Analytical approach to the generation of angular momentum in spontaneous fission fragments

N.V. Ryzhkov
T. M. Shneidman, A. Rahmatinejad, N. V. Antonenko, G.G. Adamian

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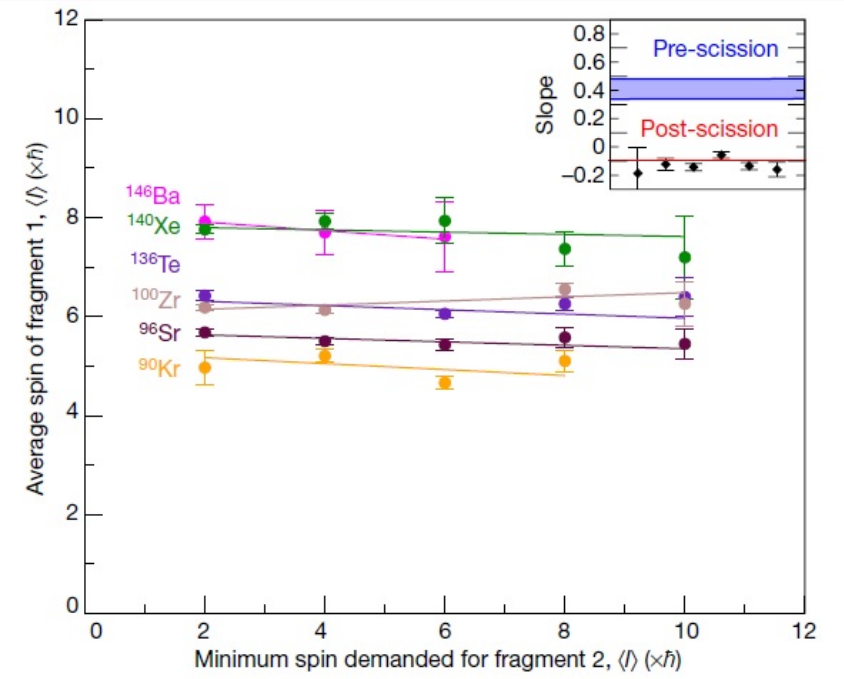
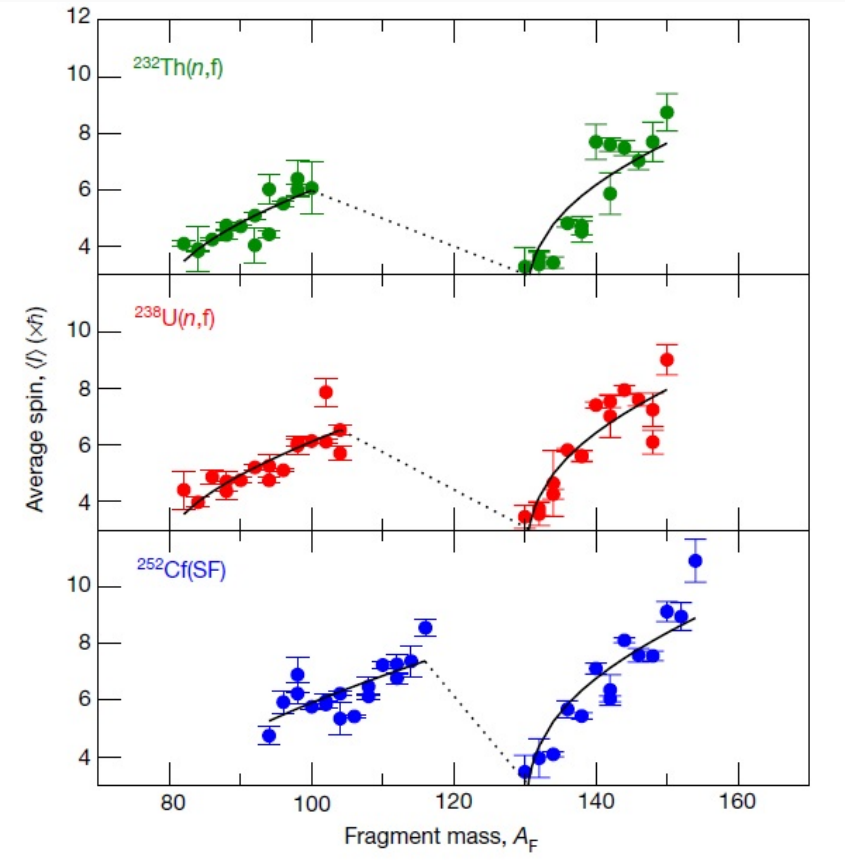
Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research

Spins of the fragments

J. N. Wilson et. al, Nature **590**, 566–570 (2021).

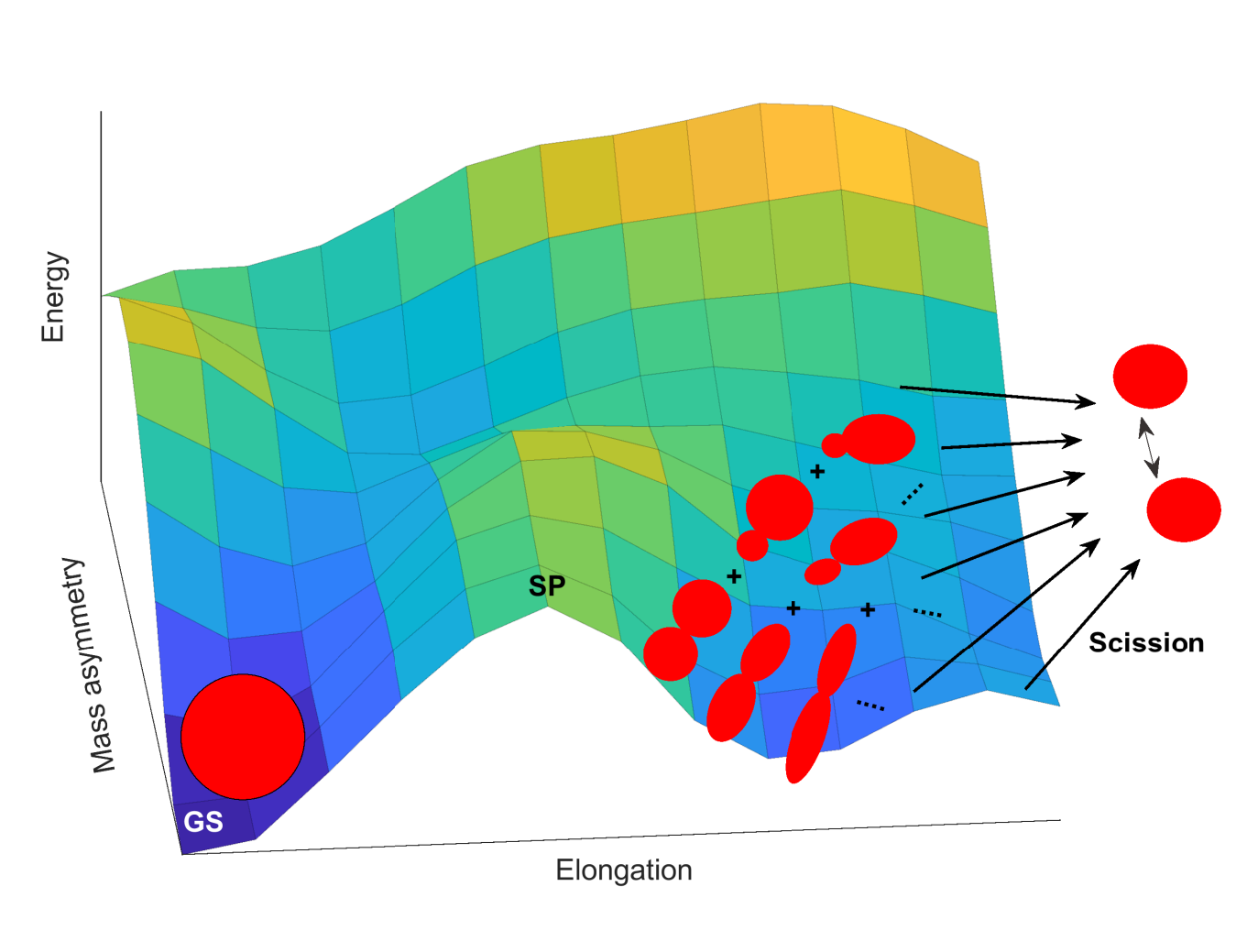


Absence correlation between fragment spins



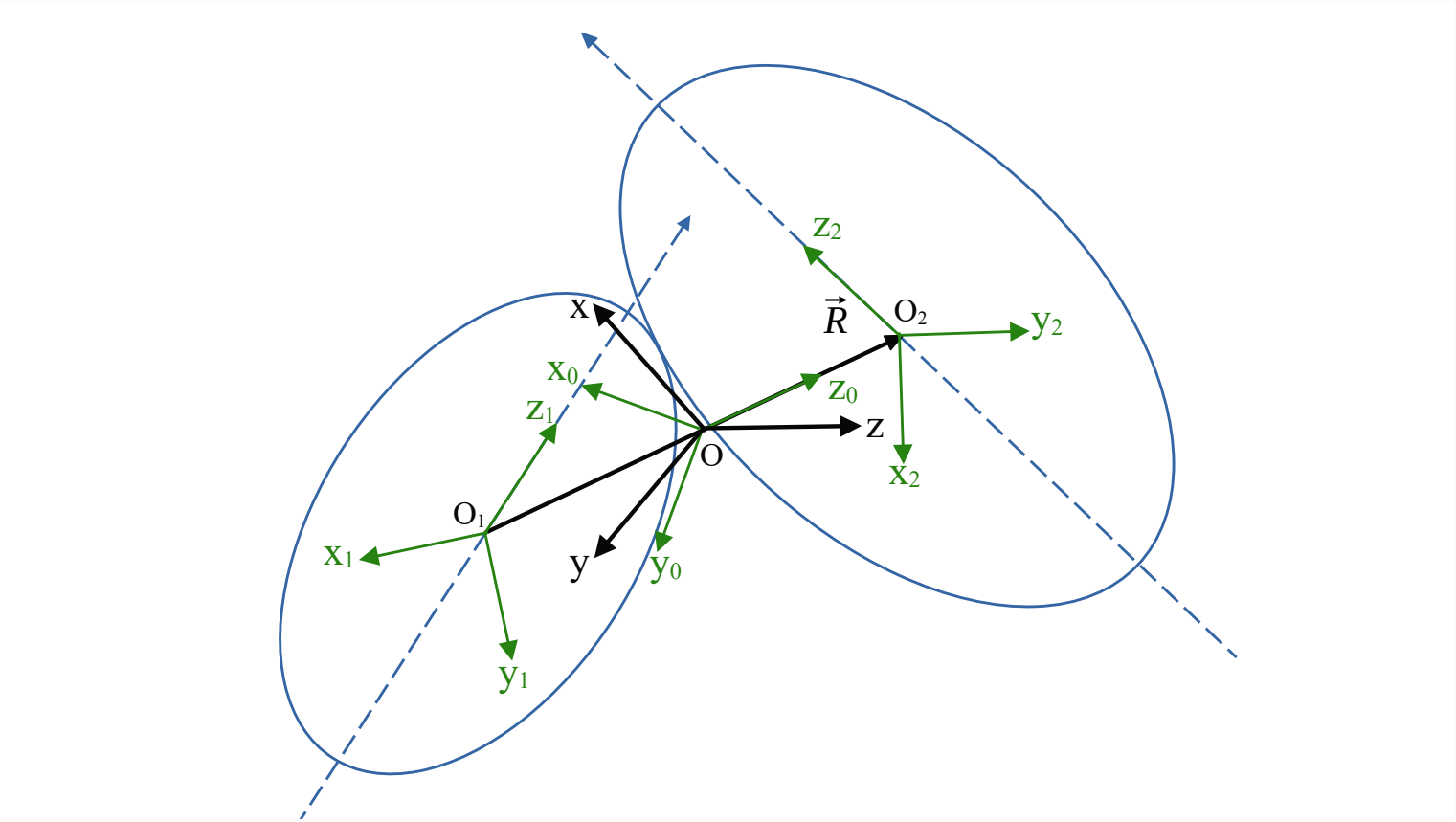
Sawtooth behavior of spin versus mass distribution

Evolution of fissioning nucleus after crossing the fission barrier



Rahmatinejad, et al., Int. J. Mod. Phys. E 33, 2441018 (2024).

Angular motion at scission



$\Omega_{1,2} = (\phi_{1,2}, \theta_{1,2}, 0)$ – orientation of fragments with respect to the lab. system

R$= (R, \theta_R, \phi_R)$ – vector connecting centers of the fragments.

Hamiltonian of angular motion in DNS

Kinetic energy:

$$T = \frac{\hbar^2 \hat{l}_0^2}{2\mu R_m^2} + \frac{\hbar^2 \hat{l}_1^2}{2\Im_1} + \frac{\hbar^2 \hat{l}_2^2}{2\Im_2}$$

Angular momentum operators and eigen function:

$$\hat{l}_i^2 = -\left(\frac{1}{sin\theta_i}\frac{\partial}{\partial\theta_i} sin\theta_i \frac{\partial}{\partial\theta_i} + \frac{1}{sin^2\theta_i}\frac{\partial^2}{\partial\phi_i^2}\right), \quad \hat{l}_i^2 Y_{l_i M_i}(\Omega_i) = l_i(l_i+1) Y_{l_i M_i}(\Omega_i)$$

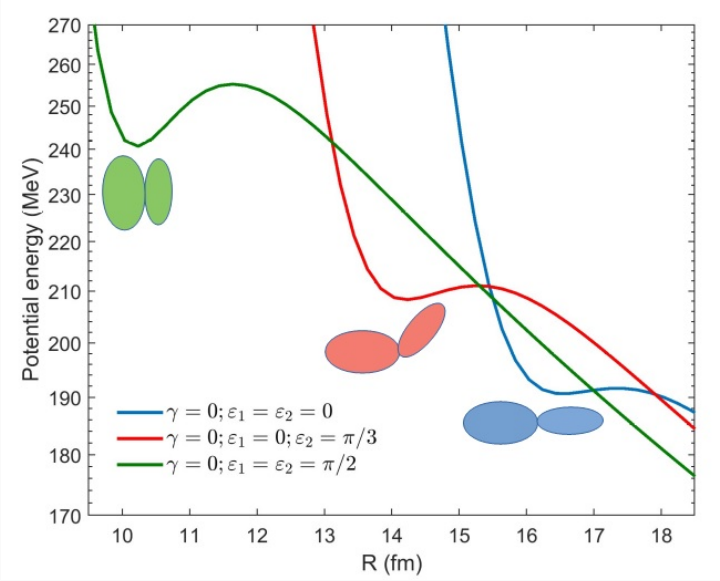
$$\left[Y_{i_0}(\Omega_0) \times [Y_{i_1}(\Omega_1) \times Y_{i_2}(\Omega_2)]_{i_{12}}\right]_{(l,M)} =$$

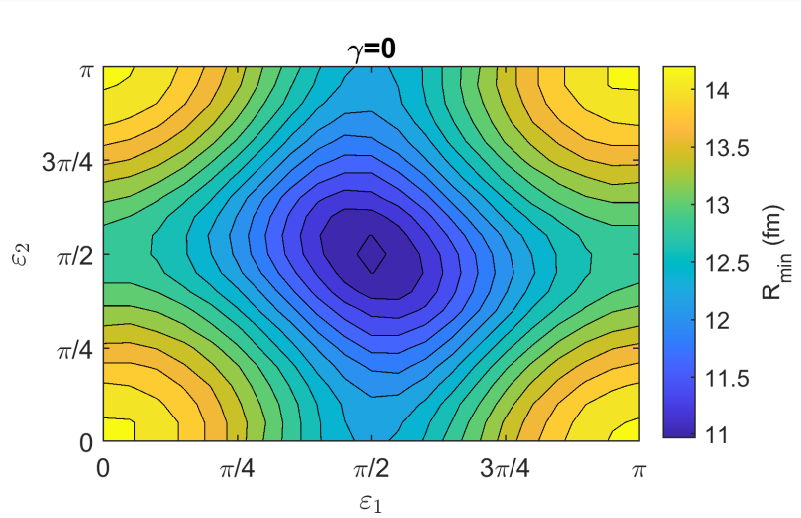
$$= \sum_{m_0 m_1 m_2 m_{12}} C^{lM}_{i_0 m_0, i_{12} m_{12}} C^{i_{12} m_{12}}_{i_1 m_1, i_2 m_2} Y_{i_0 m_0}(\Omega_R) Y_{i_1 m_1}(\Omega_1) Y_{i_2 m_2}(\Omega_2)$$

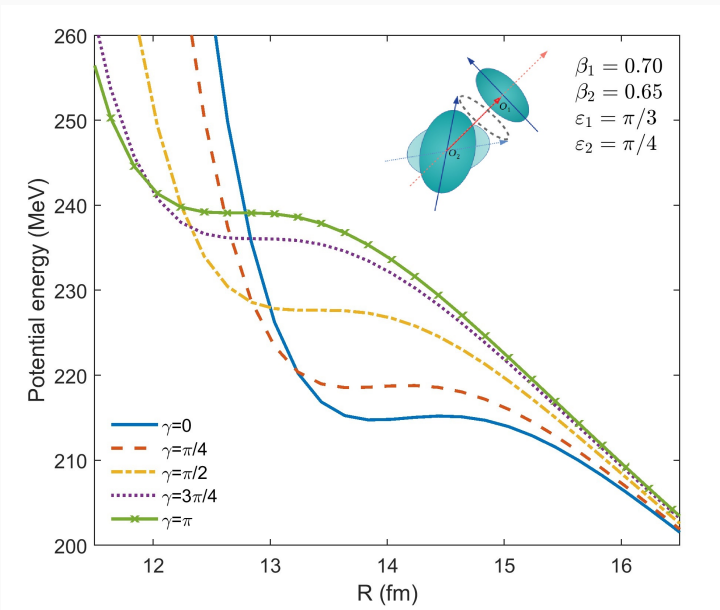
Potential energy:

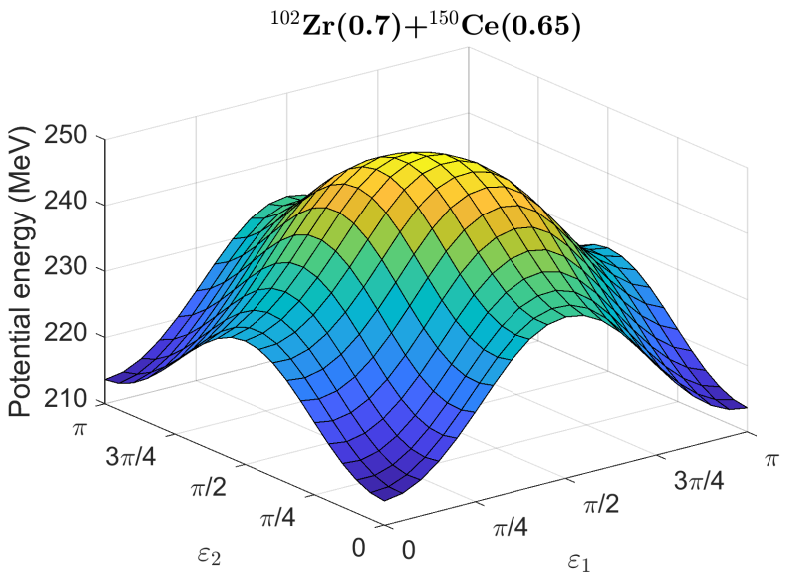
$$V(\Omega_1, \Omega_2, \Omega_0) = \sum_{\lambda_0, \lambda_1, \lambda_2} V_{\lambda_0, \lambda_1, \lambda_2} \left[Y_{\lambda_0}(\Omega_0) \times [Y_{\lambda_1}(\Omega_1) \times Y_{\lambda_2}(\Omega_2)]_{\lambda_0}\right]_{(00)}$$

Potential energy









Angular vibrations at scission

$$V(\Omega_1, \Omega_2, \Omega_0) = \sum_{\lambda_0,\lambda_1,\lambda_2} V_{\lambda_0,\lambda_1,\lambda_2} \left[Y_{\lambda_0}(\Omega_0) \times [Y_{\lambda_1}(\Omega_1) \times Y_{\lambda_2}(\Omega_2)]_{\lambda_0} \right]_{(00)}$$

$$V(\tilde{\Omega}_1, \tilde{\Omega}_2) \approx \frac{C_1}{2} \sin^2 \varepsilon_1 + \frac{C_2}{2} \sin^2 \varepsilon_2 + C_{12} \sin^2 \varepsilon_1 \sin^2 \varepsilon_2 \cos(\gamma_1 - \gamma_2)$$

For large enough deformations ($\beta_i \geqslant 0.15$), potential energy:

$$V(\varepsilon_1', \varepsilon_2') \approx \frac{C_1}{2} \varepsilon_1'^2 + \frac{C_2}{2} \varepsilon_2'^2, \qquad (\varepsilon_i' = \varepsilon_i, \pi - \varepsilon_i)$$

Kinetic energy ($\mathbf{I}_0 + \mathbf{I}_1 + \mathbf{I}_2 = 0$):

$$T = \frac{\hbar^2 (\hat{I}_1 + \hat{I}_2)^2}{2\mu R_m^2} + \frac{\hbar^2 \hat{I}_1^2}{2\Im_1} + \frac{\hbar^2 \hat{I}_2^2}{2\Im_2} = \frac{\hbar^2 \hat{I}_1^2}{2\Im_{b,1}} + \frac{\hbar^2 \hat{I}_2^2}{2\Im_{b,2}} + \frac{\hbar^2 \hat{I}_1 \hat{I}_2}{\mu R_m^2}$$

Angular momentum operators:

$$\hat{I}_i^2 = -\left(\frac{1}{\varepsilon_i'} \frac{\partial}{\partial \varepsilon_i'} \varepsilon_i' \frac{\partial}{\partial \varepsilon_i'} + \frac{1}{\varepsilon_i'^2} \frac{\partial^2}{\gamma_i^2} \right)$$

Hamiltonian of angular vibrations at scission

$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \sum_{i=1,2} \left[-\frac{1}{2\Im_{b,i}} \left(\frac{1}{\varepsilon'_i} \frac{\partial}{\partial \varepsilon'_i} \varepsilon'_i \frac{\partial}{\partial \varepsilon'_i} + \frac{1}{\varepsilon'^2_i} \frac{\partial^2}{\gamma_i^2} \right) + \frac{C_i}{2} \varepsilon'^2_i \right]$$

Eigenstates and eigenvalues:

$$E_{n_i K_i} = \hbar\omega_i (2n_i + |K_i| + 1)$$

$$\phi_{n_i K_i}(\varepsilon'_i, \gamma_i) = N_{n_i K_i} \exp\left(-\frac{\varepsilon'^2}{2\varepsilon'^2_{0i}} \right) L_{n_i}^{(|K_i|)} \left(\frac{\varepsilon'^2}{\varepsilon'^2_{0i}} \right) \exp\left(-iK_i\gamma_i\right)$$

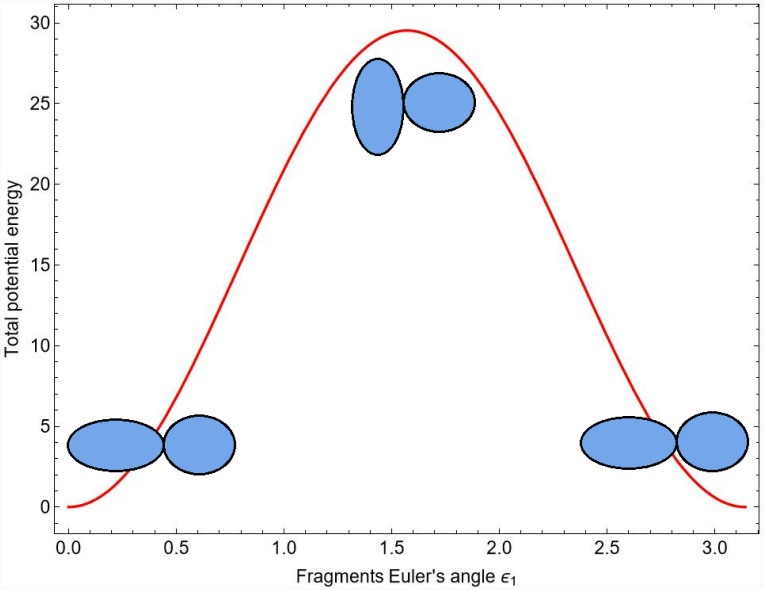
$$\omega_i = (C_i/\Im_{bi})^{1/2}, \qquad \varepsilon^2_{0i} \sim (C_i \Im_{bi})^{-1/2}$$

Symmetrized solution:

$$\Phi_{n_i K_i}(\varepsilon_i, \gamma_i) \sim \phi_{n_i K_i}(\varepsilon_i, \gamma_i) \pm \phi_{n_i K_i}(\pi - \varepsilon_i, \gamma_i + \pi)$$

Stiffness parameters and moments of inertia

Stiffness parameters C_i is a measure of restoring force for angular motion:



$$V(\varepsilon_1, \varepsilon_2) \approx \frac{C_1}{2} \sin^2 \varepsilon_1 + \frac{C_2}{2} \sin^2 \varepsilon_2$$

$$C_i = 2 \left[V(R_{touch}^{pole-neck}) - V(R_{touch}^{pole-pole}) \right]$$

Moments of inertia:

$$\Im_i \approx \frac{2}{5} m_0 A_i R_i^2 \left(1 + \sqrt{\frac{5}{16\pi}} \beta_i + \frac{20}{7\pi} \beta_i^2 + \ldots \right)$$

Expansion in spherical harmonics

$$\Phi_{n_i K_i}(\varepsilon_i, \gamma_i) = \sum_{l_i} a_{n_i K_i}(l) Y_{l_i, K_i}(\varepsilon_i, \gamma_i)$$

$$a_{n_i K_i}(l_i) = (-1)^{n_i} \left(\frac{(2l_i + 1)(l_i + |K_i|)!}{n_i!(n_i + |K_i|)!(l_i - |K_i|)!} \right)^{1/2}$$

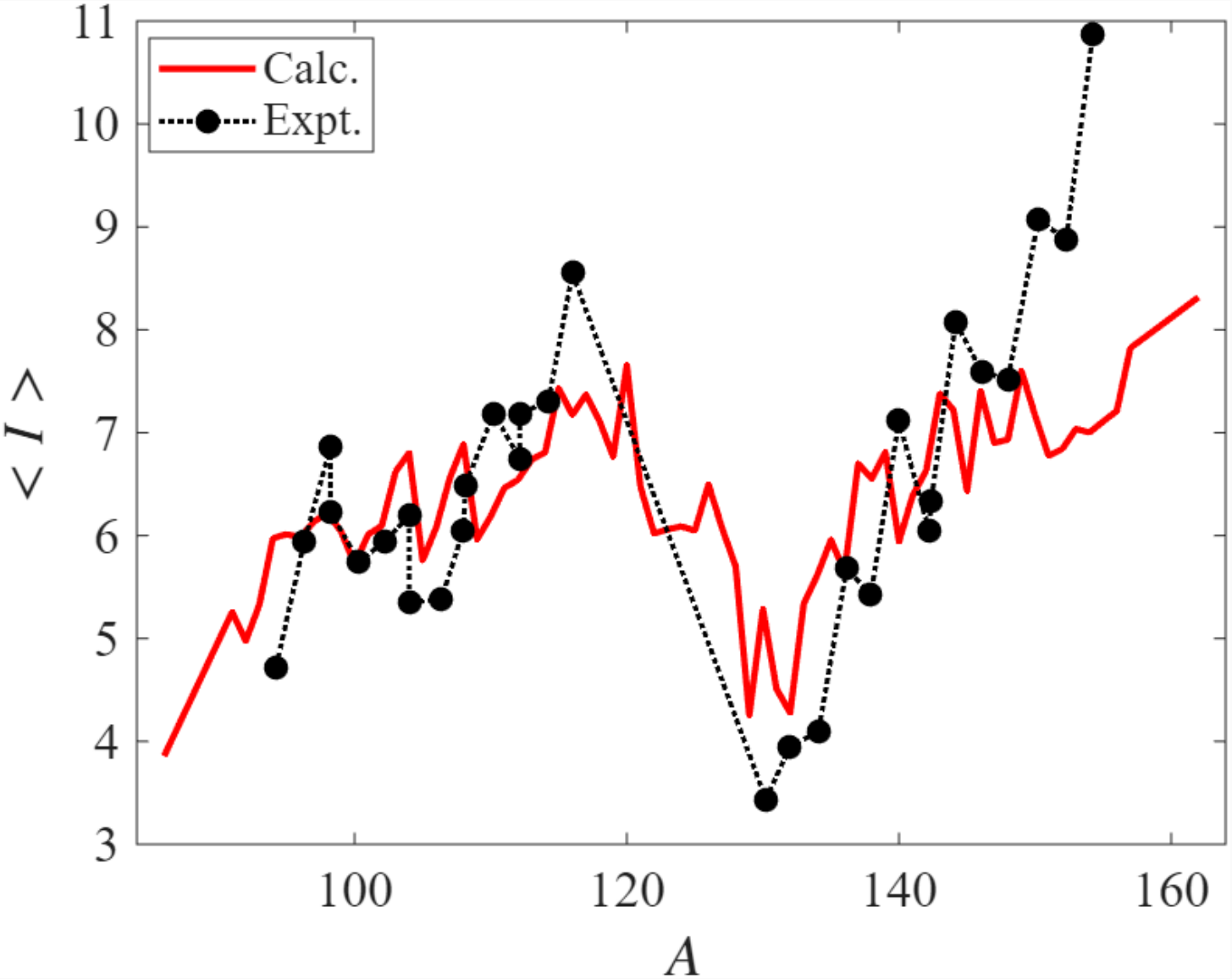
$$\times \varepsilon_0^{|K_i|+1} \exp\left(-\frac{1}{2} \left(l_i + \frac{1}{2} \right)^2 \varepsilon_0^2 \right) L_n^{|K_i|} \left[(l_i + \frac{1}{2})^2 {\varepsilon_0}^2 \right]$$

Probabilities of various angular momenta of fragments are independent on each other.

For ground-state $n_1 = n_2 = K_1 = K_2 = 0$:

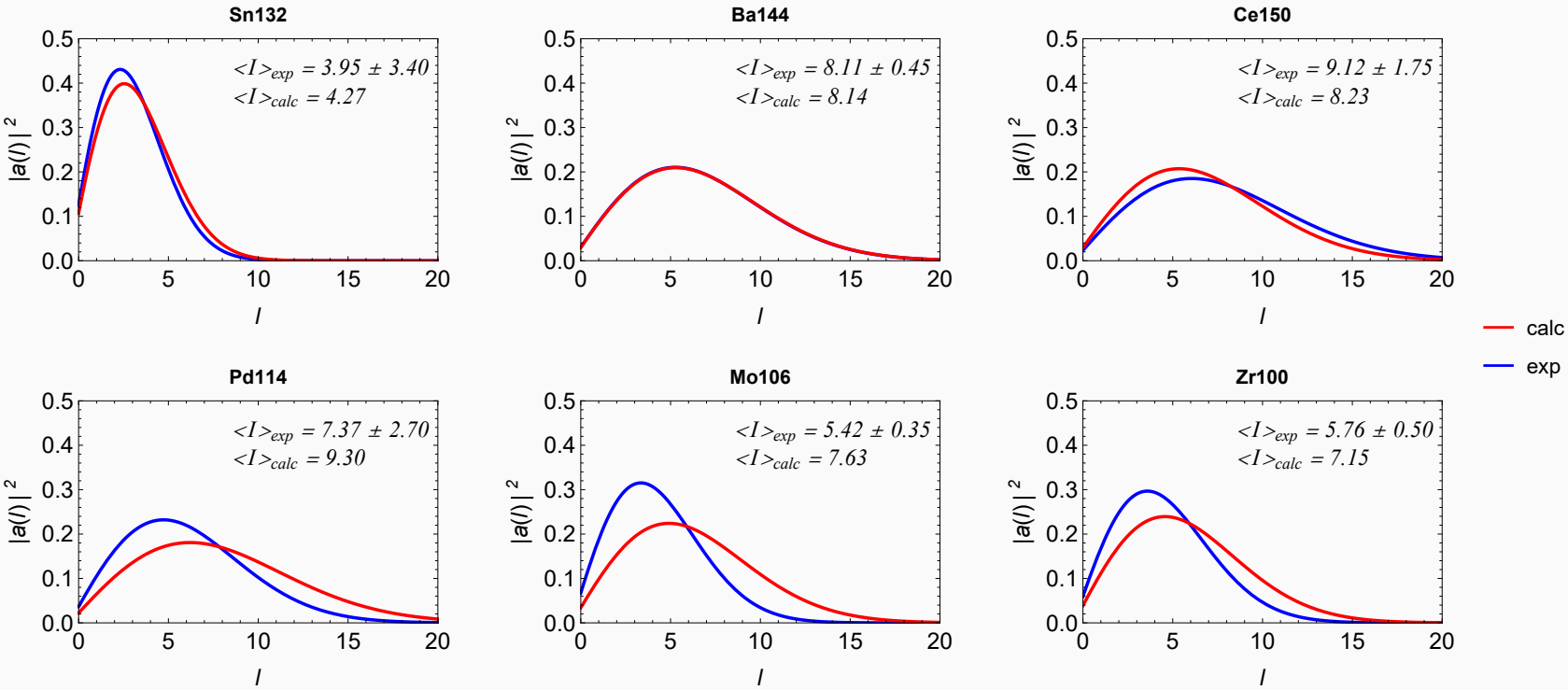
$$|a^{(i)}(l)|^2 = \varepsilon_{0i}^2 (2l + 1) \exp\left\{ -(l + \frac{1}{2})^2 \varepsilon_{0i}^2 \right\}$$

Angular momentum vs fragment mass



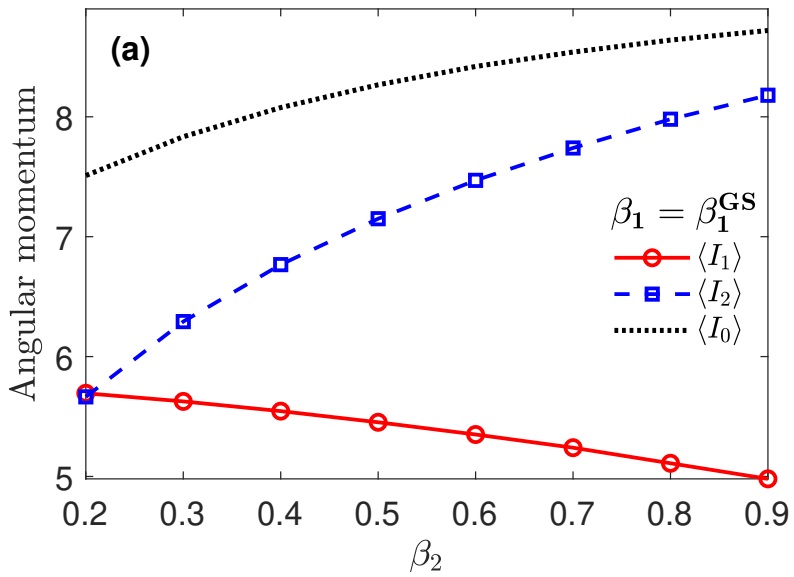
Analytical expression for angular momentum vs experiment

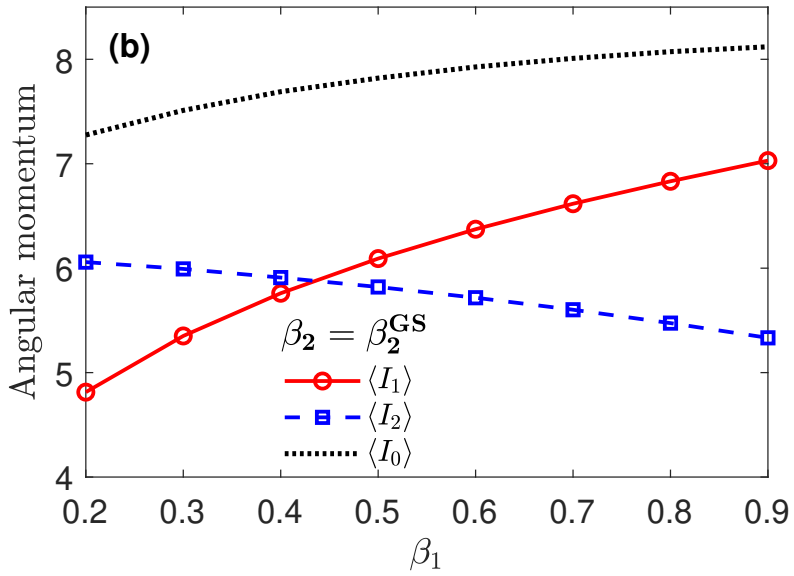
J. N. Wilson et. al, Nature **590**, 566–570 (2021).



The cause of deviation is overestimating the fragments deformations

The role of deformation





Average distortion angle:

$$\overline{\varepsilon_i} = \sqrt{\langle \varepsilon_i^2 \rangle} \sim \hbar (C_i \Im_i)^{-1/4}$$

Uncertainty relation:

$$\langle I_i \rangle \sim (C_i \Im_i)^{1/4}$$

Stiffness parameter:

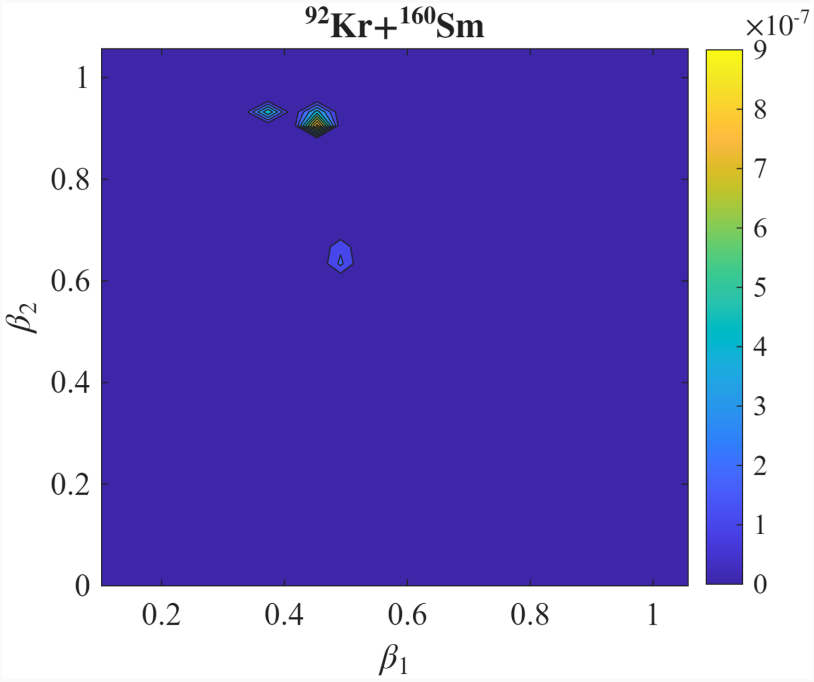
$$C_i \sim Z_1 Z_2 \beta_i / R_m^3(0)$$

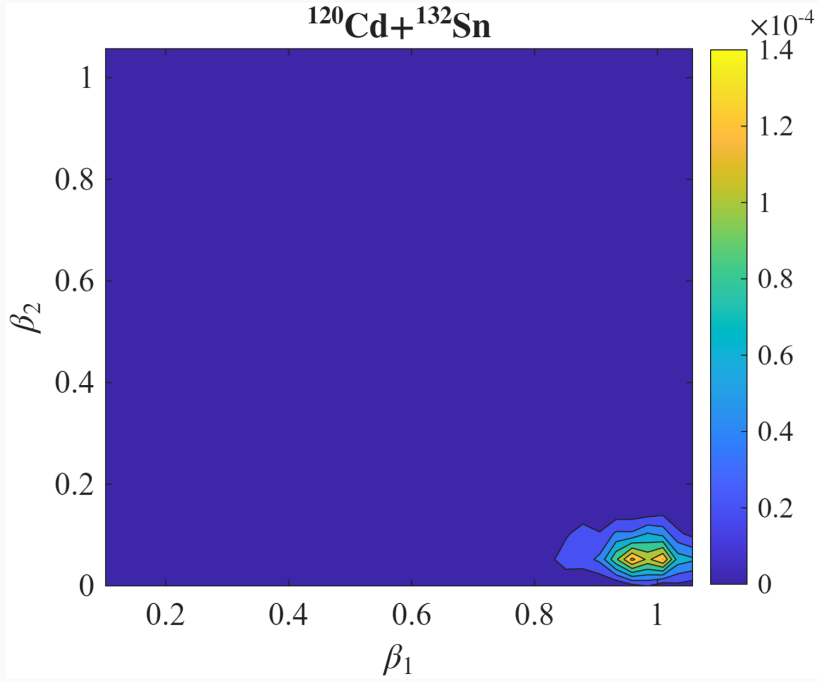
Moments of inertia:

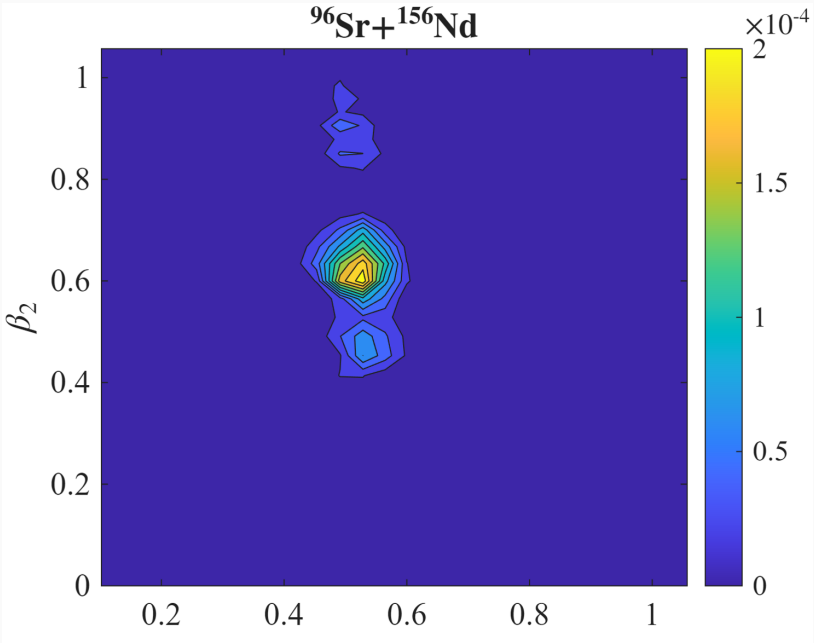
$$\Im_i \approx \tfrac{2}{5} m_0 A_i R_i^2 \left(1 + \sqrt{\tfrac{5}{16\pi}} \beta_i + \tfrac{20}{7\pi} \beta_i^2 + \ldots\right)$$

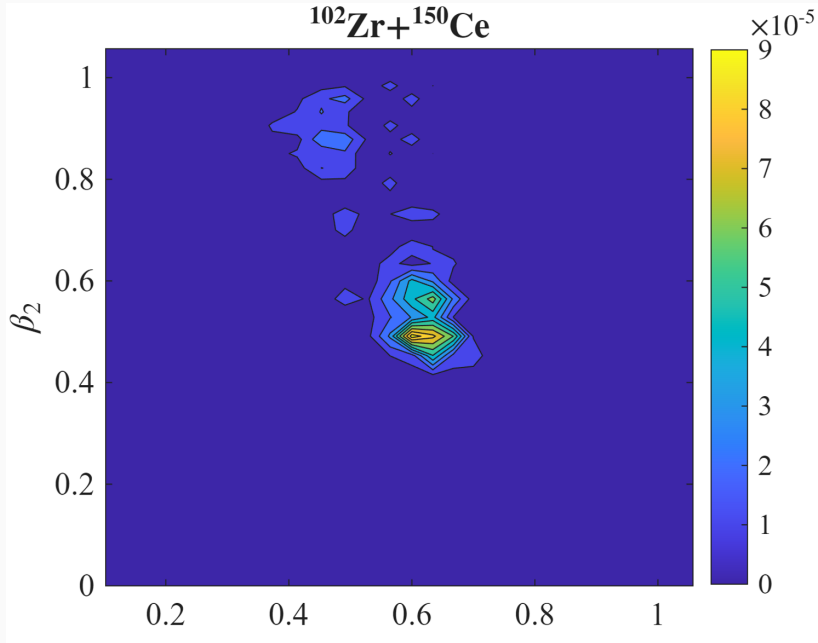
Angular momentum of the fragment increases with increase of its deformation at scission.

Decay-distributions of various DNS from SF of ^{252}Cf





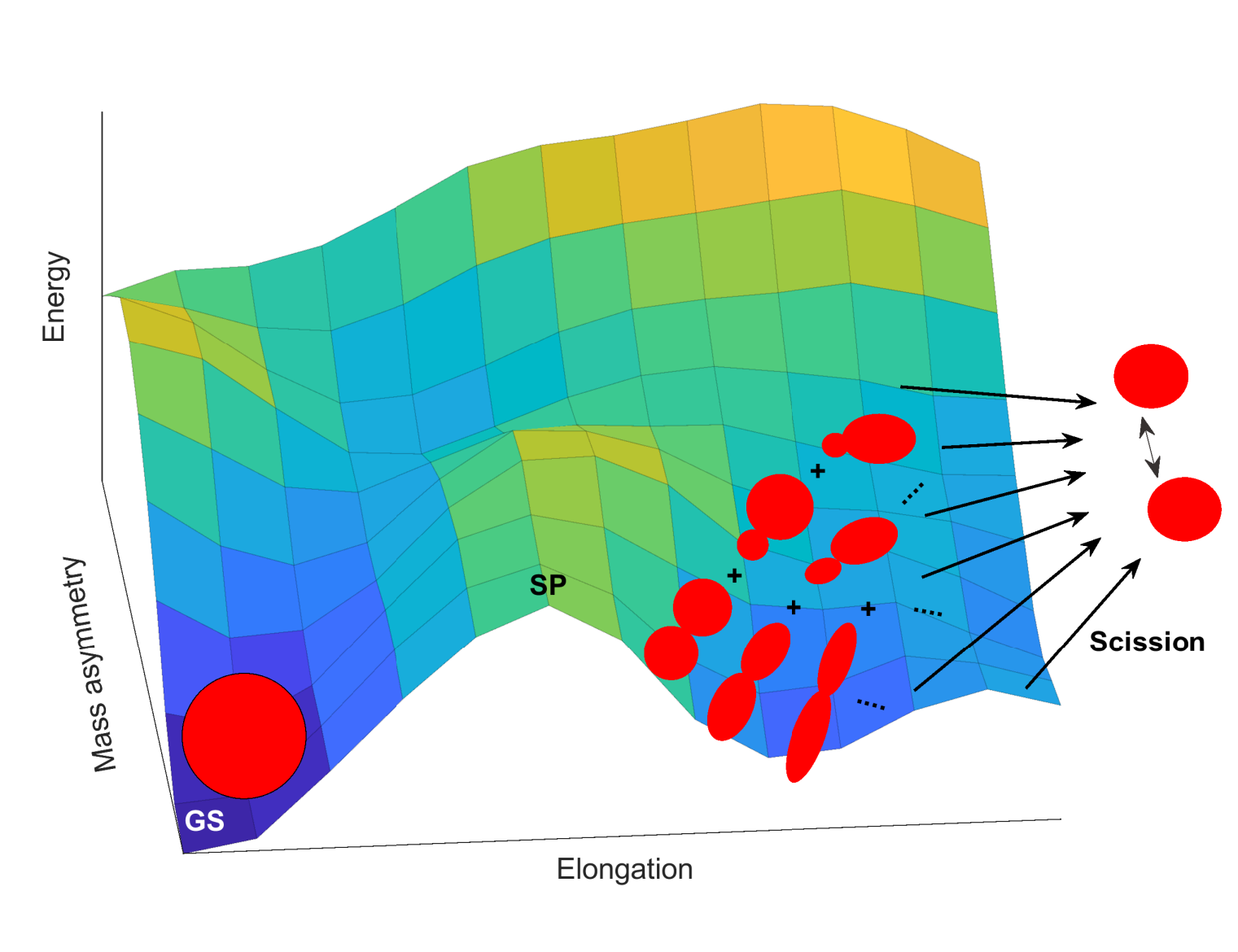




Conclusion

- The model describes the evolution of a fissioning nucleus after tunneling through the fission barrier as a random walk among various scission configurations.
- Angular motion in scission configuration of fissioning nucleus is described quantum-mechanically.
- The absence of correlation between the spins of fission fragments, the sawtooth pattern in spin vs mass distribution is explained.

Evolution of fissioning nucleus after crossing the fission barrier



Evolution of fissioning nucleus after crossing the fission barrier

The evolution of DNS distribution in time is described using the master equation:

$$n = (Z_1, A_1, \beta_1, Z_2, A_2, \beta_2).$$

$$\frac{dP(n)}{dt} = \sum_{n' \neq n} \Lambda(n'|n)P(n') - \sum_{n' \neq n} \Lambda(n|n')P(n) - \Lambda_d(n)P(n)$$

$P(n)$ – Probability that the system is in the state n.
$\Lambda(n|n')$ – Transition rates for switching from n to n'.
$\Lambda_d(n)$ – Decay rates.

The distribution of primary fission fragments

$$P_f(n[Z_1, A_1, \beta_1, Z_2, A_2, \beta_2])$$

is obtained with Monte-Carlo technique.

Transition rates

The transition rates are expressed in terms of the microscopic transition probabilities and of the level densities of the final state.

$$\Lambda(n|n') = \lambda_{nn'}\rho(E^*_{n'}, n'), \quad \Lambda(n'|n) = \lambda_{n'n}\rho(E^*_n, n)$$

$$\lambda_{n'n} = \lambda_{nn'} = \lambda^{(i)}/\sqrt{\rho(n')\rho(n)}; \quad i = (m.a., \beta)$$

$$\Lambda_d(n) = \lambda_d \rho_{s.p.}(E^* - V_B, n)$$

$$\lambda_d = \begin{cases} 1/\rho(E^*, n), & \text{if } (E^* > V_B) \\ 0, & \text{if } (E^* < V_B) \end{cases}$$

$\rho_{s.p.}$ – the level density at the top of barrier in the potential of DNS as a function of R

L.G. Moretto and J.S. Sventek, Phys. Lett. B **58**, 26 (1975).
G.G. Adamyan, A.K. Nasirov, N.V. Antonenko, R.V. Jolos , Fiz. Elem. Chastits At.Yadra **25**, 1379 (1994).

Level densities of DNS

The intrinsic level density (LD) of DNS:

$$\rho_{int}(E^*, n) = \int \rho_1(A_1, Z_1, \beta_1, \varepsilon)\rho_2(A_2, Z_2, \beta_2, E^* - \varepsilon)\mathrm{d}\varepsilon.$$

The LD of DNS fragments is calculated in superfluid formalism.

The DNS level density is obtained as a folding of intrinsic LD and the density of collective states:

$$\rho(E^*, n) = \sum_x \rho_{int}(E^* - E_x^n, n).$$

P. Decowski, et al., Nucl. Phys. A **110**, 129 (1968).

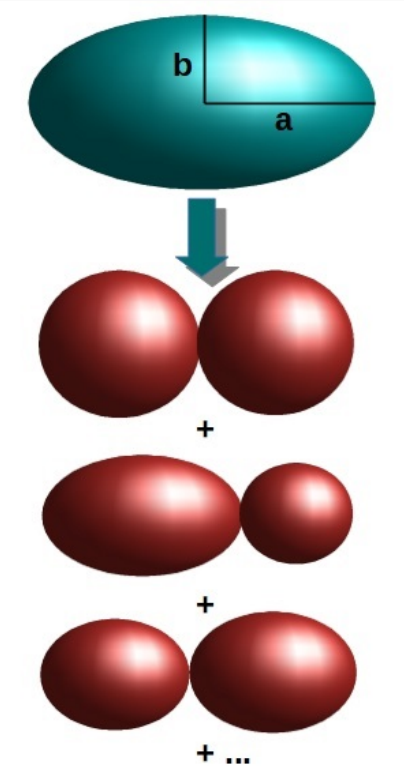
A. Bezbakh, et al., EPJA, **52**, 353 (2015).

A. Rahmatinejad, et al, Phys. Rev. C, **101**, 054315 (2020).

Choice of distribution of initial states

We choose the systems whose mass quadrupole moment lies in the interval of 10% around the value of quadrupole moment of ellipsoid with axis ratio $a : b$.

$Q_2^{ellips} \sim a : b$



$Q_2 \sim Q_2^{ellips} \pm 10\%$

- **Probability of each DNS**

$$P_0(n) \sim \rho(n, E^* = U_{comp} - U_{DNS})$$

- **Quadrupole moment of DNS**

$$Q_2(n) = 2\frac{A_1 A_2}{A} R^2 + Q_2(A_1, \beta_1) + Q_2(A_2, \beta_2)$$

Various calculations show that nucleus can be presented as a DNS around $Q_2^{ellips} \sim 3 : 1$.

S. Cwiok, W. Nazarewicz et al., PLB, **322**, 304 (1994).
T. M. Shneidman et al., NPA, **671**, 119 (2000).
A. V. Afanasjev et al., Phys. Scr. **93**, 034002 (2018).