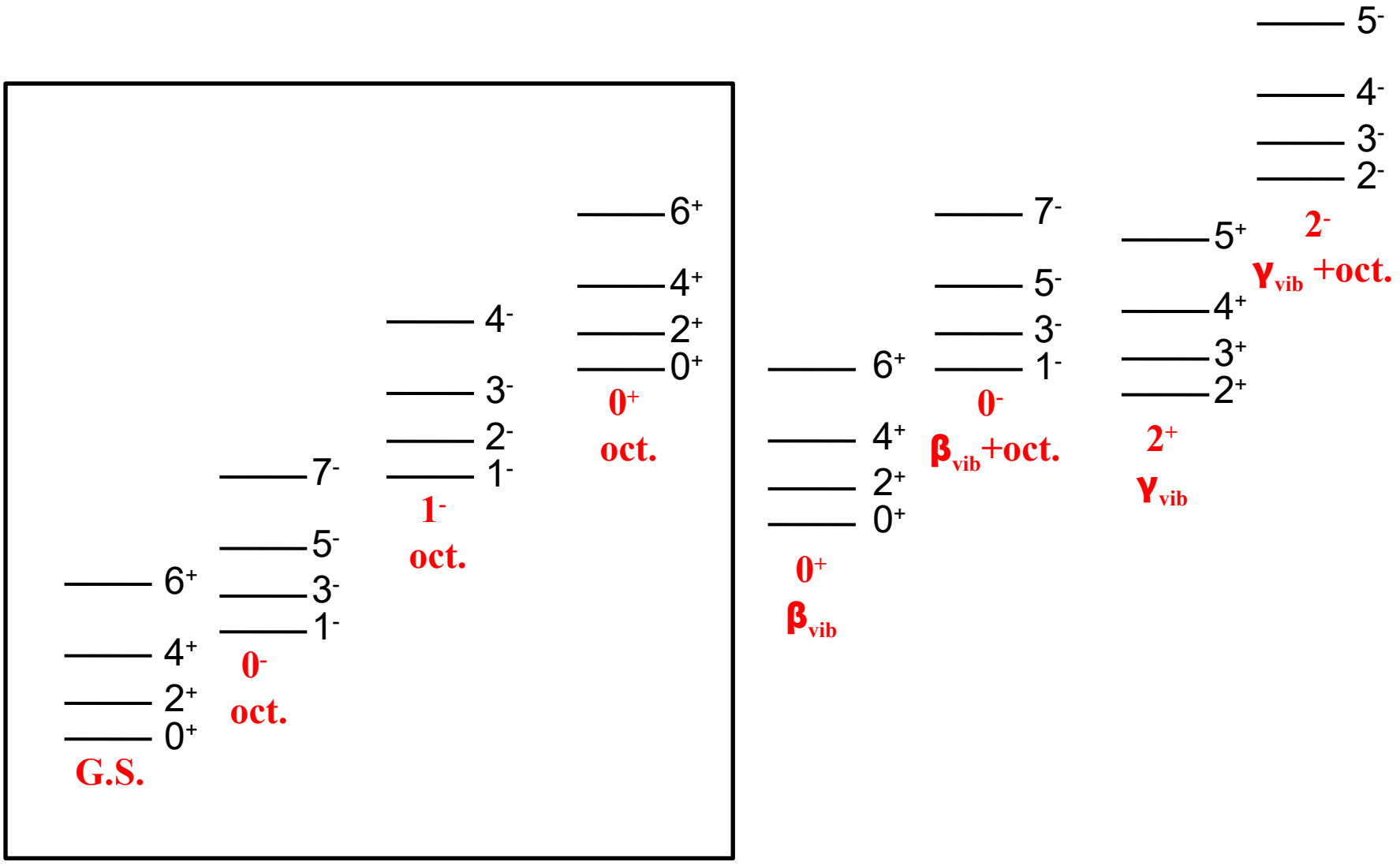
$$-\frac{\hbar^2}{2}\sum_{n=1}^{2}\frac{1}{D_n(\xi_0)}\left(\frac{1}{\beta_n^4}\frac{\partial}{\partial\beta_n}\beta_n^4\frac{\partial}{\partial\beta_n} + \frac{1}{\beta_n^2}\frac{1}{\sin 3\gamma_n}\frac{\partial}{\partial\gamma_n}\sin 3\gamma_n\frac{\partial}{\partial\gamma_n}\right)$$

$$\left(\equiv \hat{T}_{\mathrm{int}\,r}\right)$$

The potential energy of the DNS is

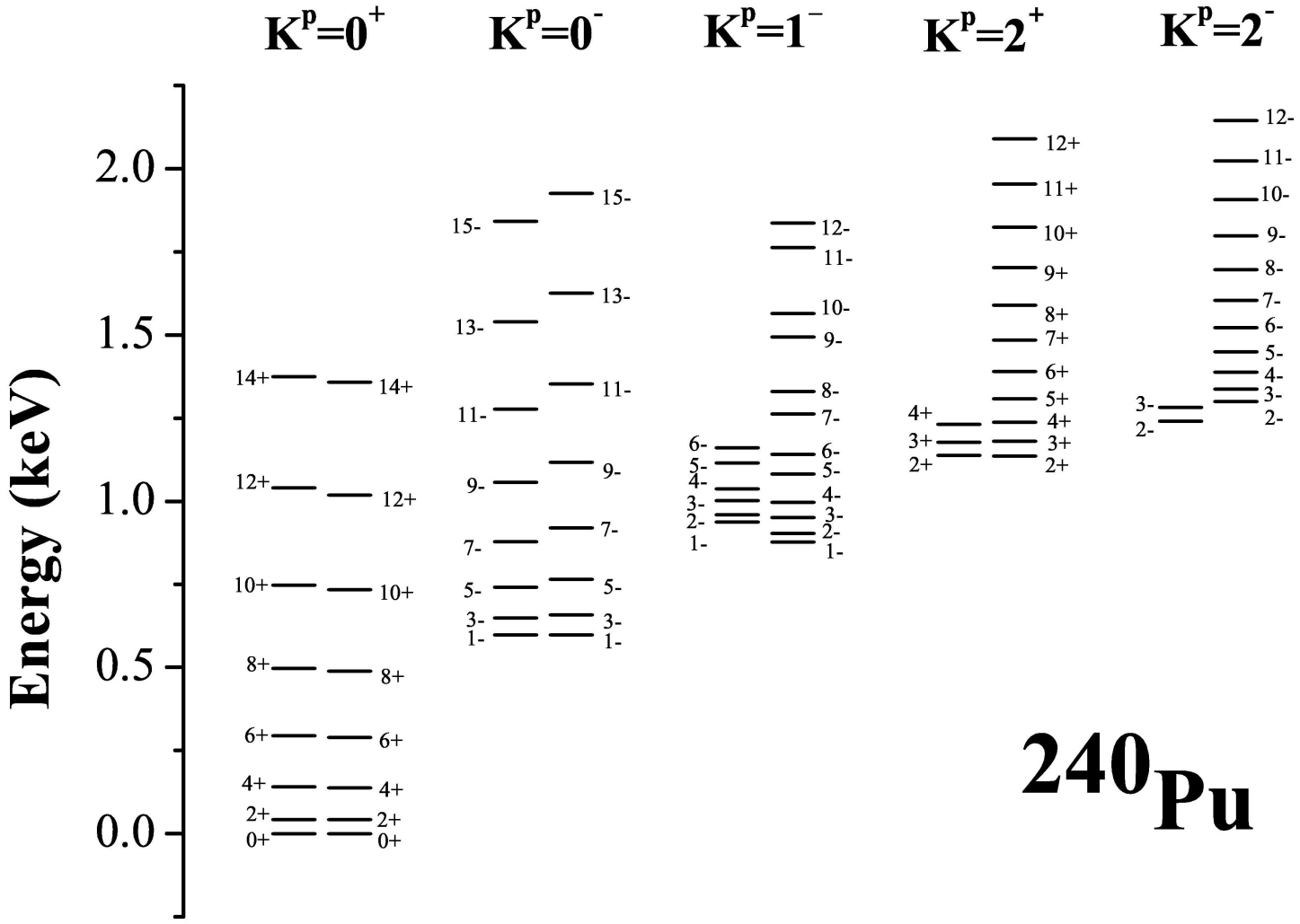
$$V(\xi) = E_1(\xi,\beta_1,\gamma_1) + E_2(\xi,\beta_2,\gamma_2) + V_N(R,\xi,\beta_{\{1,2\}},\gamma_{\{1,2\}},\Omega_{\{1,2\}}) + V_C(R,\xi,\beta_{\{1,2\}},\gamma_{\{1,2\}},\Omega_{\{1,2\}})$$

Schematic Spectrum Produced by DNS Hamiltonian



Excitation spectrum of ^{240}Pu

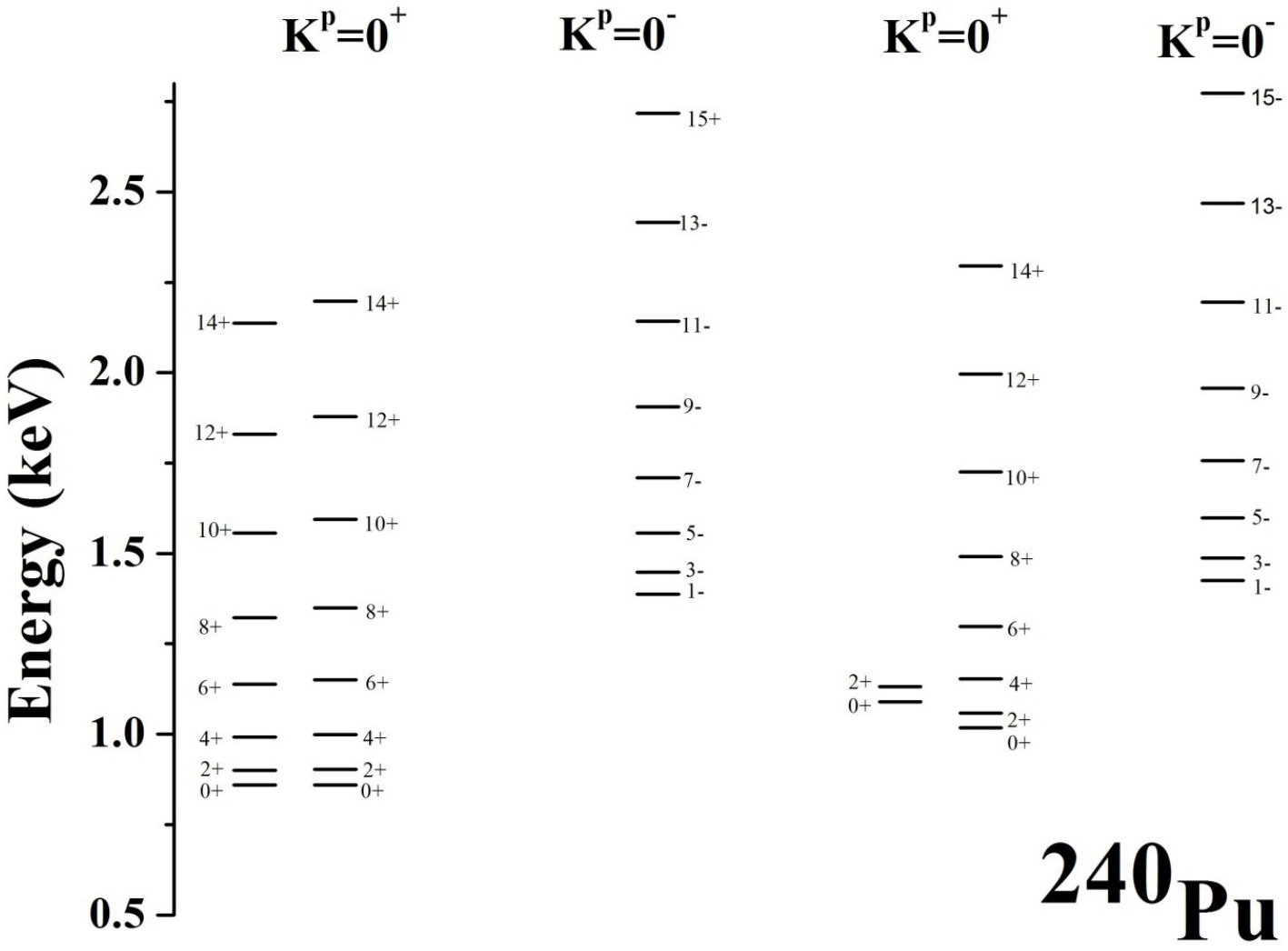
(эксп. данные: http://www.nndc.bnl.gov/ensdf/)



T. M. Shneidman et al., ***PRC92, 034302 (2015)***

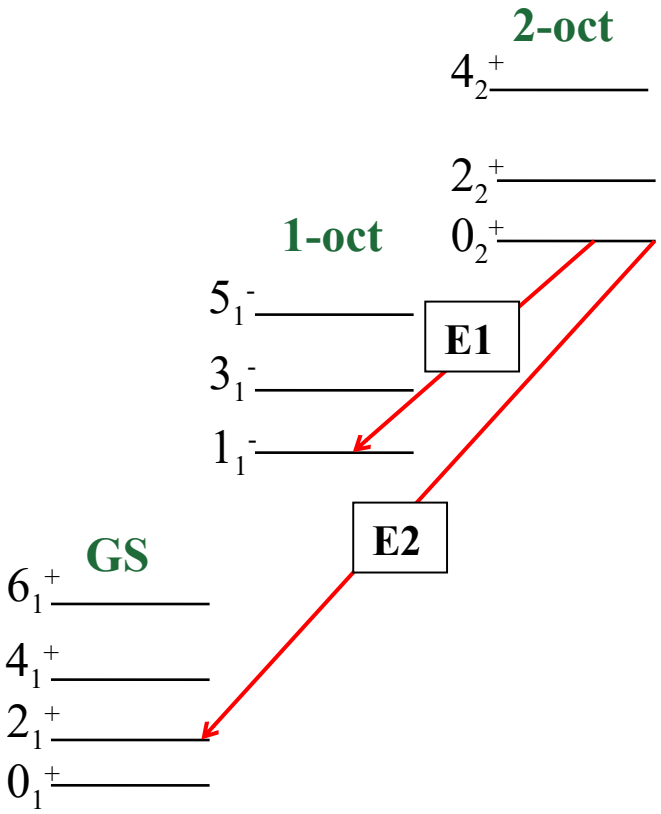
Спектр возбуждений ^{240}Pu - продолжение

(эксп. данные: http://www.nndc.bnl.gov/ensdf/)



T. M. Shneidman et al., ***PRC92, 034302 (2015)***

Electromagnetic transitions from band, built on excited 0^+ state in ^{240}Pu.

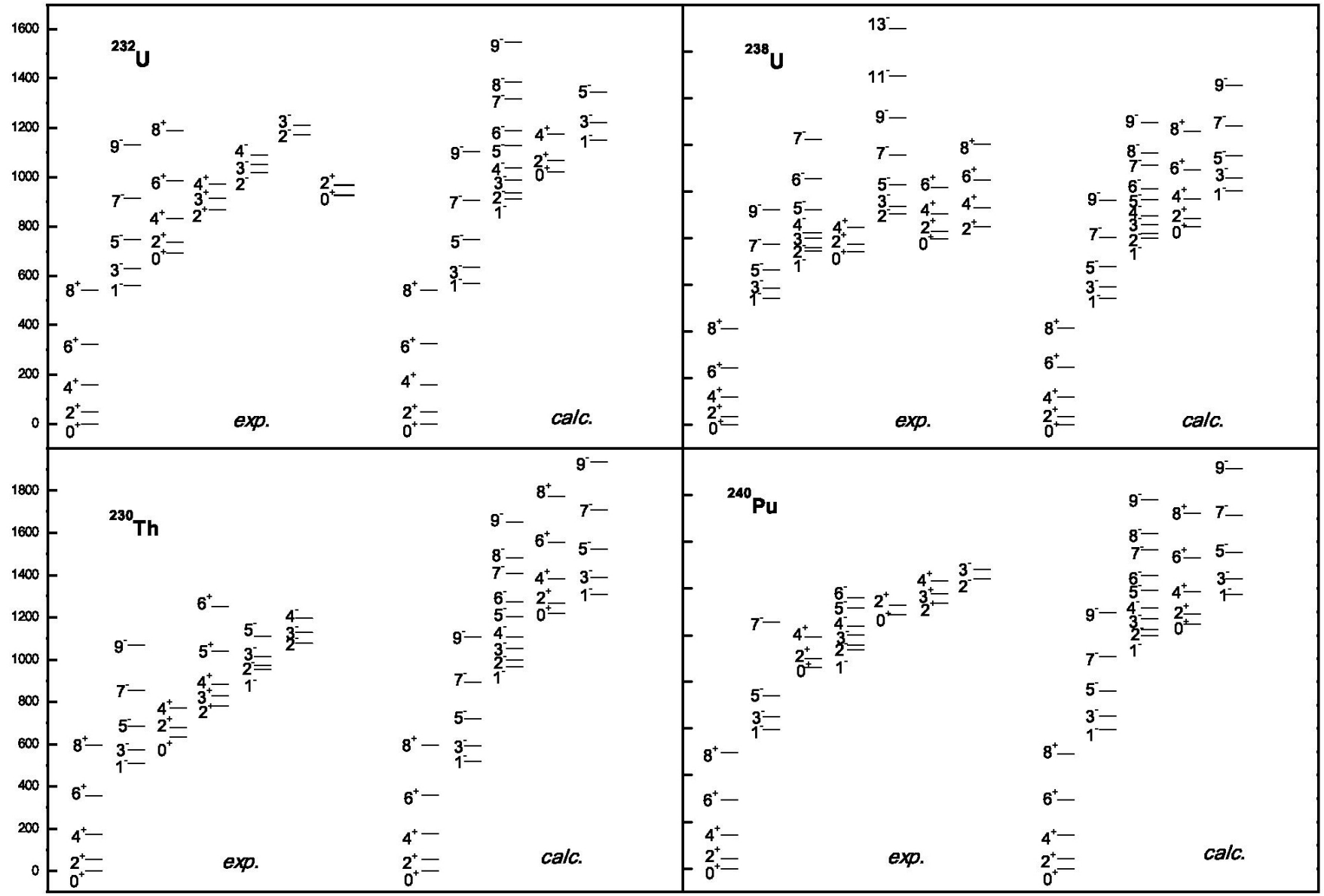


Experimental ratios R_{exp} = B (E1)/B(E2) in comparision with calculated within DNS

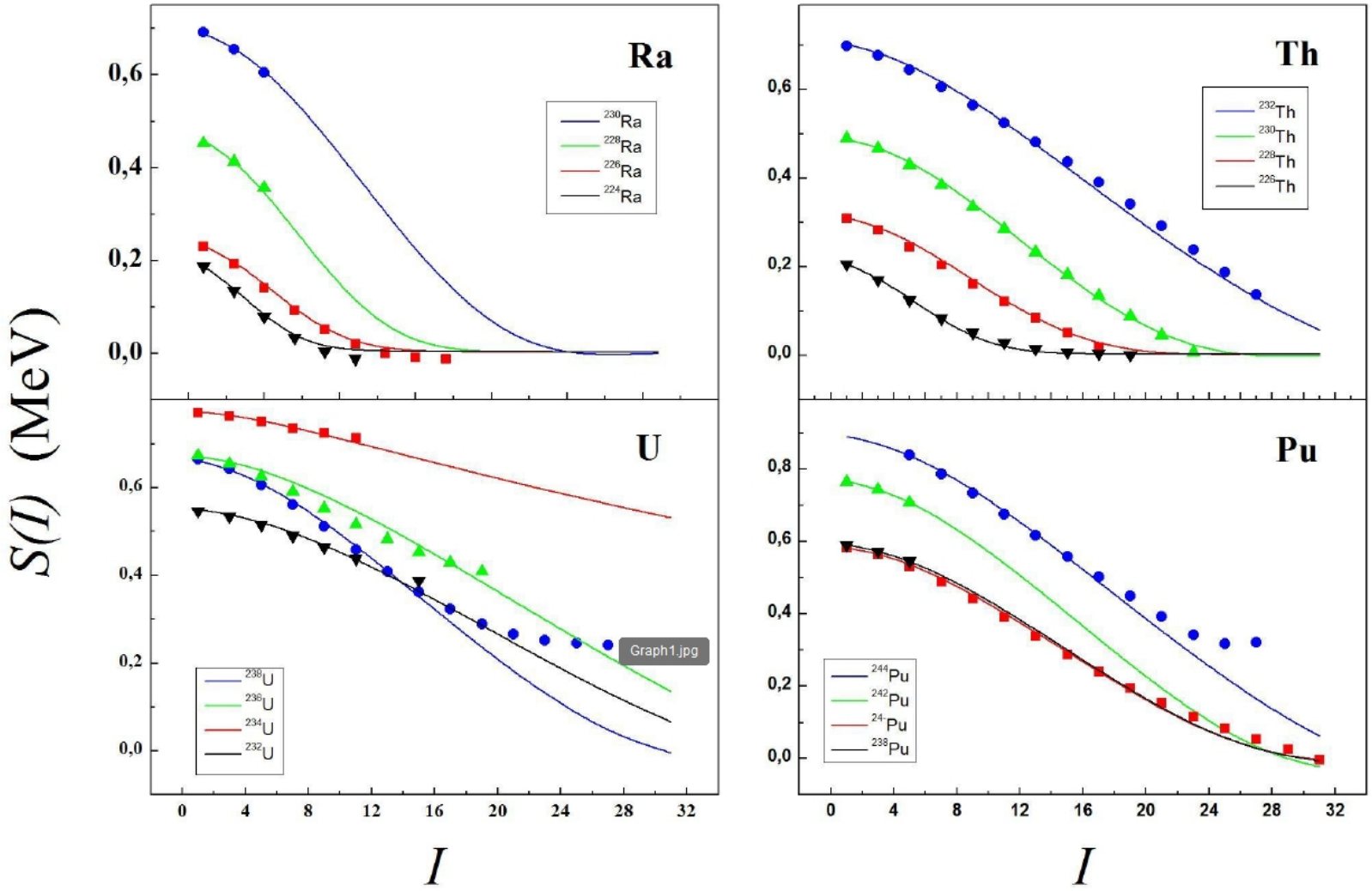
I_i^π	$I_{f,E1}^\pi$	$I_{f,E2}^\pi$	R_{exp} $(10^{-6}$ fm$^{-2})$	R_{DNS} $(10^{-6}$ fm$^{-2})$
0_2^+	1_1^-	2_1^+	13.7(3)	19.17
2_2^+	1_1^-	0_1^+	99(15)	99.95
2_2^+	1_1^-	2_1^+	26(2)	39.15
2_2^+	1_1^-	4_1^+	5.9(3)	8.57
2_2^+	3_1^-	0_1^+	149(22)	165.60
2_2^+	3_1^-	2_1^+	39(2)	64.9
2_2^+	3_1^-	4_1^+	8.9(5)	14.2
4_2^+	3_1^-	6_1^+	4.4(11)	6.9
4_2^+	5_1^-	6_1^+	4.7(13)	10.59

M. Spieker et al., **Phys.Rev. C 97, 064319 (2018)**

Results of Calculation for Heavy Actinides



Parity splitting in alternating parity bands



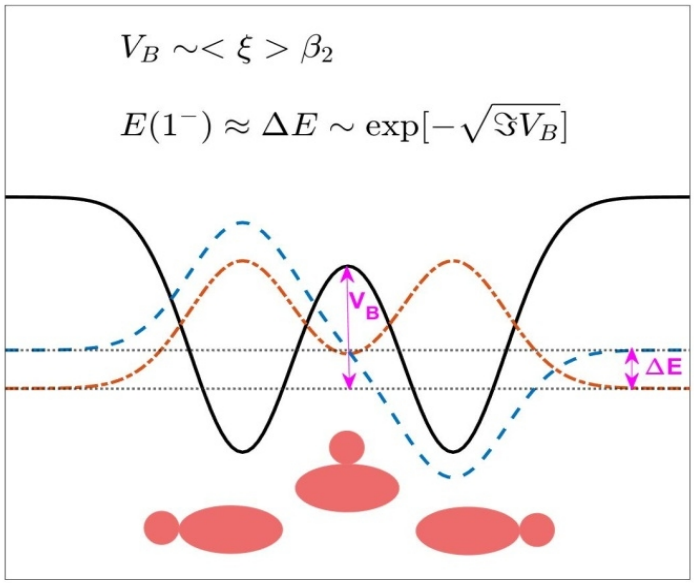
$$S(I^-) = E(I^-) - \frac{(I+1)E^+_{(I-1)} + IE^+_{(I+1)}}{2I+1}$$

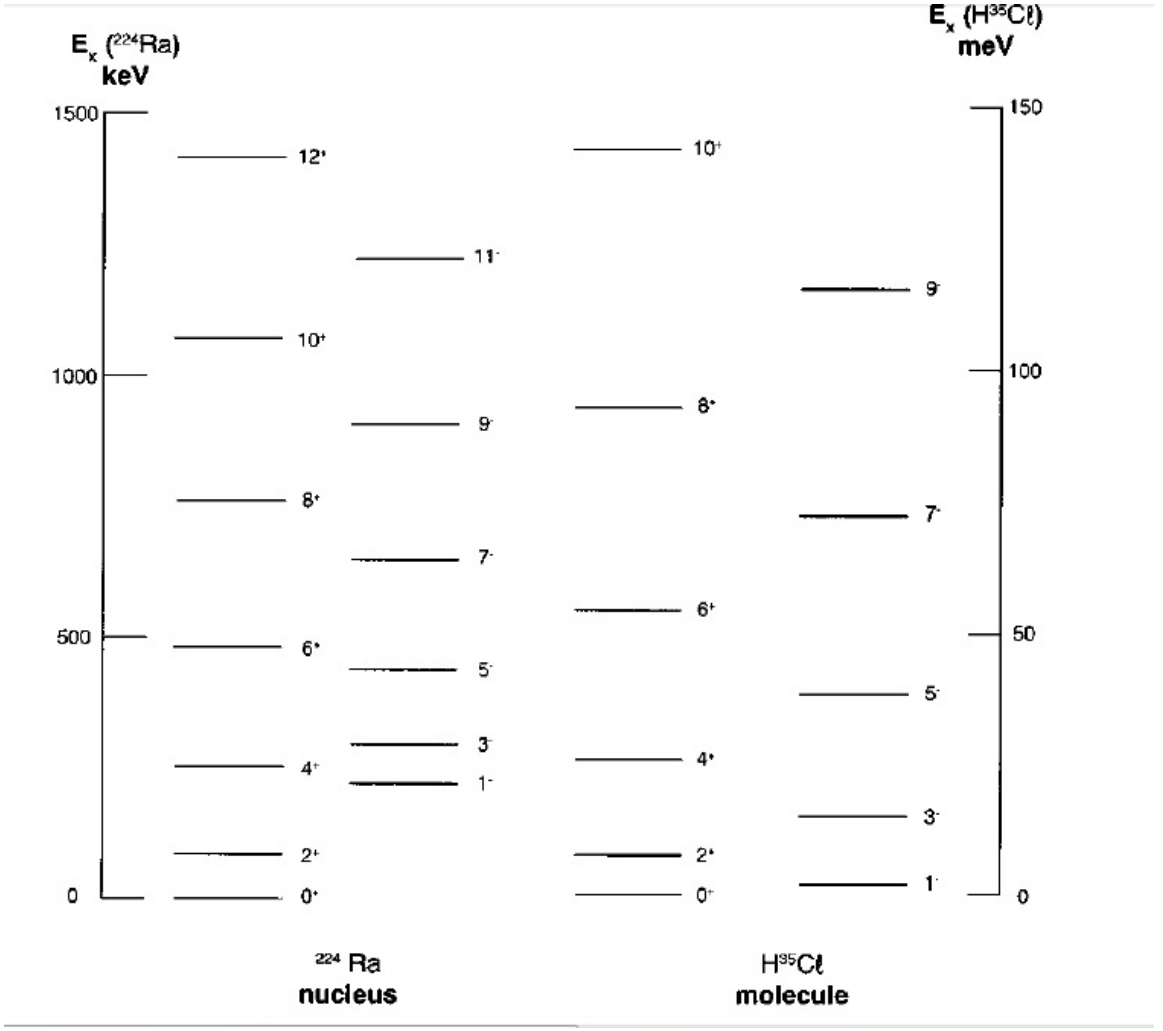
EPJ WC 107, 03009, (2016)

Расщепление по четности

$$V_B \sim < \xi > \beta_2$$

$$E(1^-) \approx \Delta E \sim \exp[-\sqrt{\Im V_B}]$$





Angular Momentum Dependence of the Parity Splitting

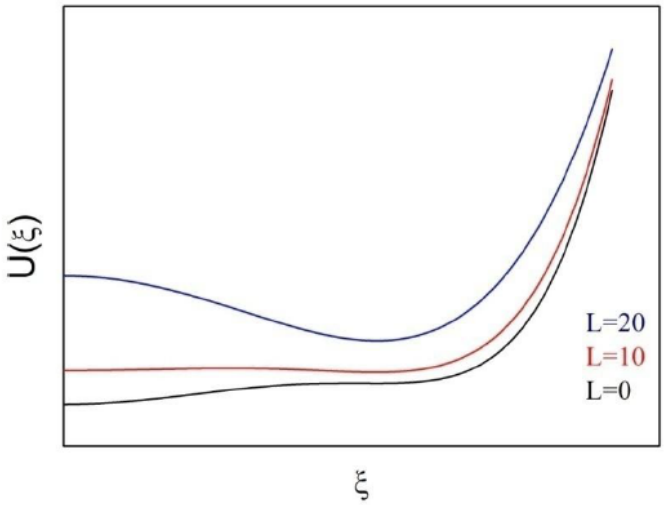
Assumption for wave function

$$\Psi_{LM}(\xi,\Omega_h,\Omega_R)=\Phi(\xi,L)g_{LM}(\xi_0,\Omega_h,\Omega_R)$$

where ξ_0 is the average mass asymmetry.

Hamiltonian in mass asymmetry

$$H(\xi,L)=-\frac{\hbar^2}{2B}\frac{1}{\xi^{3/2}}\frac{\partial}{\partial\xi}\xi^{3/2}\frac{\partial}{\partial\xi}+U_0(\xi)+\frac{\hbar^2L(L+1)}{2J(\xi)}$$



$\xi=0;$

$$U(\xi,L)=U(\xi,L=0)+\frac{\hbar^2}{2}\frac{L(L+1)}{J_h}$$

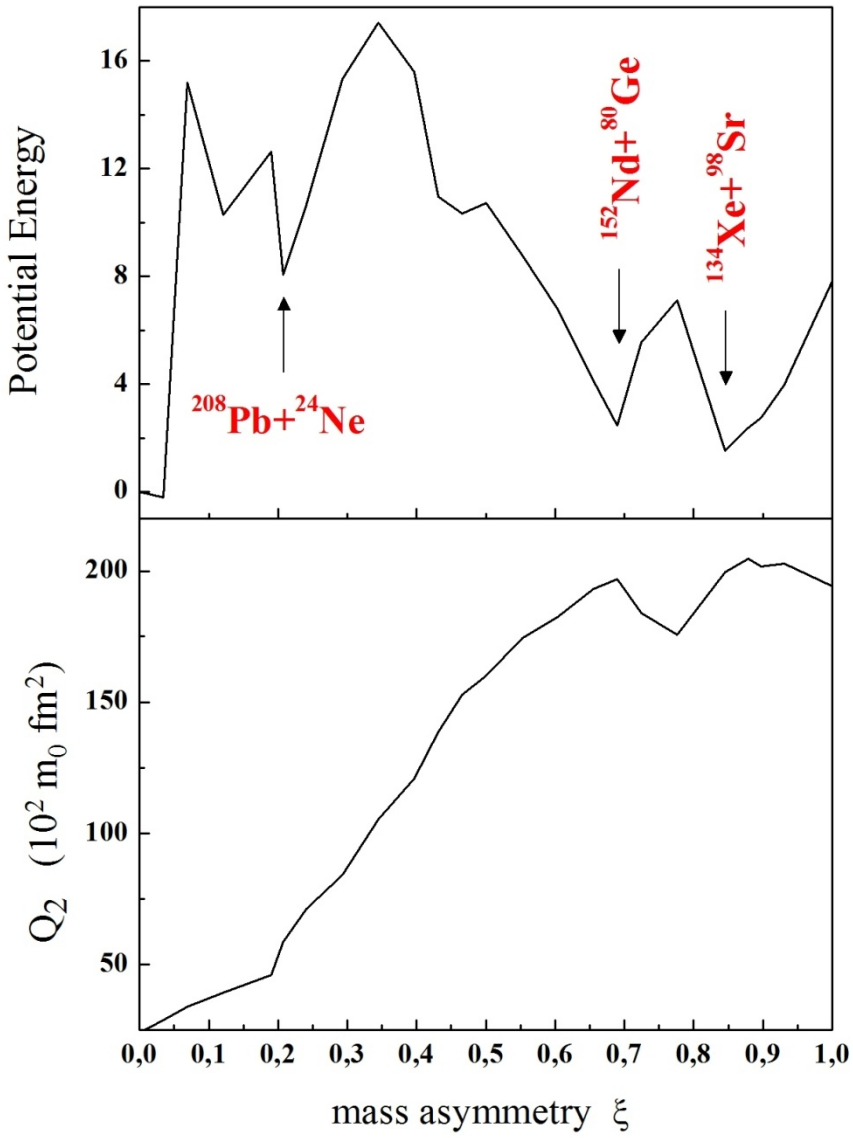
$\xi=1;$

$$U(\xi,L)=U(\xi,L=0)+\frac{\hbar^2}{2}\frac{L(L+1)}{J_{tot}}$$

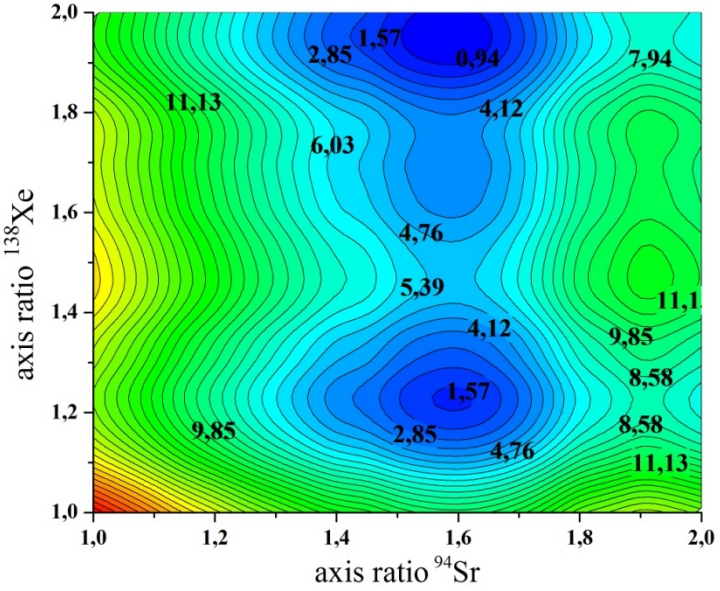
$$J_{tot}>J_h$$

As a result average mass asymmetry ξ_0 increases with angular momentum.

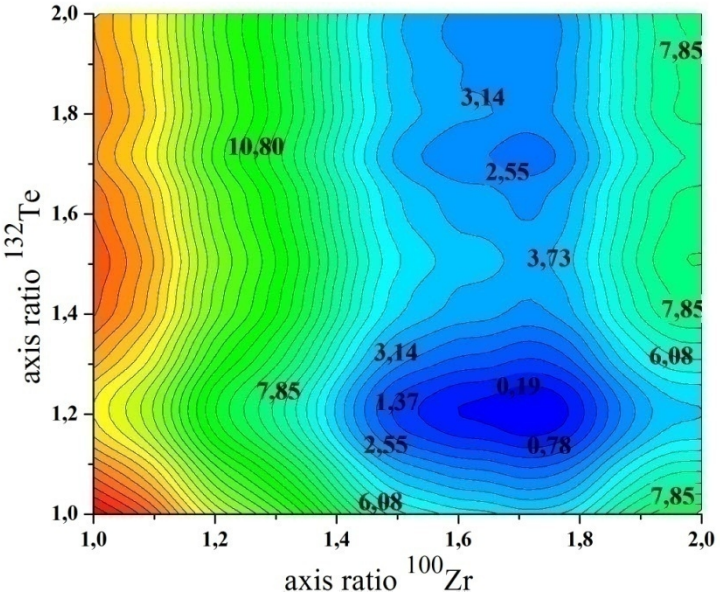
Hyperdeformed nuclear states



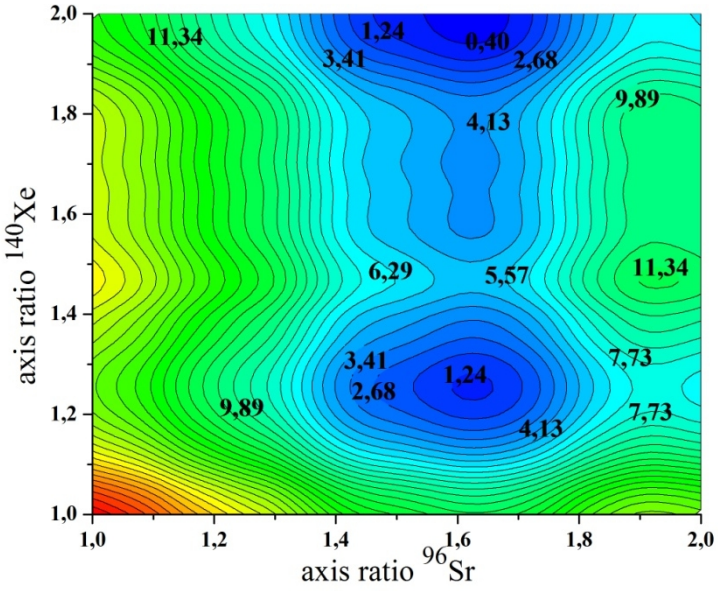
$^{232}U \rightarrow {}^{94}Sr + {}^{138}Xe$



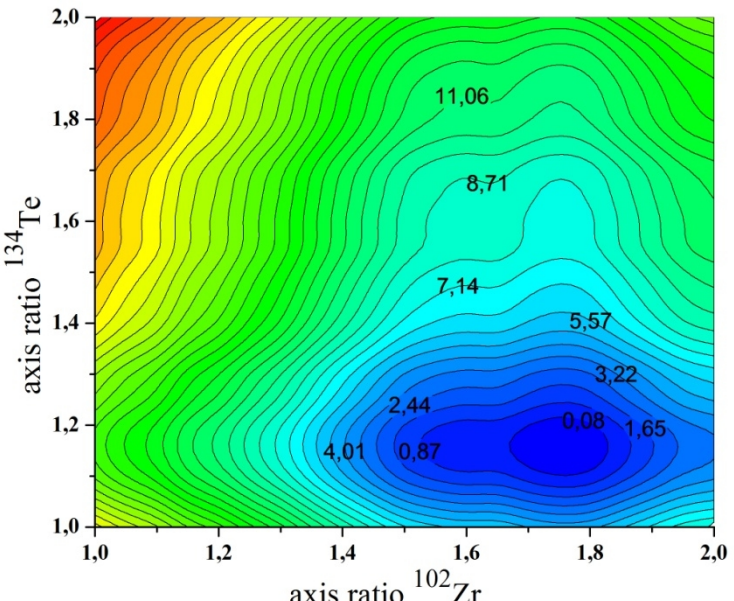
$^{232}U \rightarrow {}^{100}Zr + {}^{132}Te$



$^{236}U \rightarrow {}^{96}Sr + {}^{134}Te$



$^{236}U \rightarrow {}^{102}Zr + {}^{134}Te$

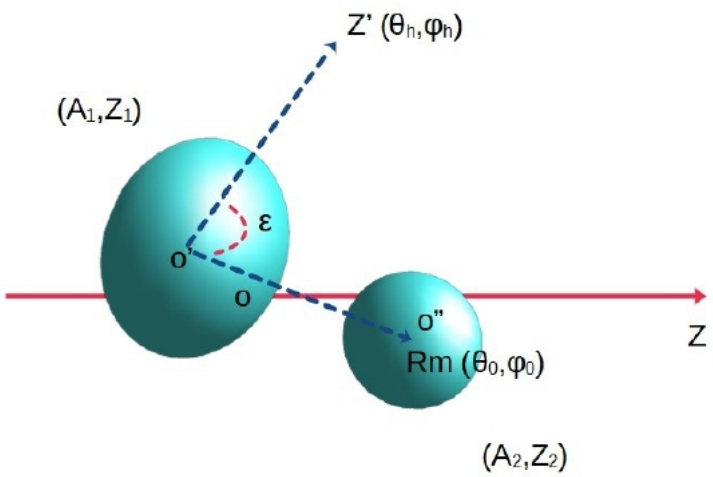


Characteristics of HD minima in U isotopes

Nucleus	^{232}U	^{234}U	^{236}U	^{238}U
DNS	^{94}Sr+^{138}Xe	^{96}Sr+^{138}Xe	^{96}Sr+^{140}Xe	^{98}Sr+^{140}Xe
Energy (MeV)	3.06	2.6 (3.1±0.4)	2.81 (2.7±0.4)	3.49
Rot. Const. (keV)	1.825 (1.96±0.11)	1.772 (2.1±0.2)	1.751 (2.4±0.4)	1.697
Q_2 (10^2 e fm^2)	92.37	93.021	93.466	96.772
Q_3 (10^3 e fm^3)	29.96	28.48	29.92	27.84

Exp: *L. Csige et al., Journal of Physics: Conference Series* ***312*** *(2011) 092022;*
A. Krasznahorkay et. al, AIP Conf. Proc. 819, 439 (2006) .
Calc. *T.M. Shneidman et al., J.Phys.:CS* ***366, 012046 (2012)***

Спектр возбуждения угловых колебаний в двойных ядерных системах



$$H = H_{\rm rot} + H_{\rm bend} + V_{\rm int},$$

$$H_{\rm rot} = \frac{\hbar^2}{2\mu R_{\rm m}^2}(L^2 - 2L_3'),$$

$$H_{\rm bend} = \frac{\hbar^2}{2\Im_b}\frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\epsilon\frac{\partial}{\partial\epsilon} + \frac{\hbar^2}{2\Im_b\epsilon^2}L_3'^2 + \frac{C}{2}\epsilon^2,$$

$$V_{\rm int} = \frac{\hbar^2}{2\mu R_{\rm m}^2}\left[\frac{1}{\epsilon}(L_1'L_3' + L_3'L_1') + 2iL_2'\frac{1}{\sqrt{\epsilon}}\frac{\partial}{\partial\epsilon}\sqrt{\epsilon}\right]$$

Базисные состояния

$$\Psi_{n,L,M,K} = \left[D^L_{M,K}(\theta_1,\theta_2,\theta_3) + \pi(-1)^{L+K}D^L_{M,-K}(\theta_1,\theta_2,\theta_3)\right]L_{n,|K|}\left(\frac{\epsilon^2}{\epsilon_0^2}\right)$$

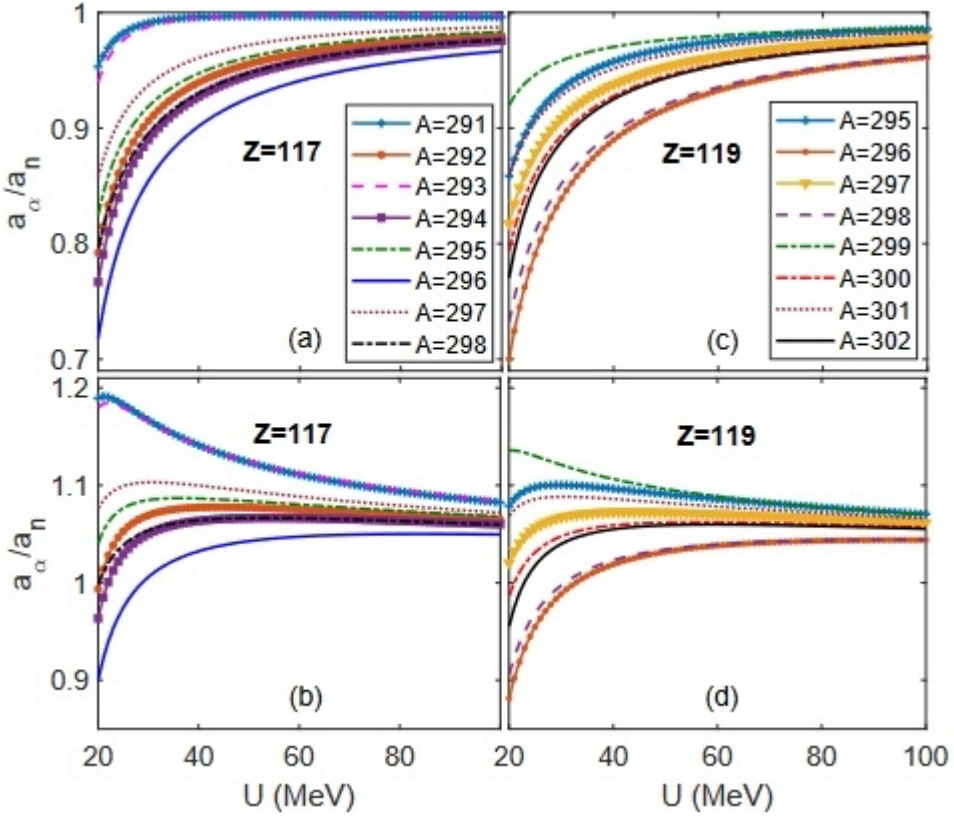
$$E_{L,M,K,n} = \frac{\hbar^2}{2\mu R_m^2}\left[L(L+1) - 2K^2\right] + \frac{\hbar^2}{\Im_b\epsilon_0^2}(2n + |K| + 1)$$

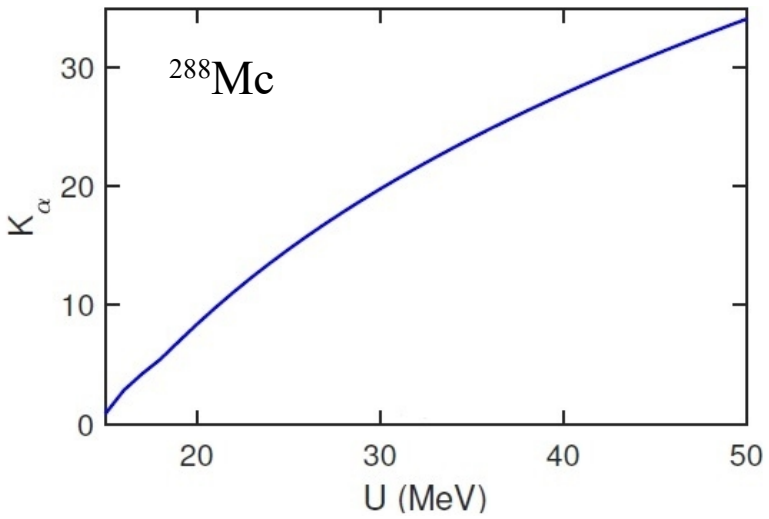
Энергетический спектр

$$E_{LMK} = \frac{\hbar^2}{2\mu R_m^2 + \Im_h}\left[L(L+1) - 2K^2\right] + \frac{\hbar^2}{\Im_b\epsilon_0^2}(2n + |K| + 1)$$

Role of collective enhancement on the emission of alpha-particles

$$\rho_{tot} = \sum_c \rho_i(U - U_c)\tau_c(U_c) \approx \rho_i(U) \underbrace{\sum_c \exp\left[-\frac{U_c}{T}\right]\tau_c(U_c)}_{K_{coll}}$$





Phys. Rev. C, **101**, 54315 (2020)
103, 034309 (2021)
105, 044328 (2022)