

Introduction to Quantum Gravity and Cosmology

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Supported by the “”BASIS” Foundation for the Advancement of Theoretical Physics and Mathematics

Plan of lecture course

Canonical formalism of the gravitational field in GR

Quantization of gauge constrained systems

Wheeler-DeWitt equation, BFV formalism and path integral

Semiclassical approximation

Quantum states as a source of initial conditions for cosmological evolution: no-boundary, tunneling and the density matrix of the Universe

Schwinger-DeWitt technique for the calculation of effective action

Inflation theory

Canonical formalism of the gravitational field in Einstein GR

$$S[\,g_{\mu\nu},\phi\,] = \frac{M_P^2}{2}\int_{\mathcal{M}} d^4x\, g^{1/2}(R-2\Lambda) + S_{\text{matter}}[\,g_{\mu\nu},\phi\,] + \text{surface terms}$$

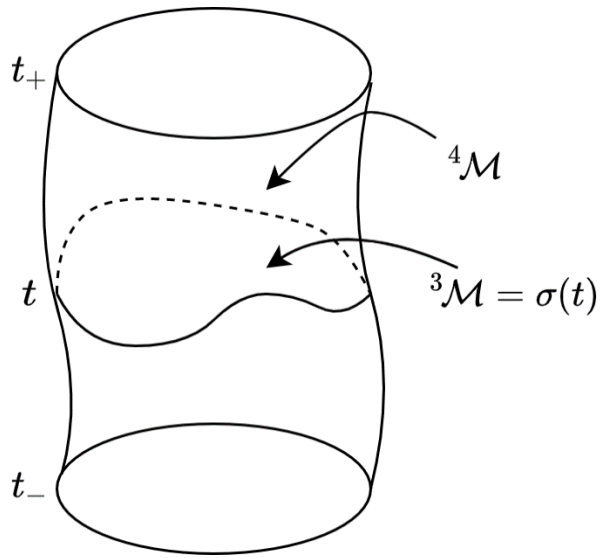
$$M_P^2 = \frac{1}{8\pi G}$$ --- *reduced Planck mass in terms of Newton constant* G

$$S_{\text{matter}}[\,g_{\mu\nu},\phi\,] = \int d^4x\, L(\phi, g_{\mu\nu}, \nabla_\mu\phi)$$

$$\nabla_\mu\phi = \partial_\mu\phi + \Gamma^\alpha_{\mu\beta}\hat{G}^\beta_\alpha\phi,$$

$\hat{G}^\beta_\alpha \in GL(4)$ *-- generator in the representation of* ϕ

Introduce foliation of spacetime by spacelike hypersurfaces



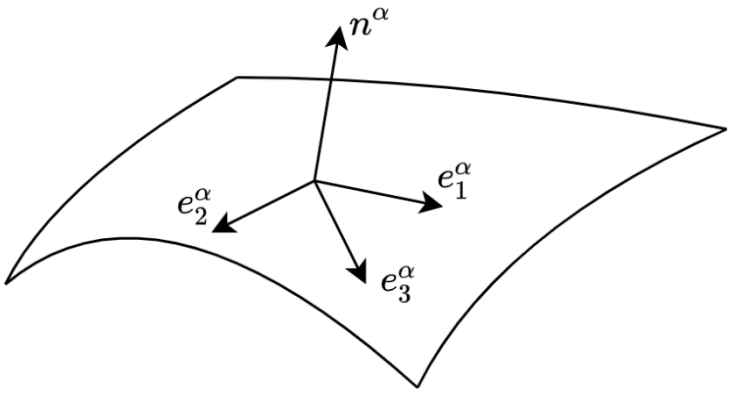
embedding functions

$$\sigma(t): \quad x^\alpha = e^\alpha(\mathbf{x}, t), \quad \alpha = 0,1,2,3,$$
$$\mathbf{x} = x^a, \quad a = 1,2,3$$

Basis of tangential and normal vectors

$$e^\alpha_a = \frac{\partial e^\alpha(\mathbf{x}, t)}{\partial x^a}, \quad n_\alpha e^\alpha_a = 0, \quad n_\alpha = g_{\alpha\beta} n^\beta$$

$$g_{\alpha\beta} n^\alpha n^\beta = \epsilon = \pm 1$$



spacelike hypersurface

$$\epsilon = -1$$

Induced metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \underbrace{e^\alpha_a g_{\alpha\beta} e^\beta_b}_{\gamma_{ab}} dx^a\, dx^b, \quad \gamma_{ab} = e^\alpha_a g_{\alpha\beta} e^\beta_b$$

Decomposition in the normal basis

$$\phi^\alpha(x)\Big|_{x=e(\mathbf{x},t)} = \phi_\perp n^\alpha + \phi^a e^\alpha_a, \quad \phi_\perp = \epsilon n_\alpha \phi^\alpha, \quad \phi^a = e^a_\alpha \phi^\alpha$$

$$e^a_\alpha \equiv \gamma^{ab} e^\beta_b g_{\beta\alpha} \quad \rightarrow \quad e^\alpha_a e^b_\alpha = \delta^b_a$$

$$g_{\alpha\beta} = \epsilon n_\alpha n_\beta + \gamma_{ab} e^a_\alpha e^b_\beta \quad \rightarrow \quad e^\alpha_a e^a_\beta = \delta^\alpha_\beta - \epsilon n^\alpha n_\beta \equiv \gamma^\alpha_\beta$$

$$\gamma^\alpha_\beta n_\alpha = n^\beta \gamma^\alpha_\beta = 0$$

projector on σ

Definition of induced 3D covariant derivative --- lift 3D vector to 4D one, covariantly differentiate and project back to 3D:

$$\phi^b \mapsto \phi^\beta \equiv e^\beta_b \phi^b, \quad \mathcal{D}_a \phi^b = e^\alpha_a \nabla_\alpha (e^\beta_c \phi^c) e^b_\beta = \partial_a \phi^b + \underbrace{e^\alpha_a (\nabla_\alpha e^\beta_c) e^b_\beta}_{{}^{(3)}\Gamma^b_{ac}} \phi^c$$

$${}^{(3)}\Gamma^b_{ac} = e^\alpha_a (\nabla_\alpha e^\beta_c) e^b_\beta$$

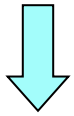
3-dimensional Christoffel symbol

Problem: prove that $\mathcal{D}_a \gamma_{bc} = 0$ ***and show that this is indeed the Christoffel symbol of*** γ_{bc}

$$^{(3)}\Gamma^b_{ac} = e^\alpha_a (\nabla_\alpha e^\beta_c) e^b_\beta$$ ***3-dimensional Christoffel symbol***

$$K_{ab} \equiv -(\nabla_\alpha n_\beta) e^\alpha_a e^\beta_b = e^\alpha_a (\nabla_\alpha e^\beta_b) n_\beta$$ ***extrinsic curvature***

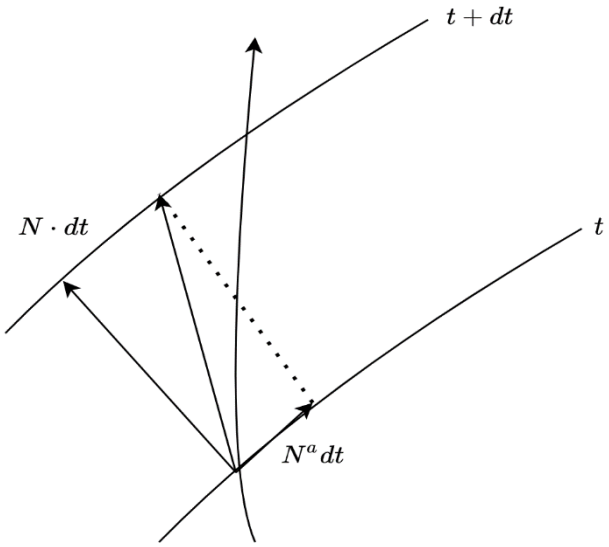
Tangential and normal projections of $e^\alpha_a (\nabla_\alpha e^\beta_b)$ ***--- tangential 4D covariant derivative of the triad***



$$e^\alpha_a (\nabla_\alpha e^\beta_b) = \epsilon n^\alpha K_{ab} + {}^{(3)}\Gamma^c_{ab} e^\alpha_c$$ ***Gauss-Weingarten equation***

Movement of spatial hypersurface in spacetime: lapse and shift functions

"velocity" vector $N^\alpha = \dfrac{\partial e^\alpha(\mathbf{x},t)}{\partial t}, \qquad N^\alpha = n^\alpha N + e^\alpha_a N^a$



Define covariant 4D derivative along the velocity vector field

$$\boldsymbol{\nabla}_N \phi \equiv \frac{\partial}{\partial t}\phi[\,e(\mathbf{x},t)\,] + N^\alpha\, \Gamma^\beta_{\alpha\lambda}\, \hat{G}^\lambda_\beta\, \phi[\,e(\mathbf{x},t)\,]$$

$$\begin{aligned}\boldsymbol{\nabla}_N e^\alpha_a &= \partial_a\left(\frac{\partial e^\alpha}{\partial t}\right) + \Gamma^\alpha_{\beta\lambda} e^\lambda_a\, N^\beta \\ &= \partial_a(n^\alpha N + e^\alpha_b N^b) + (n^\beta N + e^\beta_b N^b)\Gamma^\alpha_{\beta\lambda} e^\lambda_a \\ &= \nabla_a n^\alpha N + n^\alpha \partial_a N + (\nabla_a e^\alpha_b) N^b + e^\alpha_b \partial_a N^b\end{aligned}$$

Gauss-Weingarten equation

$$\boldsymbol{\nabla}_N e^\alpha_a = e^\alpha_b(\mathcal{D}_a\, N^b - K^b_a N) + n^\alpha(\mathcal{D}_a\, N + \epsilon K_{ab} N^b)$$

$$\frac{\partial \gamma_{ab}}{\partial t} = \partial_t (e^\alpha_a g_{\alpha\beta} e^\beta_b) = \boldsymbol{D}_N \left(e^\alpha_a e_{\alpha b} \right) = 2 \boldsymbol{D}_N e^\alpha_{(a} e_{\alpha b)} = \mathcal{D}_a N_b + \mathcal{D}_b N_a - 2 K_{ab} N$$



$$K_{ab} = \frac{1}{2N} \left(\mathcal{D}_a N_b + \mathcal{D}_b N_a - \dot{\gamma}_{ab} \right)$$

Lifting extrinsic curvature to the level of 4D tensor:

$$K_{\mu\nu} \equiv e^a_\mu e^b_\nu K_{ab} = -\gamma^\alpha_\mu \gamma^\beta_\nu \nabla_\alpha n_\beta = -\gamma^\alpha_\mu \nabla_\alpha n_\nu, \quad K_{\mu\nu} n^\nu = 0,$$

why this projector can be removed?

This allows one to relate 4D curvature tensor to 3D curvature one and extrinsic curvature

$$^{(4)}R^\mu{}_{\nu\alpha\beta} \iff {}^{(3)}R^a{}_{bcd},\ K_{ab}$$

The idea: $\gamma^{\rho}_{\alpha}\gamma^{\mu}_{\lambda}\gamma^{\nu}_{\sigma} \times \quad 2\nabla_{[\mu}\nabla_{\nu]}\phi^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}\,\phi^{\beta}, \qquad \phi^{\alpha} = e^{\alpha}_{a}\phi^{a}$

$$\gamma\gamma\gamma(\nabla\nabla\phi) \sim \underbrace{\gamma\nabla}_{\mathcal{D}}\underbrace{(\gamma\gamma\nabla\phi)}_{\mathcal{D}\phi} - 2\underbrace{(\gamma\nabla\gamma)}_{K}\underbrace{(\gamma\nabla\phi)}_{K\phi} \sim \mathcal{D}\mathcal{D}\phi - KK\phi$$

Problem: realize this idea

Gauss-Codazzi equation $\quad {}^{(4)}R_{abcd} = {}^{(3)}R_{abcd} - 2\epsilon K_{a[c}K_{d]b}$

$${}^{(4)}R^{ab}{}_{ab} = {}^{(3)}R - \epsilon(K^2 - K^2_{ab}), \qquad K = \gamma^{ab}K_{ab}$$

$$\| $$

$${}^{(4)}R_{\mu\nu\alpha\beta}(g^{\mu\alpha} - \epsilon n^{\mu}n^{\alpha})(g^{\nu\beta} - \epsilon n^{\nu}n^{\beta}) = -2\epsilon G_{\mu\nu}n^{\mu}n^{\nu}$$

Einstein tensor $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$$ {}^{(3)}R - \epsilon(K^2 - K^2_{ab}) = -2\epsilon G_{\mu\nu}n^{\mu}n^{\nu}$$

$${}^{(3)}R - \epsilon(K^2 - K_{ab}^2) = -2\epsilon G_{\mu\nu} n^\mu n^\nu$$

$$\begin{aligned}
{}^{(4)}R &= 2\epsilon(R_{\mu\nu} - G_{\mu\nu}) n^\mu n^\nu \\
&= 2\epsilon n^\nu (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n^\mu + {}^{(3)}R - \epsilon(K^2 - K_{ab}^2) \\
&= -2\epsilon(\nabla_\mu n_\nu)(\nabla^\nu n^\mu) + 2\epsilon(\nabla_\mu n^\mu)^2 + {}^{(3)}R - \epsilon(K^2 - K_{ab}^2) \\
&\quad + \text{total derivative} \\
&= {}^{(3)}R + \epsilon(K^2 - K_{ab}^2) + \text{total derivative}
\end{aligned}$$

$$S_E = \frac{M_P^2}{2} \int d^4x\, g^{1/2}(R - 2\Lambda) + \text{surface terms}$$

$$= \frac{M_P^2}{2} \int dt \underbrace{\int d^3x\, \gamma^{1/2} \left(K_{ab}^2 - K^2 + {}^{(3)}R - 2\Lambda \right)}_{L_{ADM}}$$

kinetic term $K_{ab} \sim \dot{\gamma}_{ab} + \ldots$

Arnowitt-Deser-Misner Lagrangian

$$L_{ADM} = L_{ADM}[\gamma_{ab}, \dot{\gamma}_{ab}, N, N^a]$$

From Lagrangian formalism to canonical formalism – Legendre transform with respect to $\dot{\gamma}_{ab}$

$$\pi^{ab}(\mathbf{x}) = \frac{\delta L_{ADM}}{\delta \dot{\gamma}_{ab}(\mathbf{x})} = -M_P^2 G^{ab,cd} K_{cd}, \quad G^{ab,cd} = \gamma^{1/2}\left(\gamma_{a(c}\gamma_{d)b} - \gamma_{ab}\gamma_{cd}\right)$$

No momenta for lapse and shift $\dfrac{\delta L_{ADM}}{\delta \dot{N}(\mathbf{x})} = 0, \quad \dfrac{\delta L_{ADM}}{\delta \dot{N}^a(\mathbf{x})} = 0$

DeWitt "supermetric" and its inverse

$$G^{ab,cd} G_{cd,ef} = \delta^{ab}_{ef}$$

Canonical action

$$S_{\text{canonical}}[\gamma_{ab}, \pi^{ab}, N, N^a]$$
$$= \int dt \left\{ \int_{^3\mathcal{M}} d^3x \left(\pi^{ab}\dot{\gamma}_{ab} - N H_\perp - N^a H_a\right) + \underbrace{\text{surface term}}_{-H_0} \right\}$$

0 ***in spatially closed cosmology***

Hamiltonian constraint $H_\perp = \dfrac{2}{M_P^2} G_{ab,cd}\pi^{ab}\pi^{cd} - \dfrac{M_P^2}{2}\gamma^{1/2}({}^{(3)}R - 2\Lambda)$

momentum constraint $H_a = -2\gamma_{ab} D_c \pi^{cb}$

With matter:

$$\pi^{ab}\dot{\gamma}_{ab} \to \pi^{ab}\dot{\gamma}_{ab} + \pi_\phi\dot{\phi}$$
$$H_\perp \to H_\perp + H_\perp^{\text{matter}}$$
$$H_a \to H_a + H_a^{\text{matter}}$$

Poisson brackets:

$$\{A, B\} = \int d^3x \, \frac{\delta A}{\delta\gamma_{ab}(\mathbf{x})}\frac{\delta B}{\delta\pi^{ab}(\mathbf{x})} - (A \leftrightarrow B) + \text{matter part}$$



Equations of motion with the Hamiltonian --- linear combination of constraints

$$H = \int d^3x \left(N^\perp H_\perp + N^i H_i\right)$$

Poisson brackets algebra

$$\{H_{a\mathbf{x}}, H_{b\mathbf{x}'}\} = H_{b\mathbf{x}'}\partial_a\delta(\mathbf{x}, \mathbf{x}') - (a\mathbf{x} \leftrightarrow b\mathbf{x}')$$
$$\{H_{\perp\mathbf{x}}, H_{a\mathbf{x}'}\} = -H_{\perp\mathbf{x}'}\partial_a\delta(\mathbf{x}, \mathbf{x}')$$
$$\{H_{\perp\mathbf{x}}, H_{\perp\mathbf{x}'}\} = \gamma^{ab}(\mathbf{x})H_{a\mathbf{x}}\partial_b\delta(\mathbf{x}, \mathbf{x}') - (\mathbf{x} \leftrightarrow \mathbf{x}')$$

DeWitt condensed notations

$$q^i = \gamma_{ab}(\mathbf{x}), \phi(\mathbf{x}), \quad p_i = \pi^{ab}(\mathbf{x}), p_\phi(\mathbf{x}) \quad a = 1,2,3, \;\; i \mapsto (ab\,\mathbf{x}), (\phi\,\mathbf{x})$$

$$N^\mu = (N^\perp(\mathbf{x}), N^a(\mathbf{x})), \quad H_\mu = (H_\perp(\mathbf{x}), H_a(\mathbf{x})), \;\; \mu \mapsto (\perp\mathbf{x}), (a\,\mathbf{x})$$

Summation-integration rule

$$p_i\dot{q}^i = \int d^3x \left(\pi^{ab}(\mathbf{x})\dot{\gamma}_{ab}(\mathbf{x}) + p_\phi(\mathbf{x})\dot{\phi}(\mathbf{x})\right)$$
$$N^\mu H_\mu = \int d^3x \left(N^\perp(\mathbf{x})H_\perp(\mathbf{x}) + N^a(\mathbf{x})H_a(\mathbf{x})\right)$$

$$S = \int dt \left\{p_i\dot{q}^i - H_0(q,p) - N^\mu H_\mu(q,p)\right\} \;\rightarrow\; \frac{\delta S}{\delta N^\mu} = -H_\mu(q,p) = 0$$

surface integral (0 in spatially closed cosmology)

constraints

Constraints algebra (first-class constraints)

$$\{H_\mu, H_\nu\} = U^\lambda_{\mu\nu} H_\lambda,$$
$$\{H_0, H_\nu\} = U^\lambda_{0\,\nu} H_\lambda$$

Classical theory of constrained systems

Local gauge invariance of the action:

canonical transformation

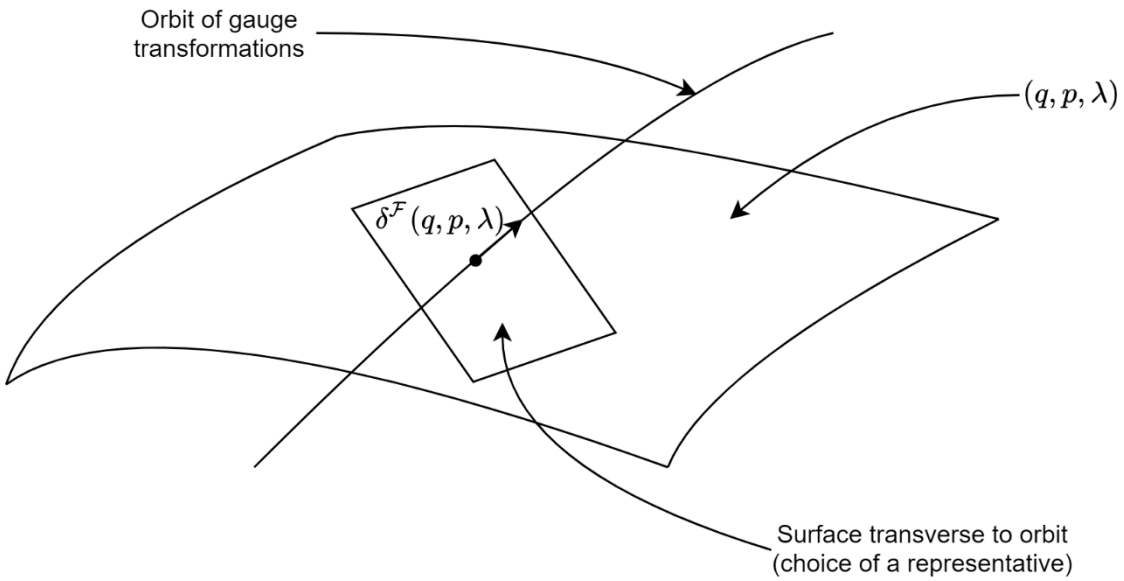
$$\delta^{\mathcal{F}} \begin{bmatrix} q \\ p \end{bmatrix} = \left\{ \begin{bmatrix} q \\ p \end{bmatrix}, H_\mu \right\} \mathcal{F}^\mu, \quad \mathcal{F}^\mu = \mathcal{F}^\mu(t)$$

non-canonical

$$\delta^{\mathcal{F}} N^\mu = \dot{\mathcal{F}}^\mu - U^\mu_{\alpha\beta} N^\alpha \mathcal{F}^\beta - U^\mu_{0\,\nu} \mathcal{F}^\nu$$

$$\delta^{\mathcal{F}} S = 0$$

local parameter



Gauge fixation – choice of the representative of the equivalence class on the orbit of gauge group by imposing the gauge

$$\chi^\mu(q, p) = 0$$

for uniqueness of the representative should be independent of the Lagrangian multipliers --- unitary gauges

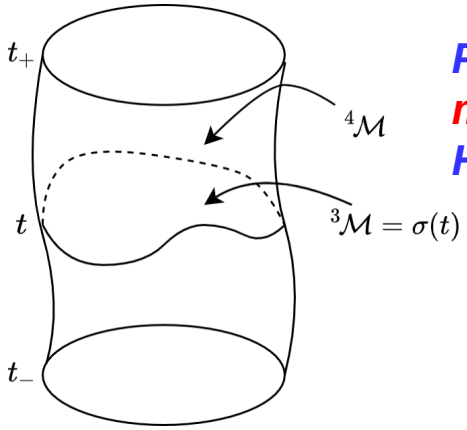
$$\delta^{\mathcal{F}}\chi^{\mu} = \{\chi^{\mu}, H_{\nu}\}\mathcal{F}^{\nu} \neq 0 \;\rightarrow\; \underbrace{J^{\mu}_{\nu} \equiv \det\{\chi^{\mu}, H_{\nu}\} \neq 0}$$

Canonical Faddeev-Popov operator is invertible

The inverse of Faddeev-Popov operator

Recovery of unique Lagrange multipliers

$$\frac{d}{dt}\chi^{\mu}(q,t) = \{\chi^{\mu}, H_0\} + \{\chi^{\mu}, H_{\nu}\}N^{\nu} = 0 \;\rightarrow\; N^{\mu} = -J^{-1}{}^{\mu}_{\nu}\{\chi^{\mu}, H_0\}$$



For $H_0 = 0$ ***lapse and shift functions should be nonzero --- moving in spacetime spatial section*** $\sigma(t)$

How to provide this?

explicitly time-dependent gauge $\chi^{\mu}(q) = 0 \;\rightarrow\; \chi^{\mu}(q,t) = 0$

$$N^{\mu} = -J^{-1}{}^{\mu}_{\nu}\frac{\partial\chi^{\nu}}{\partial t} \neq 0$$

This is the problem of frozen time formalism and its solution

Hamiltonian reduction to the physical sector

$$S = \int dt \left\{ p_i \dot{q}^i - H_0(q,p) - N^\mu H_\mu(q,p) \right\}, \qquad \underbrace{i = 1, \ldots n, \quad \mu = 1, \ldots m}$$

formal counting (in field theory always both are infinite)

Solution of first-class constraints and gauges (ultralocal in time, but nonlocal in space)

$$\frac{\delta S}{\delta N^\mu} = -H_\mu(q,p) = 0$$

$$\chi^\mu(q,p) = 0$$

$$\Rightarrow \quad q^i = e^i(\xi^A, \pi_A)$$
$$p_i = P_i(\xi^A, \pi_A), \qquad A = 1, \ldots n-m$$

$n - m = \#$ ***physical degrees of freedom***

$$S_{\text{phys}}[\,\xi, \pi\,] = \int dt \left[\pi_A \dot{\xi}^A - H_{\text{phys}}(\xi, \pi) \right]$$

$$H_{\text{phys}}(\xi,\pi) = H_0\Big(e(\xi,\pi), P(\xi,\pi)\Big) - P_i(\xi,\pi)\frac{\partial e^i(\xi,\pi,t)}{\partial t}$$

Origin oi this term?

if $\chi^\mu = \chi^\mu(q,p,t)$
$\rightarrow e^i = e^i(\xi,\pi,t)$

Problem: example of a relativistic particle as a toy model of cosmology

Action $S[\,x(t)\,] = -m\int dt\sqrt{-\dot{x}^2}, \quad \dot{x}^2 = \dot{x}^\alpha \dot{x}_\alpha$

$$q^i = x^\alpha, \quad p_i = p_\alpha, \quad \alpha = 0, 1, ...3$$

Derive the constraint, introduce it with the Lagrange multiplier N in the canonical action and obtain the analogue of the ADM canonical action. Disentangle the physical sector in the gauge

$$\chi(x^\alpha, t) = x^0 - t = 0$$

?

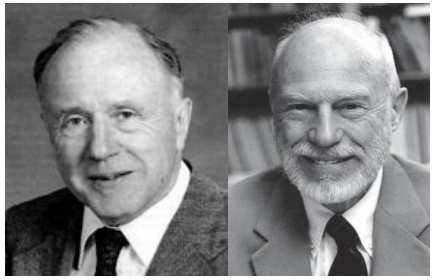
Problem: Friedmann cosmological model

Scale factor

$$ds^2 = -N^2 dt^2 + a^2(t)\sigma_{ab} dx^a\, dx^b$$

Metric of a unit radius sphere

Build the minisuperspace Lagrangian action and the Hamiltonian constraint in Einstein GR with the cosmological constant Λ



Wheeler-DeWitt equation(s)

Quantization $H_\mu(q,p) = 0 \;\rightarrow\; \hat{H}_\mu \big| \Psi \big\rangle = 0.$

Functional coordinate representation of Hamiltonian and momentum constraints

$$\pi^{ab}(\mathbf{x}), p_\phi(\mathbf{x}) \;\rightarrow\; \pi^{ab}(\mathbf{x}) = \frac{\hbar}{i}\frac{\delta}{\delta\gamma_{ab}(\mathbf{x})}, \qquad p_\phi(\mathbf{x}) = \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}$$

$$\Big\{ -\frac{2\hbar^2}{M_P^2} G_{ab,cd}(\mathbf{x}) \frac{\delta^2}{\delta\gamma_{ab}(\mathbf{x})\,\delta\gamma_{cd}(\mathbf{x})} - \frac{M_P^2}{2}\gamma^{1/2}(\mathbf{x})\Big({}^3R(\mathbf{x}) - 2\Lambda\Big) + H_\perp^{\text{matter}}\Big(\phi(\mathbf{x}), \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\Big)\Big\}\Psi[{}^3\gamma, \phi] = 0,$$

$$\Big\{ -2\gamma_{ab}(\mathbf{x})\nabla_c \frac{\delta}{\delta\gamma_{bc}(\mathbf{x})} + H_a^{\text{matter}}\Big(\phi(\mathbf{x}), \frac{\hbar}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\Big)\Big\}\Psi[{}^3\gamma, \phi] = 0$$

No Hamiltonian, no time, no Schroedinger equation, operator ordering ???

WDW equation is the ``most useless" equation in theoretical physics?

No Hamiltonian, no time, no Schroedinger equation, operator ordering ???

WDW equation is the ``most useless" equation in theoretical physics?

Dirac quantization scheme is not complete !

$$\hat{H}_\mu |\Psi\rangle = 0 \;\rightarrow\; \Psi(q) \sim \delta(\hat{H}_\mu)\Phi(q)$$

For instance, problem of physical inner product



$$\langle \Psi' | \Psi \rangle_{\text{phys}} \overset{?}{=} \int dq\, \Psi'^{*}(q)\, \Psi(q) \sim \int dq \left[\delta(\hat{H}_\mu) \right]^2 = \infty$$

Schroedinger equation from Wheeler-DeWitt equation(s)



$$i\frac{\partial}{\partial t}|\,\Psi(t)\,\rangle = \hat{H}|\,\Psi(t)\,\rangle \quad \Longleftrightarrow \quad \hat{H}_\mu|\,\Psi[\,{}^3\gamma,\phi\,] = 0$$

First attempts to extract time and Schroedinger equation from the Wheeler-DeWitt equations (DeWitt (1967), Lapchinsky-Rubakov (~1977))

$$\hat{H}_\mu = \hat{H}_\mu^{\rm grav} + \hat{H}_\mu^{\rm matter}$$

$$\Psi[\,{}^3\gamma,\phi\,] = \exp\left(\frac{i}{M_P^2}\,S\big[{}^3\gamma\big]\right)\Big|\,\Psi[\,{}^3\gamma\,]\,\Big\rangle$$

Semiclassical gravity factor

quantum state in the Hilbert space of matter parametrically depending on 3-metric

Einstein-Hamilton-Jacobi equation(s) for the phase S

$$H_\perp^{\rm grav}\left(\gamma_{ab},\frac{\delta S}{\delta\gamma_{ab}}\right) \equiv \frac{2}{M_P^2}\,G_{ab,cd}(\mathbf{x})\frac{\delta S}{\delta\gamma_{ab}(\mathbf{x})}\frac{\delta S}{\delta\gamma_{cd}(\mathbf{x})} - \frac{M_P^2}{2}\gamma^{1/2}(\mathbf{x})\left({}^3R(\mathbf{x}) - 2\Lambda\right) = 0,$$

$$H_a^{\rm grav}\left(\gamma_{ab},\frac{\delta S}{\delta\gamma_{ab}}\right) \equiv -2\gamma_{ab}(\mathbf{x})\nabla_c\frac{\delta S[\,{}^3\gamma\,]}{\delta\gamma_{bc}(\mathbf{x})} = 0$$

V.G. Lapchinsky and V.A. Rubakov,
Acta Phys. Polon. B10 (1979) 1041-1048

Quantum matter wave function in external gravitational field

$$|\Psi(t)\rangle = \left|\Psi[\,{}^3\gamma_{ab}(t)\,]\right\rangle$$

Solution of classical vacuum Einstein eqs. with ADM lapse and shift $N^{\perp}(t),\ N^i(t)$

Problem: prove this statement

$$\hat{H}_\mu \Psi[\,{}^3\gamma, \phi\,] = 0 \quad \Rightarrow$$

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}_{\text{matter}}|\Psi(t)\rangle + O\left(1/M_P^2\right)$$

graviton loops

$$\hat{H}_{\text{matter}} = \int d^3x \left(N^{\perp}\hat{H}_{\perp}^{\text{matter}} + N^a \hat{H}_a^{\text{matter}}\right)$$

Semiclassical expansion in $1/M_P^2$***, no back reaction, classical gravitational field as a whole – source of time evolution***

Dirac quantization scheme

Theory with first class constraints

$$S = \int dt \left\{ p_i \dot{q}^i - H_0(q,p) - N^\mu H_\mu(q,p) \right\} \;\rightarrow\; \frac{\delta S}{\delta N^\mu} = -H_\mu(q,p) = 0$$

Constraints algebra

$$\{H_\mu, H_\nu\} = U^\lambda_{\mu\nu} H_\lambda,$$
$$\{H_0, H_\nu\} = U^\lambda_{0\,\nu} H_\lambda$$

Quantization

$$(q, p, H_\mu, H_0) \rightarrow (\hat{q}, \hat{p}, \hat{H}_\mu, \hat{H}_0)$$

Quantum Dirac constraints

$$H_\mu(q,p) = 0 \;\rightarrow\; \hat{H}_\mu |\,\Psi\rangle = 0.$$

Consistency conditions (generalization of Poisson bracket algebra)

$$[\hat{H}_\mu, \hat{H}_\nu] = i\hbar\, \hat{U}^\lambda_{\mu\nu} \hat{H}_\lambda$$

Stands to the left of constraint operators

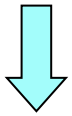
More general theories --- hierarchy of structure functions

$$H_\mu, U^\alpha_{\mu\nu} \to G = \left\{H_\mu, U^\alpha_{\mu\nu}, U^{\alpha\beta}_{\mu\nu\lambda}, ...\right\}$$

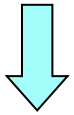
$$\{H_\mu, \{H_\sigma, H_\lambda\}\} + \text{cycle}(\mu,\sigma,\lambda) = 0$$

From Jacobi identity for the Poisson brackets

Mechanism of origin of higher order structure functions



$$\left(\{H_\mu, U^\alpha_{\sigma\lambda}\} + U^\beta_{\sigma\lambda}U^\alpha_{\mu\beta} + \text{cycle}(\mu,\sigma,\lambda)\right)H_\alpha = 0$$



$$\{H_\mu, U^\alpha_{\sigma\lambda}\} + U^\beta_{\sigma\lambda}U^\alpha_{\mu\beta} + \text{cycle}(\mu,\sigma,\lambda) = U^{\alpha\beta}_{\mu\nu\lambda}H_\beta, \qquad U^{\alpha\beta}_{\mu\nu\lambda} = -U^{\beta\alpha}_{\mu\nu\lambda}$$

Quantization $\quad G \to \hat{G} = \left\{\hat{H}_\mu, \hat{U}^\alpha_{\mu\nu}, \hat{U}^{\alpha\beta}_{\mu\nu\lambda}, ...\right\}$

$$[\,\hat{H}_\mu, \hat{U}^\alpha_{\sigma\lambda}] + i\hbar\hat{U}^\beta_{\sigma\lambda}\hat{U}^\alpha_{\mu\beta} + \text{cycle}(\mu,\sigma,\lambda) = i\hbar\hat{U}^{\alpha\beta}_{\mu\nu\lambda}\hat{H}_\beta, \qquad \hat{U}^{\alpha\beta}_{\mu\nu\lambda} = -\hat{U}^{\beta\alpha}_{\mu\nu\lambda}$$

Batalin-Fradkin-Vilkovisky (BFV) quantization formalism

Canonical pairs of relativistic phase space:

Momenta conjugated to Lagrangian multipliers

$$q^i, p_i \;\rightarrow\; Q^I, P_I = q^i, p_i;\; N^\mu, \pi_\mu;\; \underbrace{C^\mu, \mathcal{P}_\mu;\; \bar{C}_\mu, \bar{\mathcal{P}}^\mu}_{\text{ghosts}},$$

$$[Q^I, P_J] = i\,\delta^I_J, \qquad \hbar = 1$$

$$[A, B] \equiv AB - (-1)^{n(A)n(B)} BA$$

Grassmann parity: $n(q) = n(N) = 0, \quad n(C) = n(\mathcal{P}) = n(\bar{C}) = n(\bar{\mathcal{P}}) = 1$

$$||\, Q \,\rangle\rangle \equiv ||\, q, N, C, \bar{C} \,\rangle\rangle, \quad \hat{Q}^I ||\, Q \,\rangle\rangle = Q^I ||\, Q \,\rangle\rangle, \quad \boldsymbol{\Psi}(Q) = \langle\langle\, Q \,||\, \boldsymbol{\Psi} \,\rangle\rangle,$$

BFV inner product: $\langle\langle\, \boldsymbol{\Psi}_1 \,||\, \boldsymbol{\Psi}_2 \rangle\rangle = \int dQ\, \boldsymbol{\Psi}_1{}^*(Q)\, \boldsymbol{\Psi}_2(Q)$

Hermiticity: $C^{\mu\dagger} = C^\mu, \quad \mathcal{P}^\dagger_\mu = -\mathcal{P}_\mu, \quad \bar{C}^\dagger_\mu = -\bar{C}_\mu, \quad \bar{\mathcal{P}}^{\mu\dagger} = \bar{\mathcal{P}}^\mu$

Nilpotent Grassmann BRST operator

$$[\hat{\Omega}, \hat{\Omega}] \equiv 2\hat{\Omega}^2 = 0$$

$$\hat{\Omega} = \pi_\alpha \bar{\mathcal{P}}^\alpha + C^\mu \hat{H}_\mu + \frac{1}{2} C^\nu C^\mu \hat{U}^\lambda_{\mu\nu} \mathcal{P}_\lambda, \qquad \hat{\Omega}^\dagger = \hat{\Omega}$$

$$\Downarrow$$

$$\hat{H}_\mu - \hat{H}^\dagger_\mu = i\hat{U}^\lambda_{\mu\lambda}$$

Unitarizing Hamiltonian $\qquad \hat{\mathcal{H}}_\Phi = \hat{\mathcal{H}}_0 + \frac{1}{i}[\hat{\Phi}, \hat{\Omega}]$

Gauge fermion $\qquad \hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \chi^\mu(q), \qquad n(\hat{\Phi}) = 1$

Unitary evolution operator

$$i\hbar\frac{\partial}{\partial t}\hat{U}_\Phi(t,t_-)=\hat{\mathcal{H}}_\Phi\hat{U}_\Phi(t,t_-),\qquad \hat{U}_\Phi(t_-,t_-)=\mathbb{I}$$

Selection of physical states

$$\hat{\Omega}\,||\,\Psi\rangle\rangle=0$$

Main property: gauge independence of physical matrix elements

$$[\,\hat{\Omega},\hat{U}_\Phi(t,t_-)\,]=0$$



$$\hat{\Omega}\,||\,\Psi_{1,2}\rangle\rangle=0\quad\rightarrow\quad\delta_\Phi\langle\langle\,\Psi_1||\,\hat{U}_\Phi(t_+,t_-)\,||\,\Psi_2\rangle\rangle=0$$

Feynman path integral representation

$$U_\Phi(\,t_+,Q_+|\,t_-,Q_-)\equiv\langle\langle\,Q_+\,||\,\hat{U}_\Phi(t_+,t_-)\,||\,Q_-\,\rangle\rangle$$
$$=\int\limits_{Q(t_\pm)=Q_\pm}D[\,Q,P\,]\,\exp\left\{i\int_{t_-}^{t_+}dt\,\left(P_I\dot{Q}^I-\mathcal{H}_\Phi(Q,P)\right)\right\}$$

$$D[\,Q,P\,]=\prod_t dQ(t)\prod_{t^*}dP(t^*)$$
$$\mathcal{H}_\Phi(Q,P)=\mathcal{H}_0+\{\,\Phi,\Omega\,\}$$

Dirac quantization as truncation of BFV formalism

Invariance of solutions of BRST equation $\hat{\Omega}\,||\,\Psi\rangle\rangle = 0 \qquad ||\,\Psi\,\rangle\rangle \to ||\,\Psi\,\rangle\rangle' = ||\,\Psi\,\rangle\rangle + \hat{\Omega}\,||\,\Phi\,\rangle\rangle$

Batalin-Marnelius gauge fixing

$$||\,\Psi\,\rangle\rangle \to ||\,\Psi_{BM}\rangle\rangle : \quad \hat{\mathcal{P}}_\mu||\,\Psi_{BM}\rangle\rangle = 0,\ \hat{N}^\mu||\,\Psi_{BM}\rangle\rangle = 0$$



Quantum Dirac constraints from BRST equation

$$0 = \frac{1}{i}\,[\,\hat{\Omega}, \hat{\mathcal{P}}_\mu]\,||\,\Psi_{BM}\rangle\rangle = (\hat{H}_\mu + C^\nu \hat{U}^\lambda_{\nu\mu}\hat{\mathcal{P}}_\lambda)\,||\,\Psi_{BM}\rangle\rangle = \hat{H}_\mu\,||\,\Psi_{BM}\rangle\rangle\rangle$$

$$[\hat{\Omega}, N^\mu]\,||\,\Psi_{BM}\rangle\rangle = \frac{\partial}{\partial C^\mu}\Psi_{BM}(Q) = 0, \qquad \frac{\partial}{\partial \bar{C}_\mu}\Psi_{BM}(Q) = 0$$

Structure of BM wavefunction

$$\Psi_{BM}(Q) = \langle\, q\,|\,\Psi\,\rangle\,\delta(N) \equiv \Psi(q)\,\delta(N),$$

Recovery of Dirac constraints from BFV formalism

$$\hat{\Omega}\,\Psi_{BM}(Q) = 0 \ \to\ \hat{H}_\mu\Psi(q) = 0$$

Dirac wavefunction

Physical inner product

$$\int dQ\,|\,\Psi_{BM}(Q)\,|^2 \sim \int dN\,[\delta(N)]^2 \int dC\,d\bar{C} = \infty \times 0 \quad ?$$

Gauge independent regularization of inner product

$$\langle\langle\,\Psi'\,||\,\Psi\,\rangle\rangle_{\text{phys}} = \langle\langle\,\Psi'_{BM}||\,e^{[\hat{\Phi},\hat{\Omega}]}||\,\Psi_{BM}\rangle\rangle$$

$$[\hat{\Omega},[\hat{\Phi},\hat{\Omega}]] = 0, \quad \hat{\Omega}\,||\,\Psi_{BM}\rangle\rangle = 0 \quad \Rightarrow \quad \frac{\delta}{\delta\Phi}\langle\langle\,\Psi'\,||\,\Psi\,\rangle\rangle_{\text{phys}} = 0$$

Gauge fermion and Faddeev-Popov operator

$$\hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \hat{\chi}^\mu \quad \Rightarrow \quad \hat{J}^\mu_\nu = \frac{1}{i}\,[\,\hat{\chi}^\mu, \hat{H}_\nu]$$

Physical inner product

$$\langle\langle\,\Psi'\,||\,\Psi\,\rangle\rangle_{\text{phys}} = \int dq\,\Psi'^*(q)\,\hat{M}\left(q, \frac{\partial}{i\partial q}\right)\Psi(q) = \langle\,\Psi'\,|\,\hat{M}\,|\,\Psi\,\rangle$$

operator-valued measure

$$\hat{M} = \int dN\,dC\,d\bar{C}\,\delta(N)\,e^{-i\bar{C}_\mu \hat{J}^\mu_\nu C^\nu + \hat{\chi}^\mu \frac{\partial}{\partial N^\mu}}\,\delta(N)$$

$$= \int d\pi\,dC\,d\bar{C}\,e^{-i\bar{C}_\mu \hat{J}^\mu_\nu C^\nu + i\pi_\mu \hat{\chi}^\mu} = \delta(\hat{\chi})\,\det\hat{J}^\mu_\nu\left(1 + O(\,[\hat{\chi},\hat{J}\,])\right)$$

multi-loop corrections

$$\langle\,\Psi'\,|\,\Psi\,\rangle_{\text{phys}} = \int dq\,\Psi'^*(q)\,\delta(\chi(q))\,\det\hat{J}^\mu_\nu\,\Psi(q) + O(\hbar)$$

Truncation of BFV formalism to Dirac quantization

$$\Psi(q) = \int dN\, \boldsymbol{\Psi}(q, N, C, \bar{C})\,\Big|_{C=0}$$

$$U(q_+, q_-) = \int dN_+\, dN_-\, \boldsymbol{U}_{\Phi}(\, t_+, Q_+ |\, t_-, Q_-)\,\Big|_{C_\pm=0}$$

$$U(q_+, q_-) \equiv \langle\langle\, \boldsymbol{\Psi}_+ ||\, \hat{\boldsymbol{U}}_{\Phi}(t_+, t_-)\, ||\, \boldsymbol{\Psi}_- \rangle\rangle,$$

$$\boldsymbol{\Psi}_\pm(q, N, C, \bar{C}) = \delta(q - q_\pm)\delta(C)\delta(\bar{C}), \quad \hat{\pi}_\alpha ||\, \boldsymbol{\Psi}_\pm\rangle\rangle = 0, \quad \hat{\Omega}\, ||\, \boldsymbol{\Psi}_\pm\rangle\rangle = 0$$

Gauge and time Independence (spatially closed cosmology)

$$\delta_{\Phi} U(q_+, q_-) = 0$$

$$i\frac{\partial}{\partial t_+} U(q_+, q_-) = \langle\langle\, \boldsymbol{\Psi}_+ ||\, \frac{1}{i}\,[\hat{\Phi}, \hat{\Omega}\,]\, \hat{\boldsymbol{U}}_{\Phi}(t_+, t_-)\, ||\, \boldsymbol{\Psi}_- \rangle\rangle = 0$$

Quantum Dirac constraints

$$\hat{H}_\mu \hat{U} = 0, \quad \hat{U} \overleftarrow{H}{}^{\dagger}_{\mu} = 0$$



Projector on the space of physical states

$$U(q, q') = \langle\, q\,|\,\hat{U}\,|\,q'\,\rangle, \quad \hat{U} = \text{const} \times \text{''} \prod_{\mu} \delta(\hat{H}_\mu) \text{''}$$

Noncommuting!

Choice of gauge fermion for relativistic gauge:

$$\hat{\Phi} = \mathcal{P}_\mu N^\mu + \bar{C}_\mu \hat{\chi}^\mu$$

$$U(q_+, q_-) = \int D[Q,P] \exp\left[i \int_{t_-}^{t_+} dt \left(p_i \dot{q}^i - N^\mu H_\mu + \mathcal{P}_\mu \dot{C}^\mu + \bar{\mathcal{P}}_\mu \dot{\bar{C}}^\mu \right.\right.$$

$$\left.\left. + \pi_\mu (\dot{N}^\mu - \chi^\mu) - \bar{C}_\mu J^\mu_\nu C^\nu - \mathcal{P}_\alpha (\bar{\mathcal{P}}^\alpha + U^\alpha_{\mu\nu} N^\mu C^\nu) \right) \right] \Bigg|_{q(t_\pm)=q_\pm,\ C_\pm=0}$$

Boundary conditions !

$N_\pm$ *integrated over*

relativistic gauge conditions

Transition to unitary gauge: $\chi^\mu \to \dfrac{\chi^\mu}{\varepsilon}, \quad \pi_\mu \to \varepsilon \pi_\mu, \quad \bar{C}_\mu \to \varepsilon \bar{C}_\mu; \quad \varepsilon \to 0$

Canonical Faddeev-Popov integral:

$$U(q_+, q_-) = \int_{q(t_\pm)=q_\pm} D[q,p]\, DN \left(\prod_{t_+ > t > t_-} \delta(\chi)\, \det J^\mu_\nu \right) \exp\left[i \int_{t_-}^{t_+} dt \left(p_i \dot{q}^i - N^\mu H_\mu \right) \right]$$

Faddeev-Popov gauge-fixing factor (missing at $t_\pm$ ***)***

Semiclassical approximation

$$H_\mu \overset{?}{\to} \hat{H}_\mu : \quad [\hat{H}_\mu, \hat{H}_\nu] = i\hbar \hat{U}^\lambda_{\mu\nu}\hat{H}_\lambda, \quad \hat{H}_\mu - \hat{H}^\dagger_\mu = i\hbar \hat{U}^\lambda_{\mu\lambda}$$

$$\hat{H}_\mu = \mathcal{N}_W\left\{H_\mu + \frac{i\hbar}{2}U^\nu_{\mu\nu} + O(\hbar^2)\right\}, \quad \hat{U}^\lambda_{\mu\nu} = \mathcal{N}_W\left\{U^\lambda_{\mu\nu} - \frac{i\hbar}{2}U^{\lambda\sigma}_{\mu\nu\sigma} + O(\hbar^2)\right\}$$

0 *in GR*

Weyl ordering

one-loop prefactor

Semiclassical solution $\quad U(q,q') = \Big[\, P(q,q') + O(\hbar)\,\Big] \exp\left[\frac{i}{\hbar}S(q,q')\right]$

$$\hat{H}_\mu U(q,q') = 0 \;\Rightarrow\; \begin{cases} H_\mu\big(q, \partial S/\partial q\big) = 0, & \textit{Hamilton-Jacobi equation} \\ \dfrac{\partial}{\partial q^i}(\nabla^i_\mu P^2) = U^\lambda_{\mu\lambda}P^2, & \nabla^i_\mu \equiv \dfrac{\partial H_\mu}{\partial p_i}\Big|_{p=\partial S/\partial q} \end{cases}$$

"continuity" equations

important vector fields

The analogue of Pauli-Van Vleck matrix---degenerate:

$$S_{ik'} = \frac{\partial^2 S(q,q')}{\partial q^i\, \partial q^{k'}}$$

$$\frac{\partial}{\partial q^{k'}} H_\mu\big(q, \partial S/\partial q\big) = 0$$

$$\nabla^i_\mu S_{ik'} = 0, \quad S_{ik'}\nabla^{k'}_\nu = 0, \quad \nabla^{k'}_\nu \equiv \frac{\partial H_\nu(q',p')}{\partial p'_k}\bigg|_{p'=-\partial S/\partial q''}$$

$$S_{ik'} \;\to\; F_{ik'} = S_{ik'} + \underbrace{\chi^\mu_i c_{\mu\nu}\chi^\nu_{k'}}_{\text{gauge-breaking term}}, \qquad \underbrace{\chi^\mu_i \equiv \frac{\partial\chi^\mu(q)}{\partial q^i}, \quad \chi^\nu_{k'} \equiv \frac{\partial\chi^\nu(q')}{\partial q^{k'}}, \quad c_{\mu\nu}}_{\text{gauge matrices}}$$

Solution of continuity equation $$P = \left[\frac{\det F_{ik'}}{J\, J'\det c_{\mu\nu}}\right]^{1/2}$$

$$J \equiv \det J^\mu_\nu(q) \neq 0, \quad J^\mu_\nu(q) = \chi^\mu_i \nabla^i_\nu,$$
$$J' \equiv \det J^\mu_\nu(q') \neq 0, \quad J^\mu_\nu(q') = \chi^\mu_{i'} \nabla^{i'}_\nu$$

Faddeev-Popov determinants for gauges $\chi^\mu(q), \chi^\nu(q')$

Recognize in this expression the analogue of the one-loop effective action with the contribution of the Faddeev-Popov determinant.

Problem:

1) prove gauge independence of the prefactor

$$\delta_\chi P(q,q') = 0$$

2) prove closure of the Lie bracket algebra

$$\nabla^i_\mu \frac{\partial \nabla^k_\nu}{\partial q^i} - \nabla^i_\nu \frac{\partial \nabla^k_\mu}{\partial q^i} = U^\lambda_{\mu\nu} \nabla^k_\lambda$$

3) prove the ‘’continuity” equation for the prefactor

$$\frac{\partial}{\partial q^i}(\nabla^i_\mu P^2) = U^\lambda_{\mu\lambda} P^2$$

Problem:

derive semiclassical prefactor in the one-dimensional toy model – show how gauge-fixing and gauge independence explicitly work

$$H_\perp(q,p) = -p^2 + \omega^2(q)$$

$$\hat{H}_\perp = \frac{d^2}{dq^2} + \omega^2(q)$$

$$S(q,q') = \pm \int_{q'}^{q} dy\, \omega(y)$$

$$P(q,q') = ?$$

Hamiltonian reduction and semiclassical inner product

"Moving" physical subspace

$$\Sigma(t) : q^i = e^i(\xi, t), \;\; \chi^\mu(e^i(\xi, t), t) \equiv 0$$

$$q^i \to \bar{q}^i = (\xi^A, \theta^\mu), \;\; q^i = e^i(\xi^A, \theta^\mu), \;\; \theta^\mu = \chi^\mu(q),$$

$$d^{n-m}\xi = d^n q \, \delta(\chi) \, M, \;\; M = \left(\det \left[\frac{\partial e^i}{\partial \xi^A} \, \frac{\partial e^i}{\partial \theta^\mu} \right] \right)^{-1}$$

Physical Hamilton-Jacobi function

$$S(t, \xi | t', \xi') = S(q, q') \Big|_{q = e(\xi, t), \; q' = e(\xi', t')},$$

$$\frac{\partial^2 S(t, \xi | t', \xi')}{\partial \xi^A \, \partial \xi^{B'}} = S_{ik'} \frac{\partial e^i}{\partial \xi^A} \frac{\partial e^{k'}}{\partial \xi^{B'}} \Big|_{q = e(\xi, t), \; q' = e(\xi', t')}$$

Semiclassical unitary evolution operator in the physical sector (Pauli-Van Vleck formula)

$$U_{\text{phys}}(t, \xi | t', \xi') \equiv \left[\det \frac{i}{2\pi\hbar} \frac{\partial^2 S(t, \xi | t', \xi')}{\partial \xi^A \, \partial \xi^{B'}} \right]^{1/2} e^{\frac{i}{\hbar} S(t, \xi | t', \xi')} + O(\hbar)$$

Unitary map between physical and Dirac-Wheeler-DeWitt semiclassical evolution operators

$$U_{\rm phys}(t,\xi|t',\xi') = \text{const}\,\left(\frac{J}{M}\right)^{1/2} U(q,q')\left(\frac{J'}{M'}\right)^{1/2}\Bigg|_{q=e(\xi,t),\,q'=e(\xi',t')}$$

Unitary map between wavefunctions

$$\Psi_{\rm phys}(t,\xi) = \text{const}\,\left(\frac{J}{M}\right)^{1/2}\Psi(q)\Big|_{q=e(\xi,t)}$$

Unitarity --- conservation of physical inner product

$$\left(\Psi_1^{\rm phys}(t)\,|\,\Psi_2^{\rm phys}(t)\right)_{\rm phys} \equiv \int d\xi\,\Psi_1^{\rm phys\,*}(\xi,t)\,\Psi_2^{\rm phys}(\xi,t)$$
$$= \int dq\,\Psi_1^*(q)\,\underbrace{J_\chi\,\delta\Big(\chi(q,t)\Big)}_{\textit{measure}}\,\Psi_2(q) \equiv \langle\,\Psi_1\,|\,\Psi_2\,\rangle_{\rm phys}$$

Gauge independence of physical inner product

$$\frac{\delta}{\delta\chi}\int dq\,\Psi_1^*(q)\,J_\chi\,\delta(\chi)\,\Psi_2(q) = 0\ \textbf{?}$$

Exercise on Stokes theorem

$$\int dq\,\Psi_1^*(q)\,J_\chi\,\delta(\chi)\,\Psi_2(q) = \int\limits_\Sigma \omega^{(n-m)},$$

$$\omega^{(n-m)} = \frac{dq^{i_1}\wedge\ldots\wedge dq^{i_{n-m}}}{(n-m)!}\epsilon_{i_1\ldots i_n}\Psi_2^*\,\nabla_1^{i_{n-m+1}}\ldots\nabla_m^{i_n}\Psi_1$$

$m = \mathrm{codim}\,\Sigma$ -- ***# of continuity type equations:***

Lie bracket algebra of vector fields

$$\left.\begin{aligned}\frac{\partial}{\partial q^i}\left(\nabla_\mu^i\,\Psi_2^*\,\Psi_1\right) &= U_{\mu\lambda}^\lambda\Psi_2^*\,\Psi_1\\ \nabla_\mu^i\frac{\partial\nabla_\nu^k}{\partial q^i} - \nabla_\nu^i\frac{\partial\nabla_\mu^k}{\partial q^i} &= U_{\mu\nu}^\lambda\nabla_\lambda^k\end{aligned}\right\}\ \rightarrow\quad d\omega^{(n-m)} = 0$$

closed form

$\mathcal{M}_{12}$ ***forms a cobordism of*** Σ_1 ***and*** Σ_2

$$\partial\mathcal{M}_{12} = \Sigma_1\bigcup\Sigma_2,\qquad \dim\mathcal{M}_{12} = \dim\Sigma_{1,2} + 1$$



$$\int\limits_{\Sigma_2}\omega^{(n-m)} - \int\limits_{\Sigma_1}\omega^{(n-m)} = \int\limits_{\mathcal{M}_{12}} d\omega^{(n-m)} = 0$$

Problem: ***Dirac constraints operators are Hermitian with respect to physical inner product***

$$\langle \hat{H}_\mu \Psi_1 | \Psi_2 \rangle_{\text{phys}} - \langle \Psi_1 | \hat{H}_\mu \Psi_2 \rangle_{\text{phys}}$$

multi-loop orders

$$= \langle \Psi_1 | \hat{H}^\dagger_\mu J\,\delta(\chi) | \Psi_2 \rangle - \langle \Psi_1 | J\,\delta(\chi) \hat{H}_\mu | \Psi_2 \rangle = 0(\hbar)$$

Conjugation with respect to auxiliary BFV inner product $\hat{H}^\dagger_\mu = \hat{H}_\mu - i\hbar U^\lambda_{\mu\lambda}$

Try proving:
A.Barvinsky, unpublished e-print

The same is true for invariant physical observables subject to classical algebra:

$$\{\mathcal{O}_I, T_\mu\} = U^\lambda_{I\,\mu} T_\lambda, \quad \{\mathcal{O}_I, \mathcal{O}_J\} = U^L_{IJ} \mathcal{O}_L + U^\lambda_{IJ} T_\lambda, \quad U^L_{IJ} = \text{const}$$

Their semiclassical operator realization:

$$\mathcal{O}_I \to \hat{\mathcal{O}}_I, \quad [\hat{\mathcal{O}}_I, \hat{T}_\mu] = i\hbar \hat{U}^\lambda_{I\,\mu} \hat{T}_\lambda,$$

$$\hat{\mathcal{O}}_I = \mathcal{N}_W \left\{ \mathcal{O}_I + \frac{i\hbar}{2} U^\lambda_{I\lambda} + \frac{i\hbar}{2} U^J_{IJ} + O(\hbar^2) \right\}$$

All this is really not working!

$$U^{\lambda}_{\mu\lambda} = ?$$

$$U^{c\mathbf{x}''}_{a\mathbf{x}\; b\mathbf{x}'} = \delta^c_b \partial_a \delta(\mathbf{x},\mathbf{x}')\delta(\mathbf{x}',\mathbf{x}'') - (a\mathbf{x} \leftrightarrow b\mathbf{x}')$$

$$\int d^3x'\, U^{b\mathbf{x}''}_{a\mathbf{x}\; b\mathbf{x}'}\Big|_{\mathbf{x}'=\mathbf{x}''} \sim \delta(0) = \infty$$

+ other UV divergences

We need other 4D covariant formalism to be able to regularize and renormalize the theory --- path integral method

Let us begin with minisuperspace applications in which this problem does not at all arise

Closed Friedmann cosmological model

Spatially homogeneous (minisuperspace) metric

$$ds^2 = -N^2 dt^2 + a^2(t)\sigma_{ab} dx^a\, dx^b$$

Metric of a unit radius S^3

$$q^i,\ p_i,\ N^\mu \mapsto\ a(t),\ p(t),\ N(t)$$

$$S_{ADM}[a, N] = \pi^2 M_P^2 \int dt\, N a^3 \left[6\left(\frac{1}{a^2} - \frac{\dot{a}^2}{N^2 a^2} \right) - 2\Lambda \right]$$

$$H_\perp(a, p) = \frac{1}{24\pi^2 M_P^2} \left[-\frac{p^2}{a} + (12\pi^2 M_P^2)^2 a(H^2 a^2 - 1) \right]$$

$$H^2 = \frac{\Lambda}{3}$$

Hubble constant in terms of the cosmological constant

$$[\hat{a},\hat{p}]=i,\quad |\Psi\rangle=\Psi(a),\quad \hat{H}_\perp|\Psi\rangle=\hat{H}_\perp\Big(a,\frac{d}{ida}\Big)\Psi(a)=0,\quad \hbar=1$$

Weyl ordering

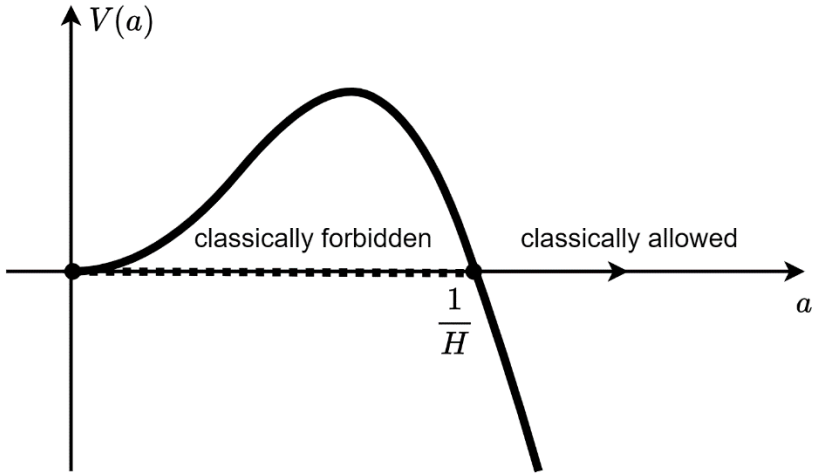
$$\mathcal{N}_W\frac{\hat{p}^2}{\hat{a}}=\frac{1}{\sqrt{\hat{a}}}\,\hat{p}^2\frac{1}{\sqrt{\hat{a}}}+O(\hbar^2)=-\frac{1}{\sqrt{a}}\,\frac{d^2}{da^2}\frac{1}{\sqrt{a}}+O(\hbar^2)$$

Problem: ***prove this relation by using***

$$\mathcal{N}_W f(q,p)=f\left(\frac{\partial}{\partial I},\frac{\partial}{\partial J}\right)\exp(I\hat{q}+J\hat{p})\Big|_{I=J=0}$$

Minisuperspace Wheeler-DeWitt equation

$$\Big[-\frac{d^2}{da^2}-(12\pi^2M_P^2)^2\,a^2(H^2a^2-1)\Big]\psi(a)=0,\quad \Psi(a)=\sqrt{a}\psi(a)$$



Stationary Schroedinger equation in the potential V(a) at the energy level E=0

Semiclassical momentum

$$p(a) = \sqrt{-V(a)} = (12\pi^2 M_P^2)\, a\sqrt{(Ha)^2 - 1}, \quad a > \frac{1}{H}$$

Semiclassical solutions in overbarrier (Lorentzian) and underbarrier (Euclidean imaginary time) domains

$$\Psi_{\pm}^{L}(a) = \frac{1}{\left(H^2a^2-1\right)^{1/4}} e^{\pm iS(a)}, \quad S(a) = -\int_{H^{-1}}^{a} da'\, p(a') = -\frac{4\pi^2 M_P^2}{H^2}\left(H^2a^2-1\right)^{3/2}, \quad a > \frac{1}{H}$$

$$\Psi_{\pm}^{E}(a) = \frac{1}{\left(H^2a^2-1\right)^{1/4}} e^{\pm I(a)}, \quad I(a) = \int_{H^{-1}}^{a} da'\, |\, p(a')\, | = -\frac{4\pi^2 M_P^2}{H^2}\left(1-H^2a^2\right)^{3/2}, \quad a < \frac{1}{H}$$

Tunneling wavefunction of the Universe (Vilenkin)

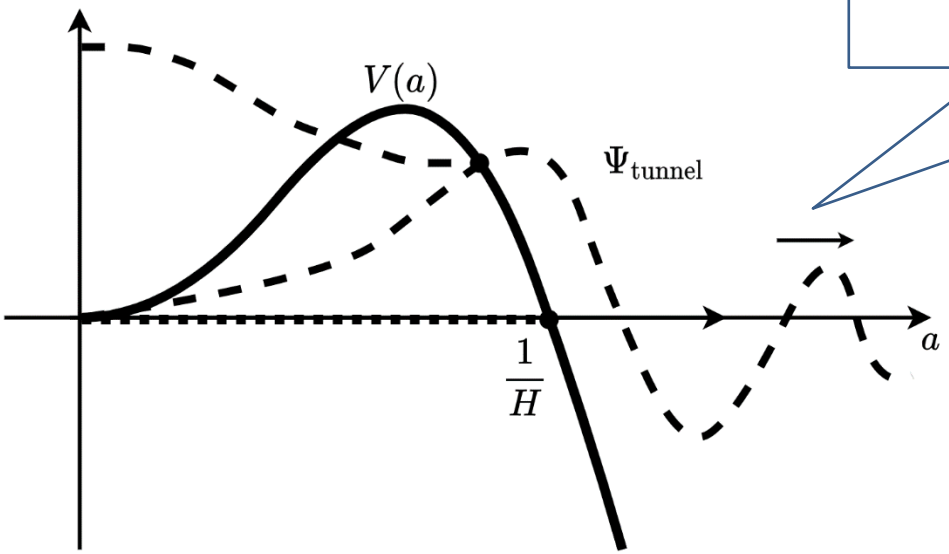
$$\Psi_{\text{tunnel}}(a) = \frac{1}{\left(H^2a^2-1\right)^{1/4}} \exp\left[-\frac{4\pi^2 M_P^2}{H^2} + iS(a)\right], \quad a > \frac{1}{H}$$

Outgoing wave (expanding Universe)

$$p(a) = \frac{dS(a)}{da} < 0$$

$$p(a,\dot{a}) = -12\pi^2 M_P^2\, a\dot{a}/N$$

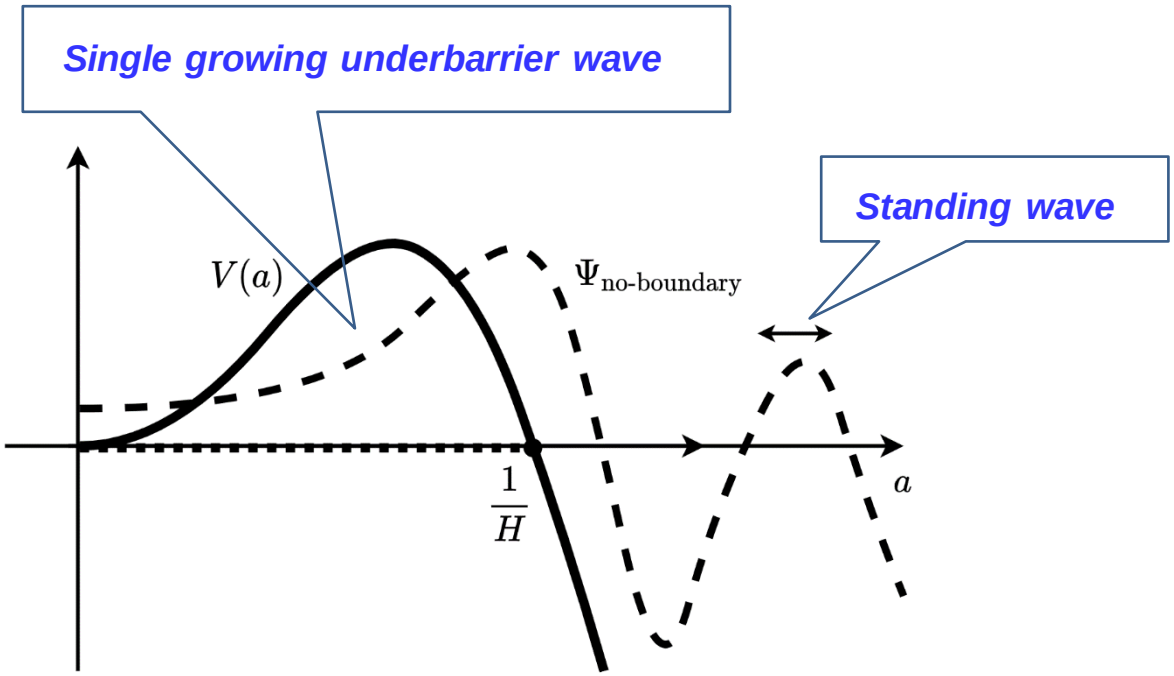
$$\hat{p}\,\Psi(a) \simeq p(a)\,\Psi(a)$$



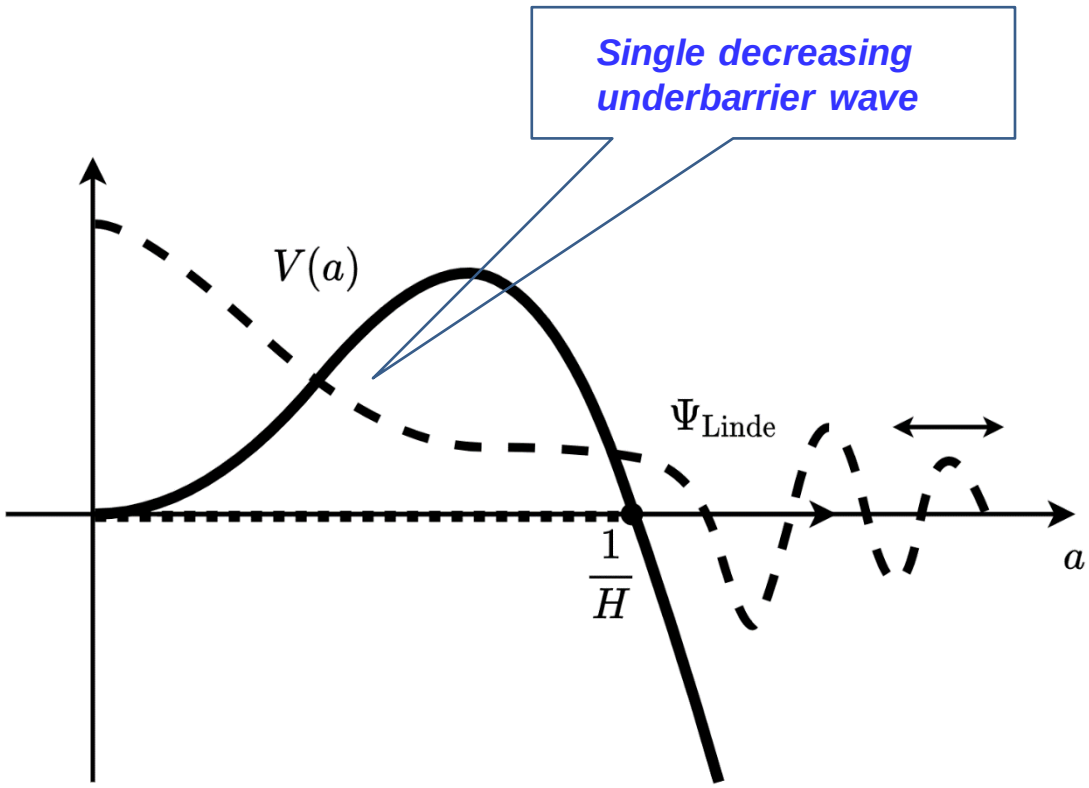
No-boundary (Hartle-Hawking) wavefunction

$$\Psi_{\text{no-boundary}}(a) = \frac{1}{\left(H^2a^2-1\right)^{1/4}} \exp\left[\frac{4\pi^2 M_P^2}{H^2}\right] \cos\left[S(a)+\frac{\pi}{4}\right], \quad a > \frac{1}{H}$$

$$\Psi_{\text{no-boundary}}(a) = \frac{1}{\left(H^2a^2-1\right)^{1/4}} \exp\left[\frac{4\pi^2 M_P^2}{H^2}\left(1-(1-H^2a^2)^{3/2}\right)\right], \quad a < \frac{1}{H}$$



Linde wavefunction



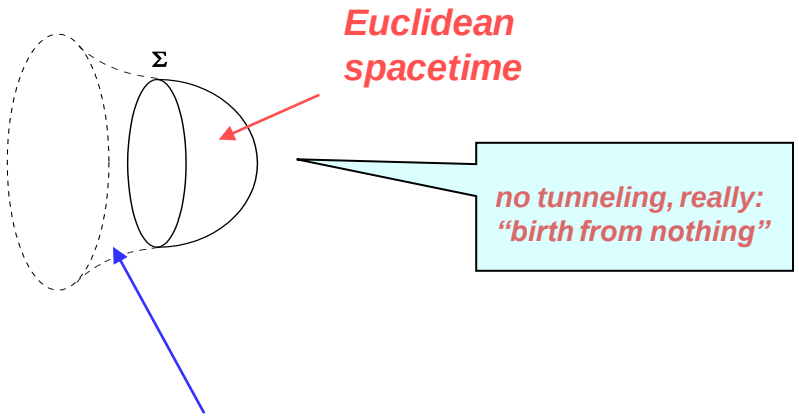
The meaning of exponential factor $\exp\left[\frac{4\pi^2 M_P^2}{H^2}\right] = ?$

Euclideanized gravitational action

in ADM parametrization

$$I[g^E_{\mu\nu}] = -\frac{M_P^2}{2}\int d^4x\, g^{1/2}(R-2\Lambda)\Big|_{\text{Euclidean metric}} = -iS[g_{\mu\nu}]\Big|_{N\to iN^E}$$

$$g_{\mu\nu}\to g^E_{\mu\nu}:\ N\to iN^E$$



$$I = -\frac{4\pi^2 M_P^2}{H^2} < 0$$

4D Euclidean hemisphere

$$ds^2 = d\tau^2 + a_E^2(\tau)\, d\Omega^2_{(3)}$$

$$a_E(\tau) = \frac{1}{H}\sin(H\tau),$$

Analytic continuation to Lorentzian signature de Sitter geometry:

$$\tau = \pi/2H + it$$

$$a_L(t) = \frac{1}{H}\cosh(Ht)$$

Application in inflation theory: quantum initial conditions for cosmological inflation

Slow roll approximation – neglecting kinetic term for inflaton scalar field, effective cosmological and Hubble constants are generated by the inflaton potential

$$\Lambda \;\rightarrow\; \frac{V(\phi)}{M_P^2} = 3H^2(\phi)$$

$$\hat{H}_\perp \,|\,\Psi\,\rangle = 0 \;\rightarrow\; \left[-\frac{\partial^2}{\partial a^2} + \frac{6M_P^2}{a^2}\frac{\partial^2}{\partial \phi^2} - (12\pi^2 M_P^2)^2\, a^2 (H^2(\phi)a^2 - 1)\right]\psi(a,\phi) = 0$$

cannot neglect at $a \to 0$

$$\Rightarrow \quad \psi(0,\phi) \sim \text{const}$$

$$\Downarrow$$

$$\psi_{\text{no-boundary/tunnel}}\Big|_{a=1/H(\phi)} \sim \exp\left[\pm\frac{4\pi^2 M_P^2}{H^2(\phi)}\right]$$

Probability of inflation:

$$P(\phi) \simeq |\,\Psi(\phi)\,|^2\Big|_{a=1/H(\phi)} \simeq \left(\exp\left[\pm\frac{4\pi^2 M_P^2}{H^2(\phi)}\right]\right)^2 = \exp\left[\pm\frac{24\pi^2 M_{\mathrm{P}}^4}{V(\phi)}\right]$$

no-boundary

tunneling

Point of nucleation of the Lorentzian spacetime from the Euclidean one

Hartle-Hawking no-boundary wavefunction

$$\Psi_{\text{no-boundary}} \sim \exp(-I) = \exp\left(12\pi^2 \frac{M_P^4}{V(\phi)}\right) \to \infty$$

$$\frac{V(\phi)}{M_P^2} = \Lambda_{\text{eff}} = 3H_{\text{eff}}^2 \to 0$$

Most probable at the minimum of inflaton potential $H_{eff} \to 0$
-- insufficient amount of inflation

Tunneling wavefunction

$$\Psi_{\text{tunnel}} \sim \exp(+I) \to 0, \quad V(\phi) \to 0$$

Cosmology debate: no-boundary vs tunneling

Questionable status of both states within unitarity approach to quantum gravity

Why the Hartle-Hawking wavefunction is called no-boundary ?

Euclidean quantum gravity path integral for no-boundary wavefunction (Hartle-Hawking)

$$\Psi_{\text{no-boundary}}[\,{}^3\gamma,\varphi\,] = \int D[\,{}^4g,\phi\,]\, e^{-I_{\mathcal{M}}[\,{}^4g,\phi\,]}\,\Big|_{\,{}^3g(\partial\mathcal{M})={}^3\gamma,\ \phi(\partial\mathcal{M})=\varphi}$$

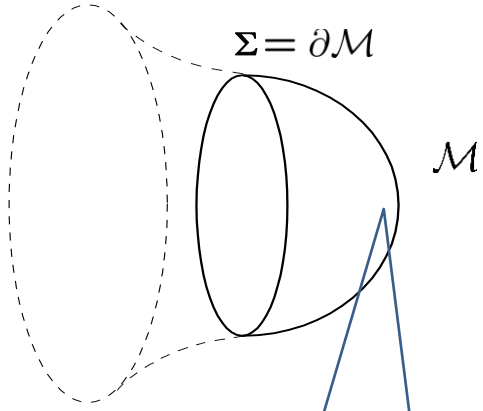
Euclidean metric and matter regular on $\mathcal{M}$, includes all gauge fixing details

Euclidean action $$I_{\mathcal{M}}[\,g_{\mu\nu},\phi\,] = -\frac{M_P^2}{2}\int_{\mathcal{M}} d^4x\, g^{1/2}(R-2\Lambda) + S_{\text{matter}}[\,g_{\mu\nu},\phi\,]$$

Semiclassically:

$$\Psi_{\text{no-boundary}}[\,{}^3\gamma,\varphi\,] = e^{-I_{\mathcal{M}}[\,{}^4g_0,\phi_0\,]}$$
$${}^4g_0\,|_{\partial M} = {}^3\gamma,\quad \phi_0\,|_{\partial M} = \varphi$$

Saddle point –- solution subject to boundary data



no initial Cauchy boundary, just regularity

Beyond minisuperspace: no-boundary wavefunction is a vacuum of linearized fields

$$\Phi\Big|_{\partial\mathcal{M}} = \varphi \to \varphi + f, \quad f \ll \varphi \quad \Longrightarrow \quad \Psi_{\text{no-boundary}}[\,{}^3\gamma, \varphi + f\,] \sim e^{-I^{(2)}_{\mathcal{M}}[\,f\,]}$$

Quasi-Gaussian state

Quadratic part of the action

$$I_{\mathcal{M}}[\,{}^4g_0, \Phi_0 + \phi\,] = I^{(0)} + I^{(2)} + O[\,f^2\,]$$

$$I^{(2)} = \frac{1}{2}\phi\vec{W}\phi\Big|_{\partial\mathcal{M}} = \frac{1}{2} f\omega f, \qquad \omega = \left[\vec{W}u\right]u^{-1}\Big|_{\partial\mathcal{M}}$$

Wronskian operator of the Hessian of the action

Hessian of the action

$$F \equiv \frac{\delta^2 I}{\delta\Phi\,\delta\Phi}\Big|_{\Phi_0}, \quad \int_{\mathcal{M}} \left(\psi\,\vec{F}\phi - \psi\overleftarrow{F}\,\phi\right) = \int_{\partial\mathcal{M}} \left(\psi\,\vec{W}\phi - \psi\overleftarrow{W}\,\phi\right)$$

Full set of regular basis functions of the wave operator

$$Fu = 0, \quad u\Big|_{\mathcal{M}} = \text{regular}$$

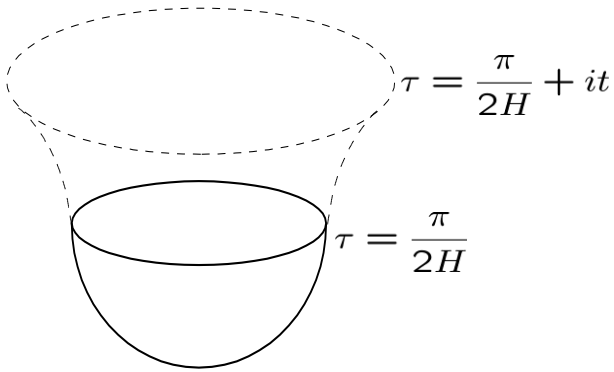
Euclidean wavefunction

prefactor from one-loop functional determinant

$$\Psi_E(\tau, f) = \Psi_{\text{no-boundary}}[\,{}^3\gamma(\tau), f\,] \simeq \frac{1}{\sqrt{u(\tau)}} \exp\left[-\frac{1}{2} f\omega(\tau) f\right]$$

Transition to Lorentzian spacetime

$$\tau \quad \rightarrow \quad \tau = \frac{\pi}{2H} + it$$



negative frequency basis function

$$u\left(\tfrac{\pi}{2H} + it\right) = v^*(t)$$

positive frequency basis function

$$u\left(\tfrac{\pi}{2H} - it\right) = v(t)$$

Lorentzian wavefunction

$$\Psi_L(t, f) \simeq \frac{1}{\sqrt{u(\tau)}} \exp\left[-\frac{1}{2} f\omega(\tau) f\right]\Bigg|_{\tau=\frac{\pi}{2H}+it}$$

solves the Schroedinger equation

Conservation of probability

$$\int df \left|\,\Psi_L(t, f)\,\right|^2 \sim \frac{1}{\sqrt{|v(t)|^2(\omega + \omega^*)}} \sim \sqrt{\langle\, v(t), v(t)\,\rangle} = \text{const}$$

Conserved inner product in the space of solutions of the Lorentzian wave equation

Microcanonical density matrix of the Universe

$$\hat{\rho} = \frac{1}{Z} \sum_{\text{all } |\Psi\rangle} w_\Psi |\Psi\rangle\langle\Psi|, \qquad w_\Psi = 1$$

sum over "everything" that satisfies the Wheeler-DeWitt equation $\hat{H}_\mu |\Psi\rangle = 0$

Motivation:

A simplest analogy in unconstrained system with a conserved Hamiltonian $\hat{H}$ ***Is the microcanonical density matrix with a fixed energy*** ***E***

$$\hat{\rho} \sim \delta(\hat{H} - E)$$

Spatially closed cosmology does not have freely specifiable constants of motion. The only conserved quantities are the Hamiltonian and momentum constraints H_μ***, all having a particular value --- zero.***

An ultimate equipartition in the full set of states of the theory --- "Sum over Everything". Creation of the Universe from Everything is conceptually more appealing than creation from Nothing, because the democracy of the microcanonical equipartition better fits the principle of Occam razor, preferring to drop redundant assumptions, than the selection of a concrete state.

Projector onto the subspace of quantum gravitational constraints

Partition function

$$\hat{\rho} = \frac{1}{Z}\prod_{\mu}\delta(\hat{H}_\mu), \qquad Z = \mathrm{Tr}_{\mathrm{phys}}\prod_{\mu}\delta(\hat{H}_\mu)$$

See Lecture 3

$$\begin{cases} \langle\, q \,|\, \prod_{\mu}\delta(\hat{H}_\mu) \,|\, q' \,\rangle = U(q,q') \\ \mathrm{Tr}_{\mathrm{phys}}\hat{U} = \int dq\, \hat{M}_+\, U(\underbrace{q_+, q_-}_{q_\pm = (\gamma^{\pm}_{ab},\, \phi_\pm)})\Big|_{q_\pm = q} \end{cases}$$

Faddeev-Popov gauge-fixing factors (missing at $t_\pm$)

$$Z = \int dq\, \hat{M}_{t_+} \int\limits_{q(t_\pm)=q_\pm} D[\,q,p\,]\, DN \Big(\prod_{t_+>t>t_-} M_t\Big)\, \exp\Big[\, iS[\,q,p,N\,]\Big]\Big|_{q_\pm=q}$$

$$= \int\limits_{\text{periodic}} D[\,q,p\,]\, DN \Big(\prod_{\text{all } t} M_t\Big)\, \exp\Big[\, iS[\,q,p,N\,]\Big]$$

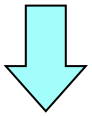
$$= \int\limits_{\text{periodic}} D[\,g_{\mu\nu},\phi\,]\; e^{iS[\,g_{\mu\nu},\phi\,]}$$

inner product measure is absorbed into path imtegral measure

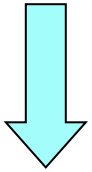
Lorentzian and covariant in relativistic gauge

No need for inner product and its measure – path integral takes care of it automatically!

Absence of periodic Lorentzian histories



rotation of integration contours over fields (or time)



$$Z = \int_{\text{periodic}} D[\, g_{\mu\nu}, \Phi\,]\; e^{-S[\, g_{\mu\nu}, \Phi\,]}$$

Problem:

euclideanization of the theory

Prove that euclideanization of the metric and its gravitational action can be attained by the transition

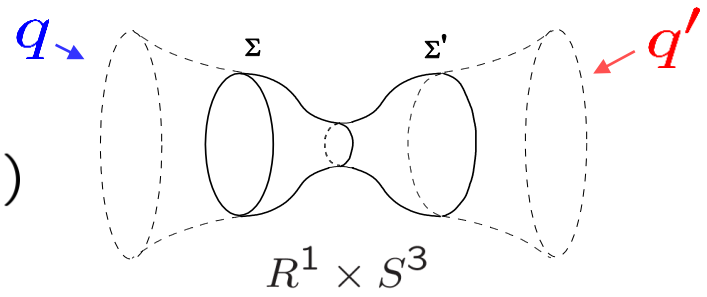
$$N \to -iN$$

$$iS \to -S_{\text{Euclidean}} \equiv -I$$

Euclidean path integral and its saddle points

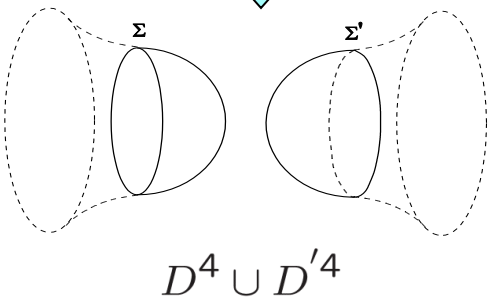
Hartle-Hawking state as a vacuum member of the microcanonical ensemble:

$$\hat{\rho}_{\text{mixed}} = \rho_{\text{mixed}}(q, q')$$

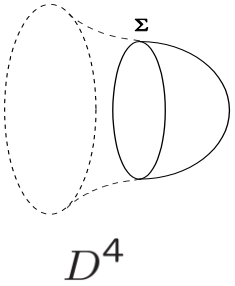


pinching a tubular spacetime

$$|\Psi_{HH}\rangle\langle\Psi_{HH}| = \rho_{HH}(q, q')$$

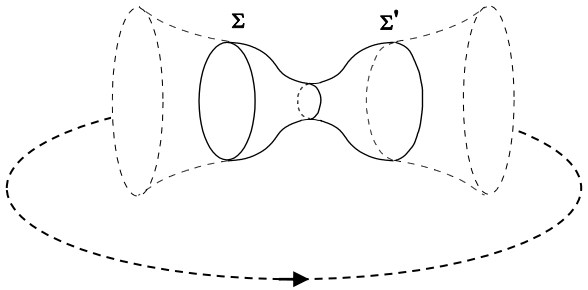


$$\Longleftrightarrow \quad |\Psi_{HH}\rangle = \Psi_{HH}(q)$$

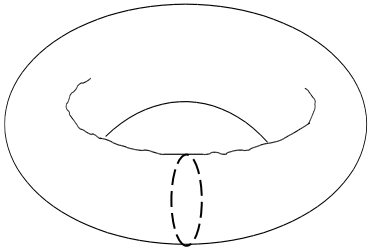


density matrix representation of a pure Hartle-Hawking state

Transition to statistical sums



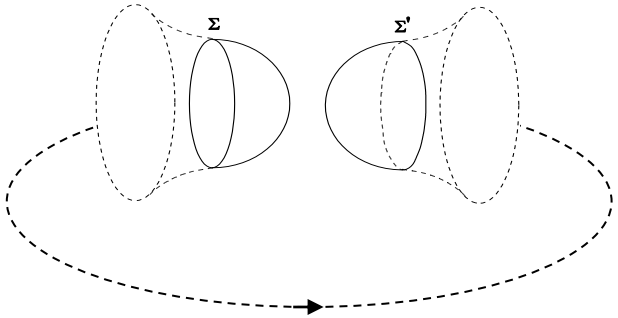
$R^1 \times S^3$



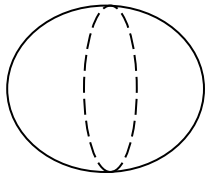
$\Sigma = \Sigma'$

$S^1 \times S^3$

thermal instantons



$D^4 \cup D'^4$



$\Sigma = \Sigma'$

S^4

Hartle-Hawking (vacuum) instanton

How to calculate the path integral $Z = \int\limits_{\text{periodic}} D[g_{\mu\nu}, \Phi]\, e^{-S[g_{\mu\nu},\Phi]}$ ***?***

Choice of the physical model (local or global symmetries of the action of physical fields)

Choice of the approximation (semiclassical $\hbar$-expansion or saddle-point approximation)

Einstein gravity theory with a cosmological constant + conformal matter field theory

$$S[g_{\mu\nu}, \Phi] = -\frac{M_P^2}{2}\int d^4x\, g^{1/2}\,(R - 2\Lambda) + S_{CFT}[g_{\mu\nu}, \Phi] = S_E[g_{\mu\nu}] + S_{CFT}[g_{\mu\nu}, \Phi]$$

Local conformal (Weyl) invariance $S_{CFT}[e^{2\sigma(x)} g_{\mu\nu}(x), e^{-d_\varphi \sigma(x)}\varphi(x)] = S_{CFT}[g_{\mu\nu}(x), \varphi(x)]$

Example: conformal scalar field in curved spacetime

$$S_{CFT}[g_{\mu\nu}(x), \varphi(x)] = \frac{1}{2}\int d^4x g^{1/2}\Big(-\varphi\Box\varphi + \frac{1}{6}R\varphi^2 + \frac{\lambda}{12}\varphi^4\Big),$$

$$g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}, \quad \varphi \to e^{-\sigma}\varphi, \quad \sigma = \sigma(x)$$

$$Z = \int D[\,g_{\mu\nu}, \Phi\,]\; e^{-S[\,g_{\mu\nu},\Phi\,]} = D[\,g_{\mu\nu}\,]\; e^{-S_E[\,g_{\mu\nu}\,]} \left(\int D[\,\Phi\,]\; e^{-I_{CFT}[\,g_{\mu\nu},\Phi\,]} \right)$$

$$= \int D[\,g_{\mu\nu}\,]\; e^{-\Gamma_{\text{eff}}[\,g_{\mu\nu}\,]}$$

$$\int D[\,\Phi\,]\; e^{-I_{CFT}[\,g_{\mu\nu},\Phi\,]} = e^{-\Gamma_{CFT}[\,g_{\mu\nu}\,]}$$

$$\Gamma_{\text{eff}}[\,g_{\mu\nu}\,] = -\frac{M_P^2}{2} \int d^4x\, g^{1/2} (R - 2\Lambda) + \Gamma_{CFT}[\,g_{\mu\nu}\,]$$

Effective action of matter on the background of curved spacetime metric $g_{\mu\nu}$

Saddle-point method (omission of graviton loops)

$$Z = \int_{\text{periodic}} D[\,g_{\mu\nu}\,]\; e^{-\frac{1}{\hbar}\Gamma_{\text{eff}}[\,g_{\mu\nu}\,]} = \text{const} \times e^{-\frac{1}{\hbar}\Gamma_{\text{eff}}[\,g^0_{\mu\nu}\,]} + \mathcal{O}(\hbar)$$

$$\frac{\delta \Gamma_{\text{eff}}[\,g_{\mu\nu}\,]}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}=g^0_{\mu\nu}} = 0$$

Saddle-point configuration---solution of effective (quantum) Einstein equations

Effective action --- how to calculate $\Gamma[\,g_{\mu\nu}\,]$*?*

$$e^{-\Gamma[\,\phi\,]/\hbar} = \int D\varphi\, \exp\frac{1}{\hbar}\{-S[\,\varphi,\phi\,]\}$$

quantum field — φ

external field — ϕ

$$\frac{\delta S[\,\varphi_0]}{\delta\varphi_0} = 0 \quad + \quad \textit{physical boundary conditions},$$

Stationary (saddle) point of the integral

$$\varphi = \varphi_0 + \Delta,$$

$$S[\,\varphi\,] = S[\,\varphi_0\,] + \frac{1}{2}\frac{\delta^2 S}{\delta\varphi\delta\varphi}\Delta^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta\varphi^3}\Delta^3 + \frac{1}{4!}\frac{\delta^4 S}{\delta\varphi^4}\Delta^4 + \cdots$$

$$e^{-\Gamma/\hbar} = \text{const} \times e^{-S[\,\varphi_0\,]/\hbar} \int D\Delta\, \exp\left\{-\frac{1}{2\hbar}\frac{\delta^2 S}{\delta\varphi\,\delta\varphi}\Delta^2\right\}\left(1 + \mathcal{O}(\Delta^4/\hbar)\right)$$

$$= \text{const} \times e^{-S[\,\varphi_0\,]/\hbar}\left(\text{Det}\,\frac{\delta^2 S}{\delta\varphi\,\delta\varphi}\right)^{1/2}\left(1 + \mathcal{O}(\hbar)\right)$$

functional determinant

Finite dimensional Gaussian integral

$$\int d^n x \, \exp\left\{-\frac{1}{2} S_{ij} x^i x^j\right\} = \left(\frac{1}{2\pi} \det S_{ij}\right)^{-1/2}$$

$$i \mapsto x, \quad x^i \mapsto \varphi(x), \quad \int d^n x \mapsto \int D[\varphi(x)],$$

Hessian of the action --- wave operator of physical modes and its functional determinant

$$S_{ij} \mapsto \frac{\delta^2 S}{\delta\varphi(x)\,\varphi(x')} = F(\nabla)\,\delta(x,x'),$$

$$\det S_{ij} \mapsto \mathrm{Det}\left[\, F(\nabla)\,\delta(x,x')\,\right] \equiv \mathrm{Det}\, F(\nabla)$$

Variational definition of the functional determinant

$$\delta \ln \det F_{ij} = \left(F^{-1}\right)^{ij} \delta F_{ji} = \mathrm{tr}\left(F^{-1}\delta F\right) \mapsto$$

$$\delta \ln \mathrm{Det}\, F(\nabla) = \mathrm{Tr}\left(F^{-1}(\nabla)\delta F(\nabla)\right)$$

$$\equiv \int dx\, \delta F(\nabla_x) G(x,x') \Big|_{x'=x}, \quad G \equiv \frac{1}{F(\nabla)}$$

Green's function of $F(\nabla)$

$$\ln \mathrm{Det}\, F(\nabla) = \mathrm{Tr}\, \ln F(\nabla) + \mathrm{const}$$

Functional trace $\mathrm{Tr} K(x,x') = \int dx\, \mathrm{tr}\, K(x,x)$

multi-loop terms

Loop expansion

$$\Gamma[\phi] = S[\phi] + \hbar\,\Gamma_{1-\text{loop}}[\phi] + \hbar^2\Gamma_{2-\text{loop}}[\phi] + \ldots$$

One-loop order

$$\Gamma_{1-\text{loop}} = \frac{1}{2}\,\text{ln Det}\,F(\nabla) = \frac{1}{2}\,\text{Tr ln}\,F(\nabla)$$

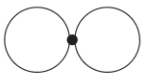
$$F(\nabla)\,\delta(x,y) = \frac{\delta^2 S[\phi]}{\delta\phi(x)\,\delta\phi(y)}$$

Feynman diagrams representation---lines are Green's functions in external field $G(x,y)$

$$F(\nabla)G(x,y) = -\delta(x,y)$$

Vertices---functional Derivatives of the action

$$\Gamma_{1-\text{loop}} = \frac{1}{2}$$ 

$$\Gamma_{2-\text{loop}} = \frac{1}{8}$$  $$+ \frac{1}{12}$$ 

Heat kernel method

Heat kernel (proper time) representation

$$G \equiv -F^{-1}(\nabla) = \int_0^\infty ds\, K(s), \quad K(s) = e^{sF(\nabla)},$$

$$\Gamma_{\text{one-loop}} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}\,K(s) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\int dx\,\mathrm{tr}\,\hat{K}(s|\,x,x)$$

Heat equation

$$\frac{\partial}{\partial s}\hat{K}(s\,|\,x,y) = \hat{F}(\nabla)\,\hat{K}(s\,|\,x,y), \quad \hat{K}(0\,|\,x,y) = \hat{1}\,\delta(x,y)$$

Action Hessian $\hat{F}(\nabla) = F^A_B(\nabla)$ **acting in the space of fields** $\varphi = \varphi^A(x)$

Minimal 2-nd order operator

$$\hat{F}(\nabla) = \Box + \hat{P} - \frac{\hat{1}}{6}R, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Generic metric and fibre bundle curvatures

$$[\nabla_\mu, \nabla_\nu]\,V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho, \quad [\nabla_\mu, \nabla_\nu]\,\varphi = \hat{\boldsymbol{R}}_{\mu\nu}\,\varphi$$

Heat kernel (Schwinger-DeWitt) expansion for minimal 2-nd order operator

$$e^{s\widehat{F}(\nabla)}\delta(x,y)=\frac{\mathcal{D}^{1/2}(x,y)}{(4\pi s)^{d/2}}\,g^{1/2}(y)\,e^{-\frac{\sigma(x,y)}{2s}}\sum_{n=0}^{\infty}s^n\,\widehat{a}_n(x,y)$$

Synge world function---one half geodetic distance between the points in curved space and Van-Vleck determinant

$$\frac{1}{2}\,|x-y|^2\mapsto\sigma(x,y),\quad \mathcal{D}(x,y)=g^{-1/2}(x)\det\frac{\partial^2\sigma(x,y)}{\partial x^\mu\partial y^\nu}\,g^{-1/2}(y)$$

Schwinger – DeWitt (Gilkey-Seely) coefficients: $\widehat{a}_n(x,y)=a^A_{n\,B}(x,y)$

Coincidence limits for $\widehat{a}_n(x,y),\ \nabla...\nabla\widehat{a}_n(x,y)$ ***--- local tensor powers in curvatures and their covariant derivatives (EXACTLY CALCULABLE!)***

$$\widehat{a}_0(x,x)=\widehat{1},\quad \widehat{a}_1(x,x)=\widehat{P},$$

$$\widehat{a}_2(x,x)=\frac{1}{180}\,(R^2_{\alpha\beta\gamma\delta}-R^2_{\mu\nu}+\Box R)\,\widehat{1}+\frac{1}{12}\widehat{\boldsymbol{R}}^2_{\mu\nu}+\frac{1}{2}\widehat{P}^2+\frac{1}{6}\Box\widehat{P},\cdots$$

Theory with mass parameter: $\widehat{F}(\nabla) = \Box + \widehat{P} - \frac{\widehat{1}}{6} R - M^2 \widehat{1}$

Local curvature and gradient expansion in d-dimensional spacetime

$$\Gamma_{\text{one-loop}} = \Gamma_{\text{div}} + \Gamma_{\log} - \frac{1}{2}\left(\frac{M^2}{4\pi}\right)^{d/2} \sum_{n=d/2+1}^{\infty} \frac{\Gamma(n-\frac{d}{2})}{(M^2)^n} \int dx\, g^{1/2} \text{tr}\, \widehat{a}_n(x,x)$$

Origin of ultraviolet (UV) divergences: $\Gamma_{\text{one-loop}} = -\frac{1}{2}\int_0^{\infty} \frac{ds}{s^{d/2+1}} \left(\cdots \right)$

dimensional regularization

Divergent and logarithmic parts

$$\Gamma_{\text{div}} = \frac{1}{2(4\pi)^{d/2}} \sum_{n=0}^{d/2} \left[-\frac{1}{\frac{d}{2}-\omega} - \psi\left(\frac{d}{2}-n+1\right) \right] \frac{(-M^2)^{\frac{d}{2}-n}}{(\frac{d}{2}-n)!} \int dx\, g^{1/2} \text{tr}\, \widehat{a}_n(x,x), \qquad \omega \to \frac{d}{2}$$

pole in dimension

$$\Gamma_{\log} = \frac{1}{2(4\pi)^{d/2}} \sum_{n=0}^{d/2} \frac{(-M^2)^{\frac{d}{2}-n}}{\left(\frac{d}{2}-n\right)!} \ln\frac{M^2}{\mu^2} \int dx\, g^{1/2} \text{tr}\, \widehat{a}_n(x,x)$$

$$\Gamma_{\text{CW}} = \int d^4x \frac{\varphi^4}{64\pi^2} \ln\frac{\varphi^2}{\mu^2}$$

Coleman-Weinberg potential in φ^4---theory in flat spacetime

$$S[\varphi] = \frac{1}{2}\int d^4x \left(-\varphi \Box \varphi + \frac{\lambda}{12}\varphi^4 \right)$$

Renormalization of UV divergences and origin of quantum anomalies

One-loop divergences

$$\Gamma^{\rm div}_{\rm one-loop} = -\frac{1}{32\pi^2\varepsilon}\int dx\, g^{1/2}{\rm tr}\,\hat{a}_2(x,x), \qquad \varepsilon = 2-\frac{d}{2}\to 0$$

Subtraction---renormalized effective action

$$\Gamma^{\rm renorm} = \Gamma - \Gamma^{\rm div}, \qquad \varepsilon = 2-\frac{d}{2}\to 0$$

Classically conformally invariant theory

$$S_{CFT}[\,e^{2\sigma(x)}g_{\mu\nu}(x)\,] = S_{CFT}[\,g_{\mu\nu}(x)\,] \Rightarrow \delta_\sigma S_{CFT} = 2g_{\mu\nu}\frac{\delta S_{CFT}}{\delta g_{\mu\nu}} = 0$$

$$\Gamma^{\rm renorm}[\,e^{2\sigma(x)}g_{\mu\nu}(x)\,] \neq \Gamma^{\rm renorm}[\,g_{\mu\nu}(x)\,]$$

$$\Downarrow$$

Quantum conformal (trace) anomaly

$$\langle\, T^\mu_\mu \,\rangle \equiv \frac{2}{g^{1/2}}\,g_{\mu\nu}\frac{\delta\Gamma^{\rm renorm}}{\delta g_{\mu\nu}} = -\frac{1}{(4\pi)^2}\,{\rm tr}\,\hat{a}_2(x,x)$$

Conformal anomaly of massless fields conformally coupled to gravity

$$\langle T^{\mu}_{\mu} \rangle = \frac{2}{g^{1/2}} g_{\mu\nu} \frac{\delta \Gamma}{\delta g_{\mu\nu}} = -\frac{1}{16\pi^2} \operatorname{tr} \hat{a}_2(x,x) = \frac{1}{32\pi^2} \left(\alpha \Box R + \beta E + \gamma C^2_{\mu\nu\alpha\beta} \right)$$

spin-dependent coefficients

Topological Gauss-Bonnet density and Weyl tensor

$$E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2,$$
$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \cdots$$

Integration along the orbit of the conformal group

$$g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}, \quad \frac{\delta \Gamma[e^{2\sigma}\bar{g}]}{\delta \sigma} = g^{1/2} \langle T^{\mu}_{\mu} \rangle \Big|_{g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}}$$

Difference of effective actions on conformally related metrics (Wess-Zumino action) $g_{\mu\nu},\ \bar{g}_{\mu\nu}$

$$\Gamma[g_{\mu\nu}] - \Gamma[\bar{g}_{\mu\nu}] = \frac{1}{16\pi^2} \int d^4x g^{1/2} \left\{ \left[\alpha C^2_{\mu\nu\alpha\beta} + \beta \left(E - \frac{2}{3} \Box R \right) \right] \sigma - 2\beta \sigma \Delta_4 \sigma \right\}$$
$$- \frac{1}{32\pi^2} \left(\frac{\gamma}{6} + \frac{\beta}{9} \right) \int d^4x \left(g^{1/2} R^2(g) - \bar{g}^{1/2} R^2(\bar{g}) \right)$$

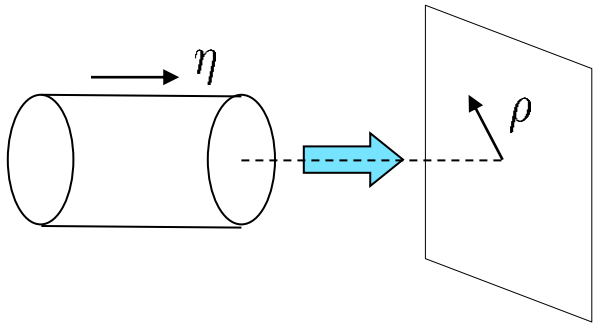
Paneitz operator

$$\Delta_4 = \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\mu R) \nabla_\mu$$

Riegert , Fradkin, Tseytlin Antoniadis,Mottola, Mazur

Applications

Casimir vacuum energy $E_{\rm vac}$ ***on static Einstein Universe --- conformal map from the rasius*** a_0 ***cylinder onto flat plane***



$$ds^2_{EU} = a_0^2\left(d\eta^2 + d\Omega^2_{(3)}\right) = e^{2\sigma}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right) \equiv e^{2\sigma} ds^2_{\rm flat}, \quad \sigma = -\eta = \ln\frac{a_0}{\rho},$$

$$\Gamma_{EU} - \Gamma_{\rm flat} = \frac{3}{4}\left(\beta - \frac{\gamma}{2}\right)\int \frac{d\tau}{a_0} \;\rightarrow\; E_{\rm vac} = \frac{3}{4a_0}\left(\beta - \frac{\gamma}{2}\right)$$

$\Gamma_{\rm flat} = 0$

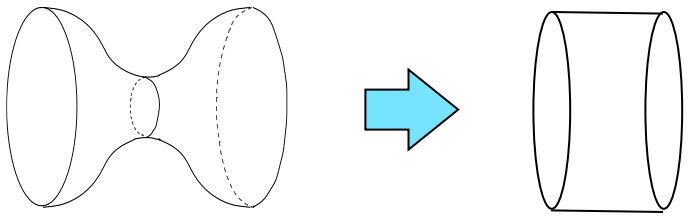
in terms of the parameters of the trace anomaly

Minisuperspace: conformal transform from Friedmann (FRW) to static Einstein Universe metric

$$g_{\mu\nu}dx^\mu dx^\nu = N^2(\tau)\,d\tau^2 + a^2(\tau)\,d^2\Omega^{(3)} = a^2(\eta)(d\eta^2 + d^2\Omega^{(3)}),$$

$$\Gamma_{\rm eff}[\,g_{\mu\nu}\,] = \Gamma_{\rm eff}[\,a(\tau), N(\tau)\,]$$

conformal time $\eta = \int d\tau \, \frac{N}{a}$



Starobinsky (1980); Fischetty,Hartle,Hu; Riegert; Tseytlin; Antoniadis, Mazur & Mottola; ……

$$ds^2 = a^2(\eta)(d\eta^2 + d^2\Omega^{(3)}) \quad \Rightarrow \quad d\bar{s}^2 = d\eta^2 + d^2\Omega^{(3)}$$

$$\Gamma_{Friedmann}[\,a, N\,] - \Gamma_{EU}[\,a, N\,] = B \oint d\tau\, N \left(\frac{a'^2}{a} - \frac{a'^4}{6a}\right), \qquad a' \equiv \frac{1}{N}\frac{da}{d\tau}$$

$$B = \frac{\beta}{8\pi^2}$$

Inflationary model driven by the trace anomaly of Weyl invariant fields --- CFT driven cosmology

$$\Gamma_{\rm eff}[\,a,N\,]=\oint d\tau\, N\,\mathcal{L}(a,a')+F(\eta)$$

Full quantum effective action on Friedmann background

$$\mathcal{L}(a,a')=6\pi^2M_P^2\Big(-aa'^2-a+\frac{\Lambda}{3}a^3\Big)+B\left(\frac{a'^2}{a}-\frac{a'^4}{6a}+\frac{1}{2a}\right)$$

classical part

conformal anomaly part

vacuum (Casimir) energy – from static EU

$B=\dfrac{\beta}{8\pi^2}$ --- *overall coefficient of the Gauss-Bonnet term in the conformal anomaly*

Recovery of the action from the conformal anomaly and the action on a static Einstein Universe

Free energy of equilibrium radiation on EU---nonlocal (thermal) part

$$F(\eta)=\pm\sum_{\omega}\ln\left(1\mp e^{-\omega\eta}\right),$$

Inverse (comoving) temperature $\eta=\oint d\tau\,\dfrac{N}{a}$

energies of field oscillators on S^3

Assumption of numerous conformal fields---their domination over omitted graviton loops:

$\beta=\sum_s \beta_s\,\mathbb{N}_s,\ \mathbb{N}_s$ ***---number of spin s fields,*** β_s ***--- spin dependent coefficients***

Effective Friedmann equation for saddle points of the path integral:

$$\frac{\delta \Gamma_{\text{eff}}[\,a, N\,]}{\delta N(\tau)} = 0$$

$$\frac{1}{a^2} - \frac{\dot{a}^2}{a^2} = \frac{\varepsilon}{3M_{\pm}^2(\varepsilon)},$$

Friedmann equation

$$M_{\pm}^2(\varepsilon) = \frac{M_P^2}{2}\left(1 \pm \sqrt{1 - \frac{\beta}{6\pi^2 M_P^4}\,\varepsilon}\,\right),$$

Effective Planck mass

$$\varepsilon = M_P^2 \Lambda + \frac{1}{2\pi^2 a^4} \sum_{\omega} \frac{\omega}{e^{\eta\,\omega} - 1},$$

Energy density = Λ + radiation of CFT particles -- sum over field oscillators with frequencies (eigenvalues of Laplacian on S^3)

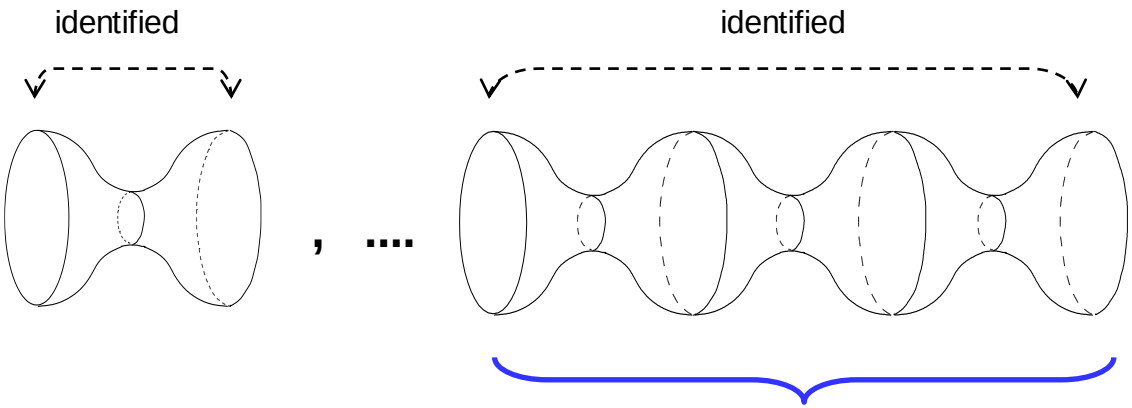
$$\eta = \oint_{S^1} \frac{d\tau N}{a}$$

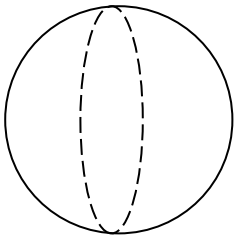
Inverse temperature in units of conformal time period on S^1



Existence of the quasi-thermal stage preceding the inflation

Saddle point solutions --- set of periodic (thermal) garland-type instantons with oscillating scale factor (S^1 X S^3) and the vacuum Hartle-Hawking instantons (S^4)





1- fold, k=1

k- folded garland, k=1,2,3,…

S^4

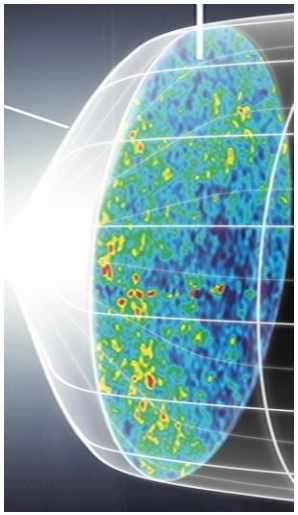
does not contribute: ruled out by infinite positive Euclidean action (effect of conformal anomaly)

UV bounded cosmological constant range:

$$\Lambda_{min} < \Lambda < \Lambda_{max} = \frac{12\pi^2}{\beta} M_P^2$$

Initial thermal state with the primordial temperature T_{prim} of matter

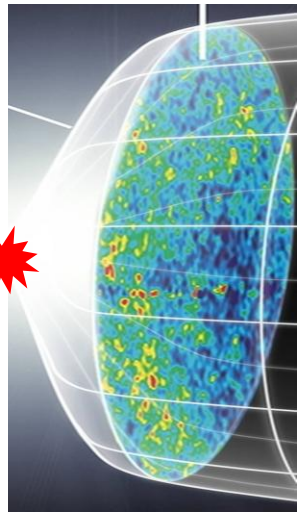
Standard inflation scenario versus Density matrix scenario



Vacuum, absolute zero temperature

Inflation, hot big-bang → relic radiation

Thermal state, primordial temperature T_{prim}



Inflation, $T_{prim} \to 0$, hot big-bang → relic radiation

“SOME LIKE IT HOT” (SLIH) scenario



Known inflation paradigm retracted the BB concept by replacing it with the initial vacuum state.

“SOME LIKE IT HOT” (SLIH) scenario recovers a new incarnation of Hot Big Bang -- it incorporates effectively thermal state at the onset of the cosmological evolution.

So how does SLIH scenario matches with inflation?

Cosmology: inflation theory

$\varepsilon \sim 10^{-29} \frac{\text{g}}{\text{cm}^3}$ — average density

$H = \frac{\dot{a}}{a} \sim 70 \frac{\text{km/s}}{\text{Mpc}}$ — Hubble parameter (expansion rate)

$T \sim 2.7\text{K}$ —cosmic microwave background radiation (CMBR) temperature

$\frac{\Delta T}{T} \sim 10^{-5}$ —CMBR anisotropy

Friedmann equation

$$H^2 + \frac{\kappa}{a^2} = \frac{\varepsilon}{3M_P^2}, \qquad \kappa = 0, \pm 1$$

(flat 0, closed +1, open -1)

Cosmological density parameter and critical density

$$\Omega = \frac{\varepsilon}{\varepsilon_0}, \quad \varepsilon_0 = 3M_P^2 H^2,$$

$$\Omega^{-1} - 1 = -\frac{3M_P^2}{\varepsilon a^2}\kappa$$

Stress tensor conservation

$$\nabla_\mu T^\mu_\nu = 0 \quad \rightarrow \quad \dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0$$

pressureless dust

$$p = 0, \ \ \varepsilon \propto a^{-3}, \ \ a \propto t^{2/3}, \qquad \Omega^{-1} - 1 \propto a,$$

black body radiation

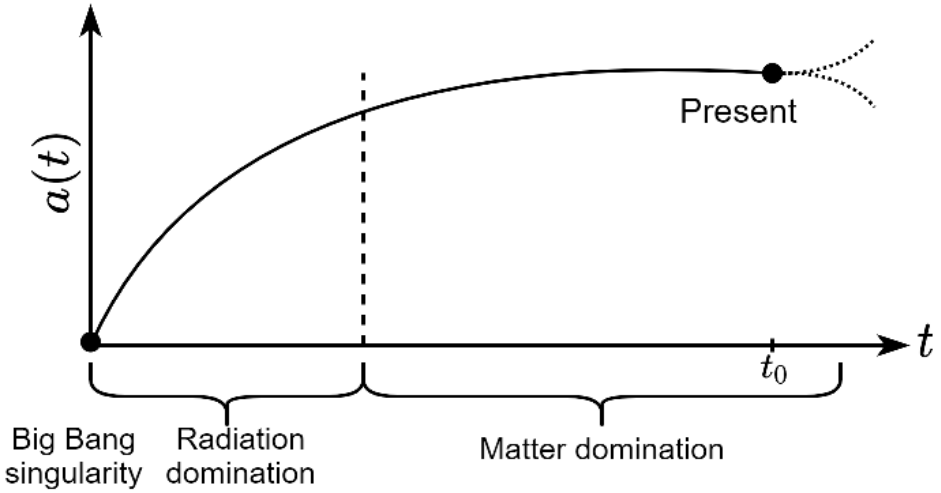
$$p = \frac{1}{3}\varepsilon, \ \ \varepsilon \propto a^{-4} \propto T^4, \ \ a \propto t^{1/2}, \qquad \Omega^{-1} - 1 \propto a^2,$$

cosmological constant --- inflation or dark energy

$$p = -\varepsilon < 0, \ \ \varepsilon = \text{const}, \ \ a \propto e^{Ht}, \qquad \Omega^{-1} - 1 \propto a^{-2}$$

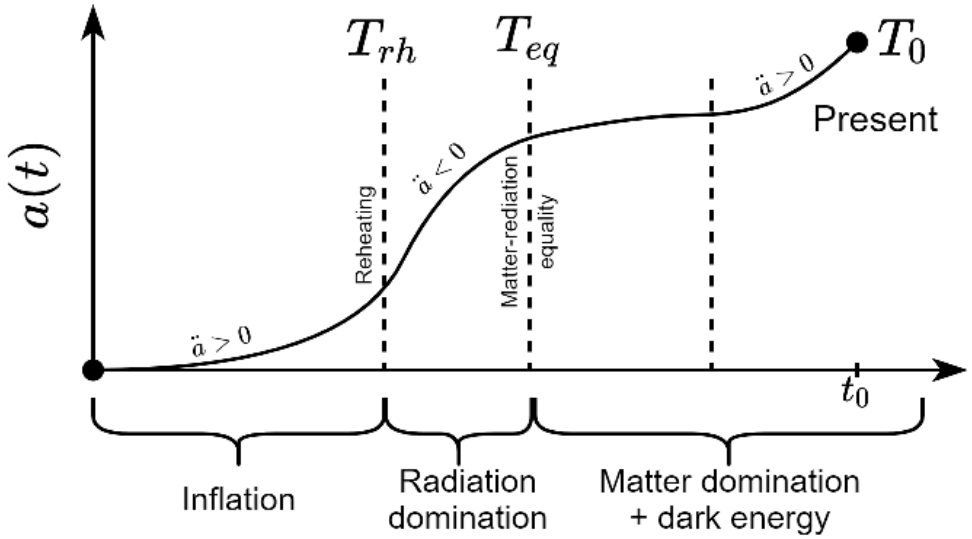
Big Bang vs Inflation

Big Bang scenario

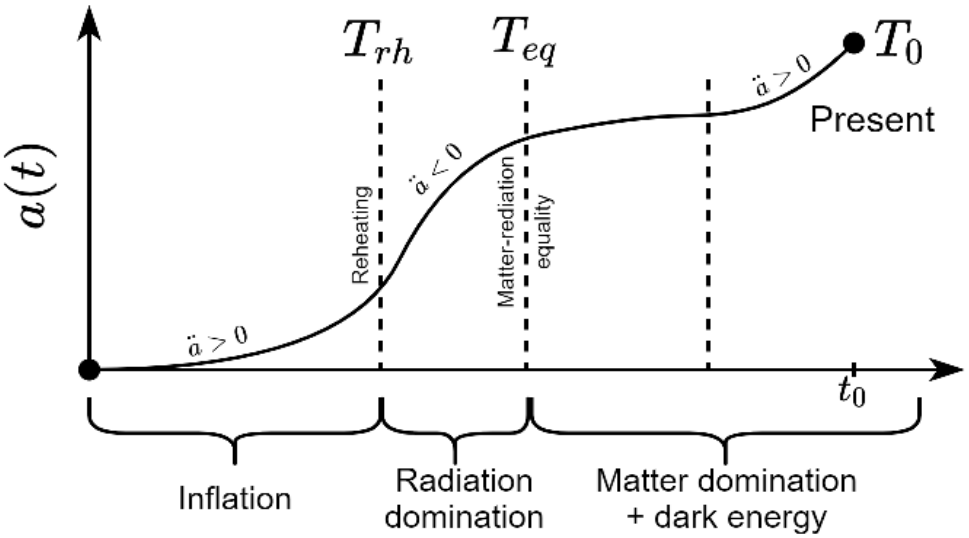


Problems of horizon, flatness, causality

Inflation scenario



A.Guth, early eighties



vacuum (inflation), $\ddot{a} > 0$

particle creation and (pre)heating,

\radiation stage (cooling down), $\ddot{a} < 0$

recombination into neutral atoms, origin of relic radiation --- transparent Universe, $\ddot{a} > 0$

matter-dominated stage: star\&galaxies --- large-scale structure (LSS) formation, $\ddot{a} < 0$

Cosmological acceleration due to dark energy, $\ddot{a} > 0$

$$\Omega^{-1}-1 \sim \frac{1}{\varepsilon a^2} \sim \begin{cases} 1/a^2, & \text{inflation (decrease)} \\ a^2, & \text{radiation (growth)} \\ a, & \text{matter (growth)} \end{cases}$$

Matching stages $\Omega_0^{-1} - 1 \sim \underbrace{\frac{T_{eq}}{T_0}\left(\frac{T_{rh}}{T_{eq}}\right)^2}_{\text{grows}} \underbrace{(\Omega^{-1}-1)_{\text{inflation}}}_{\text{decreases}}$

T_{rh} *--- reheating (preheating) temperature*

T_{eq} *--- at equality balance of radiation and matter*

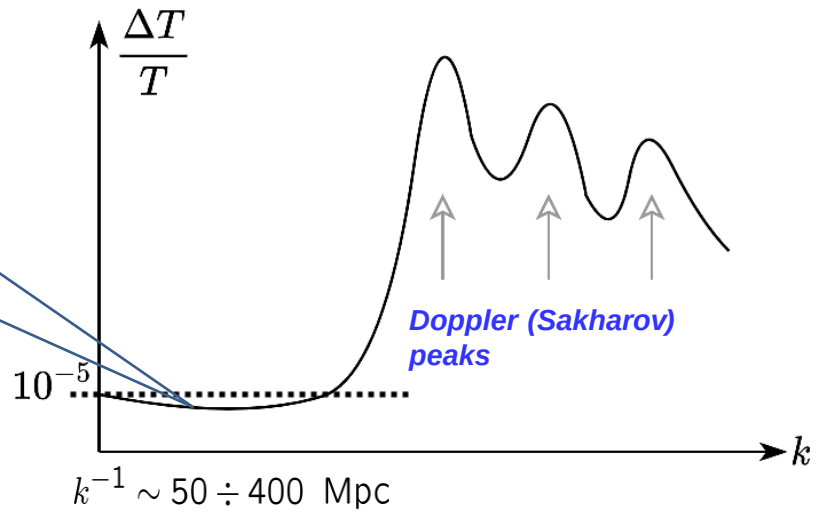
T_0 *--- present temperature of relic CMB*

Formation of CMBR at the recombination moment, a_{rec} , $a_{rec}/a_0 = z_{rec} \sim 1000$

$T_0 \sim 2.7\text{K}, \quad \frac{\Delta T}{T_0} \sim 10^{-5}$ —CMBR temperature anisotropy

Red tilted long wavelengths spectrum of CMBR (small wave number k)

$$\left(\frac{\Delta T}{T_0}\right)_k \sim k^{n_s-1}, \quad n_s \simeq 0.06$$



COBE (1991), WMAP, Planck satellites

Mechanism of inflation

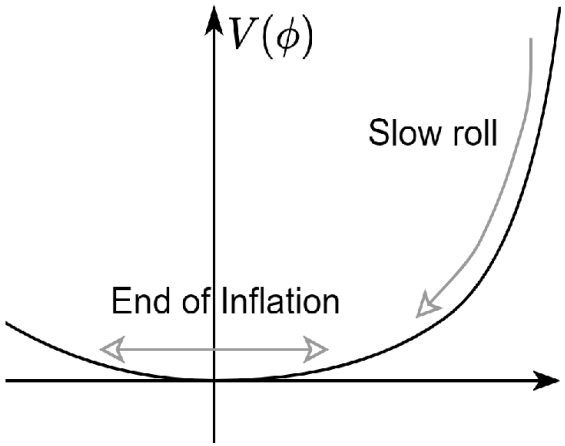
Spatially homogeneous inflaton scalar field on Friedmann background

$$S[\phi] = \int d^4x\, g^{1/2}\left(-\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right)$$

Slow roll approximation (A.Linde)

$$\varepsilon = \frac{1}{2}\dot\phi^2 + V(\phi), \qquad p = \frac{1}{2}\dot\phi^2 - V(\phi)$$

$$\dot\phi^2 \ll V(\phi) \Rightarrow \boxed{p \simeq -\varepsilon}$$



$$\frac{\dot a^2}{a^2} + \frac{\kappa}{a^2} = \frac{1}{3M_P^2}\left(V + \frac{\dot\phi^2}{2}\right)$$

$$\ddot\phi + 3\frac{\dot a}{a}\dot\phi + \frac{\partial V}{\partial\phi} = 0$$

small

$$\Rightarrow \begin{cases} H \equiv H(\phi) \simeq \sqrt{\dfrac{V(\phi)}{3M_P^2}} \\ \dot\phi \simeq -\dfrac{V'}{3H} \ll H\dot\phi \\ V' = \dfrac{\partial V}{\partial\phi} \end{cases}$$

$$a(t) = a_{\text{in}} \exp\left(\int_{t_{\text{in}}}^{t} dt H\right) = e^N, \qquad \phi \simeq \text{const}$$

e-folding number--- duration of inflation

$$N = \int_{t_{\text{in}}}^{\{V=0\}} dt\, H \simeq \frac{1}{M_P^2}\int_0^{\phi_{\text{in}}} d\phi\, \frac{V}{V'} \simeq \frac{\phi_{\text{in}}^2}{2M_P^2 n}$$

$V(\phi) \propto \phi^n$

Theory of cosmological perturbations --- formation of primordial CMBR spectrum in inflation scenario

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Friedmann}} + h_{\mu\nu}(t,\mathbf{x})$$
$$\phi = \phi(t) + \varphi(t,\mathbf{x})$$

$$h_{ij}(t,\mathbf{x}) = a^2(\eta)\Big(-2\delta_{ij}\Psi + \partial_i\partial_j E + \textbf{transverse-traceless and vector parts}\Big)$$

$$\eta = \int \frac{dt}{a(t)}, \quad a(t) \sim e^{Ht} \Leftrightarrow a(\eta) \sim -\frac{1}{H\eta}, \quad -\infty < \eta < 0$$

conformal time

end of inflation

Scalar sector gauge $\quad E = 0, \quad \varphi = 0$

New variable $\quad \Psi \to \theta = z\Psi, \quad z = \dfrac{a\dot{\phi}}{H}$

$$\theta'' - \Delta\theta - \frac{z''}{z}\theta = 0, \quad z'' \equiv \frac{d^2 z}{dz^2},$$
$$\theta_{\mathbf{k}}''(\eta) + \left(\mathbf{k}^2 - \frac{z''}{z}\right)\theta_{\mathbf{k}}(\eta) = 0$$

Mukhanov-Sasaki equation

Quantization $\quad \theta_{\mathbf{k}} \mapsto \hat{\theta}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} u_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^{\dagger} u_{\mathbf{k}}^{*}, \quad [\,\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{p}}^{\dagger}\,] = \delta(\mathbf{k}-\mathbf{p})$

Basis functions of the Mukhanov-Sasaki equation

$$u''_{\mathbf{k}}(\eta) + \left(\mathbf{k}^2 - \frac{z''}{z}\right) u_{\mathbf{k}}(\eta) = 0, \quad u_{\mathbf{k}}(\eta) \to \frac{1}{\sqrt{2|\mathbf{k}|}} e^{-i|\mathbf{k}|\eta}, \quad |\mathbf{k}|\eta \to -\infty,$$

$$i\left(u^*_{\mathbf{k}} u'_{\mathbf{k}} - u^{*'}_{\mathbf{k}} u_{\mathbf{k}}\right) = 1$$

Red shifting of the physical scale during cosmological expansion

Physical momentum scale $\mathbf{k}_{\mathsf{phys}} = \dfrac{\mathbf{k}}{a}, \quad \mathbf{k}_{\mathsf{phys}} \to \infty, \; a \to 0$

$$\left\langle \frac{\Delta T(\mathbf{x})}{T} \frac{\Delta T(\mathbf{y})}{T} \right\rangle \sim \langle\, \mathsf{vac} \,|\, \hat{\Psi}(\mathbf{x}) \hat{\Psi}(\mathbf{y}) \,|\, \mathsf{vac} \,\rangle = \int\limits_0^\infty \frac{dk}{k} \frac{k^3 |u_{\mathbf{k}}(\eta)|^2}{2\pi^2 z^2} \frac{\sin\left(k|\mathbf{x}-\mathbf{y}|\right)}{k|\mathbf{x}-\mathbf{y}|}$$

$$\delta^2_\Psi(k,\eta) = \frac{k^3 |u_{\mathbf{k}}(\eta)|^2}{2\pi^2 z^2}$$ ***power spectrum --- what we indirectly observe now at*** $t \to \infty \quad \eta \to 0$

Two regimes

$$\begin{cases} k^2 \gg \dfrac{z''}{z}, & u_{\mathbf{k}}(\eta) \simeq \dfrac{1}{\sqrt{2|\mathbf{k}|}} e^{-i|\mathbf{k}|\eta} \\ k^2 \ll \dfrac{z''}{z}, & u_{\mathbf{k}}(\eta) \simeq C_1 z(\eta) + C_2 z(\eta) \displaystyle\int_{\eta_0}^{\eta} \frac{d\eta'}{z^2(\eta')} \end{cases}$$

Matching at $k^2 = \dfrac{z''}{z} \sim (Ha)^2$ ***--- horizon crossing by the wavemode with a given comoving momentum scale***

Scalar sector contribution to the amplitude of CMBR

$$\delta_\Psi^2(k,\eta) = \frac{k_*^2}{4\pi^2 z_*^2} = \frac{H^4}{4\pi^2\dot\phi^2}\Big|_* = \frac{H^2}{12\pi^2(1+w)M_P^2}\Big|_*$$

Mukhanov-Chibisov, 1980

$$1+w \equiv 1+\frac{p}{\varepsilon} \ll 1$$

$$p \simeq -\varepsilon$$ --- quation of state at horizon crossing

horizon crossing $k = H_* a_*$

spectral tilt

$$n_s - 1 = \frac{d\ln\delta_\Psi^2(k,\eta)}{d\ln k} = 2\eta_* - 6\epsilon_*,$$

slow roll smallness parameters

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V_\phi}{V}\right)^2 \ll 1, \qquad V_\phi \equiv \frac{dV}{d\phi}, \quad V_{\phi\phi} \equiv \frac{d^2V}{d\phi^2},$$

$$\eta = M_P^2\frac{V_{\phi\phi}}{V} \ll 1$$

Tensor (graviton) sector contribution

$$\delta_t^2 = \frac{2H^2}{\pi^2 M_P^2}\Big|_* \ll \delta_\Psi^2, \qquad r = \frac{\delta_t^2}{\delta_\Psi^2} = 24(1+w)\Big|_* \ll 1$$

A.Starobinsky, 1978

$$n_t \simeq -3(1+w)\Big|_*$$

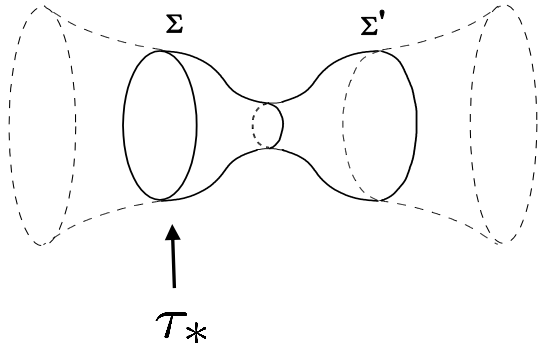
Back to SLIH inflation scenario

1) Generalization to Λ as a composite operator – inflaton potential in "slow roll" regime

$$\Lambda \to \frac{\rho_\phi}{M_P^2}, \qquad \rho_\phi = V(\phi) - \frac{\dot{\phi}^2}{2} \simeq V(\phi)$$

2) Lorentzian Universe with initial conditions set by the instanton. Analytic continuation of the instanton solutions:

$$\tau = \tau_* + it, \;\; a_L(t) = a(\tau_* + it)$$

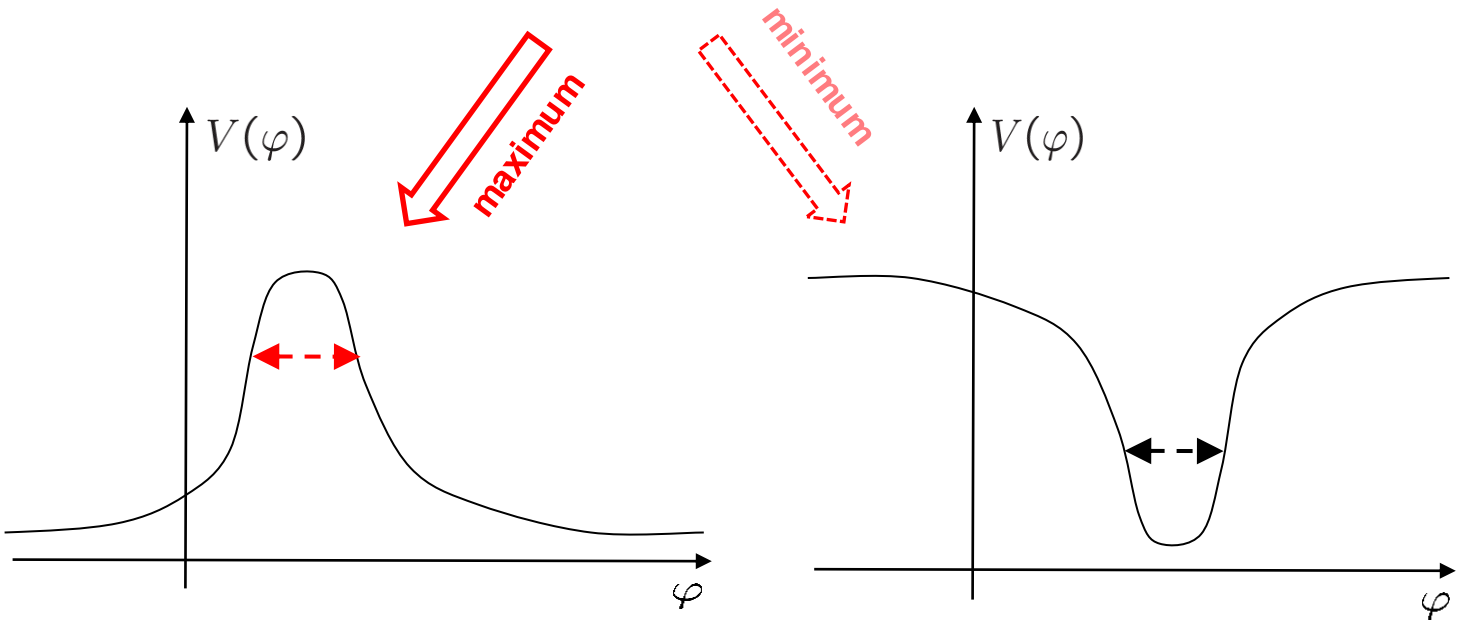


3) Expansion and quick dilution of primordial radiation, decay of a composite Λ, exit from inflation and particle creation of conformally non-invariant matter and its thermalization

Selection of inflaton potential maxima as initial conditions for inflation

Critical feature: $$\frac{d}{d\tau}a^3\dot{\phi} = a^3\frac{\partial V}{\partial \phi} \;\Rightarrow\; \oint d\tau\, a^3\frac{\partial V}{\partial \phi} = 0 \;\Rightarrow\; \frac{\partial V}{\partial \phi} \gtrless 0$$

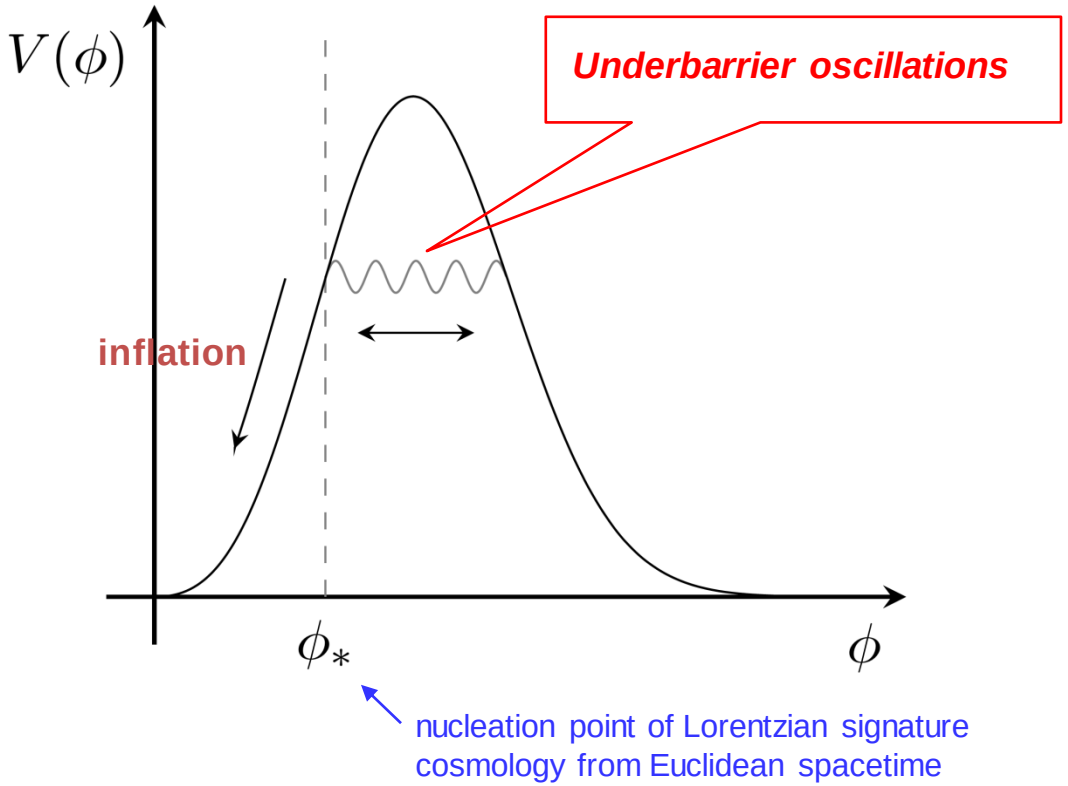
Potential extremum ``inside" instanton



classically forbidden (underbarrier) oscillation

classically allowed (overbarrier) oscillation --- ruled out because of underbarrier oscillations of scale factor

Hill-top inflation



Approximation of two coupled oscillators → slow roll parameters typical of Higgs and R^2 inflation:

$$P_\zeta(k) = 2.2 \times 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}, \quad k_0 = 0.05 Mpc^{-1}, \quad n_s = 0.965 \pm 0.005$$

$$n_s = 1 - 6\epsilon + 2\eta, \quad \epsilon = \frac{1}{2}\left(M_P \frac{V_*'}{V_*}\right)^2, \quad \eta = M_P^2 \frac{V_*''}{V_*}$$

$$\epsilon \sim \eta^2 \ll |\eta|, \quad \eta < 0$$

Mechanism of hill-top potential: origin of non-minimal Higgs inflation and R^2 gravity

Higgs field H non-minimally coupled to curvature:

B.Spokoiny 1986, A.Kamenshchik & A.B 1991,
Bezrukov,Shaposhnikov 2008
A.Kamenshchik, A.Starobinsky & A.B 2008

$$\varphi^2 \equiv H^\dagger H$$

$$S_{EH+SM}[\,g_{\mu\nu}, H, ...\,] = \int d^4x\, g^{1/2} \left(\frac{\lambda\varphi^4}{4} - \frac{M_P^2 + \xi\varphi^2}{2} R + \frac{1}{2}(\nabla\varphi)^2 + ... \right)$$

Starobinsky model of R^2 gravity:

$$S_\xi^{\rm Star}[\,g_{\mu\nu}, \varphi\,] = \int d^4x\, g^{1/2} \left\{ -\frac{M_P^2}{2} R - \frac{\xi}{4} R^2 \right\}$$

$$\Updownarrow$$

$$S_\xi^{\rm Star}[\,g_{\mu\nu}, \varphi\,] = \int d^4x\, g^{1/2} \left\{ -\frac{M_P^2}{2} \left(1 + \xi \frac{\varphi^2}{M_P^2} \right) R + \frac{\xi\varphi^4}{4} \right\}$$

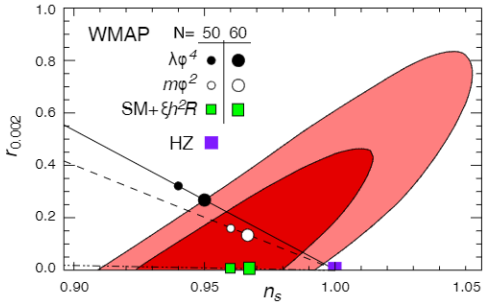


Fig. 2. The allowed WMAP region for inflationary parameters (r, n). The green boxes are our predictions supposing 50 and 60 e–foldings of inflation. Black and white dots are predictions of usual chaotic inflation with $\lambda\phi^4$ and $m^2\phi^2$ potentials, HZ is the Harrison-Zeldovich spectrum.

$$\xi \sim 10^4 \gg 1 \quad \Longrightarrow \quad \textit{Higgs inflation with} \quad \frac{\Delta T}{T} \sim 10^{-5},\ n_s \simeq 0.96,\ r \simeq 0.003$$

$$M_{\rm Higgs} \simeq 126 GeV$$

Mechanism of hill-top inflaton potential– quantization in the Jordan frame and transition to Einstein frame:

Not in Einstein frame, no shift symmetry, IR instability and breakdown of grad. expansion!

non-minimal coupling

$$\Gamma[\,g_{\mu\nu},\varphi\,]=\int d^4x\,g^{1/2}\left(V(\varphi)-U(\varphi)\,R(g_{\mu\nu})+\frac{1}{2}\,G(\varphi)\,(\nabla\varphi)^2+...\right)$$

$$V_{\rm loop}(\varphi)\sim\varphi^4\ln\frac{\varphi^2}{\mu^2},\qquad U_{\rm loop}(\varphi)\sim\varphi^2\ln\frac{\varphi^2}{\mu^2},$$

$$G_{\rm loop}(\varphi)\sim\ln\frac{\varphi^2}{\mu^2}$$

Transition to the Einstein frame:

$$V(\varphi)\to V_{EF}(\phi)=\frac{M_P^4}{4}\,\frac{V(\varphi)}{U^2(\varphi)}\sim\frac{\ln\frac{\varphi}{\mu}}{\ln^2\frac{\varphi}{\mu}}\sim\frac{1}{\ln\frac{\varphi}{\mu}}\to 0,\qquad\varphi\to\infty$$

Any l-th loop order:

$$\frac{\ln^l\frac{\varphi}{\mu}}{\ln^{2l}\frac{\varphi}{\mu}}\sim\frac{1}{\ln^l\frac{\varphi}{\mu}}\to 0,\qquad\varphi\to\infty$$

Resummation by RG confirms this.

Justification of semiclassical expansion and hierarchy problem

Starobinsky R^2-model and non-minimal Higgs inflation model

$$10^{-11} M_P^4 \simeq V_{\text{inflation}} \sim \Lambda_{max} = \frac{12\pi^2}{\beta} M_P^2 \quad \Rightarrow \quad \boxed{\beta \simeq 10^{13}}$$

Impossible in Standard model with low spins s=0,1/2,1 and N_s » 100

$$\beta = \frac{1}{180}\left(\mathbb{N}_0 + 11\mathbb{N}_{1/2} + 62\mathbb{N}_1\right)$$

Hidden sector of conformal higher spin (CHS) fields

$$S_{CHS}^{(s)} = \int d^4x \left(h^{\mu_1\dots\mu_s}\Box^s h_{\mu_1\dots\mu_s} + \dots\right), \qquad \beta_s \sim s^6$$

Giombi, Klebanov, Pufu, Safdi, and Tarnopolsky 2013; Tseytlin 2013 arXiv:1309.0785

$$\beta = \sum_{s=1}^{S} \beta_s \simeq S^7$$

$$\mathbb{N} = \sum_{s=1}^{S} N_s \sim S^3 \ - \text{total number of polarizations (species)}$$

1/N-expansion and effective field theory below the gravitational cutoff $\Lambda_{grav} = \dfrac{M_P}{\sqrt{\mathbb{N}}}$

$$\Lambda_{max} \sim \frac{M_P}{\sqrt{\beta}} \sim \frac{M_P}{S^3} \ll \Lambda_{grav} = \frac{M_P}{\sqrt{\mathbb{N}}} \sim \frac{M_P}{S^{3/2}}$$

Thermal corrections to primordial power spectrum

$$n_s(k) = n_s^{\text{vac}}(k) + \Delta n_s^{\text{thermal}}(k)$$

additional red tilt of the CMB spectrum

This number of hidden sector fields gives a red tilted thermal correction to CMB spectral index in the third (potentially observable) decimal order:

$$\Delta n_s^{\text{thermal}} \sim -0.001$$

A.B, arXiv:1308.4451
JCAP 1310 (2013) 059

Microcanonical state of CFT driven cosmology scenario works only for closed Universe with k=+1

99% C.L. evidence for positive spatial curvature (k=+1) of the closed Universe with $\Omega_k \sim$ - 0.04 --- Hubble tension discordances

E. Di Valentino, A. Melchiorri and J. Silk, Nature Astron. 4, 196 (2019);
W. Handley, Phys. Rev. D 103 (2021) L041301, arXiv:1908.09139

Conclusions

Dirac-BFV quantization

Cosmological initial conditions: *microcanonical density matrix of the Universe*

CFT driven cosmology: *suppression of no-boundary instantons; quasi-thermal stage preceding inflation and UV bounded range of its energy scale*

New type of hill-top inflation, $\Lambda \rightarrow V(\phi)$ *– selection of inflaton potential* $V(\phi)$ *maxima*

Mechanism of hill-top potential: *origin of non-minimal Higgs inflation and* R^2 *gravity*

Conformal higher spin fields (CHS): *solution of hierarchy problem -- origin of the Universe is the subplanckian phenomenon; justification of semiclassical expansion and 1/N-expansion*

Thermally corrected CMB spectrum: *observable signature of the primordial thermal epoch*