Supersymmetry and Superfields: Elementary Introduction

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Summary

Lecture I

Symmetries play the central role in physics: they underlie all theories of interest known to date. Their basis is the group theory.

- Gravity: based on the local diffeomorphism group of the space-time, *Diff* R^4, $x^m \Rightarrow x^{m'}(x)$;
- Maxwell theory and its generalization, Yang-Mills theory: based on the gauge groups $U(1)$ and $SU(n)$, with group parameters being arbitrary functions of the space-time point;
- Standard model, the unification of the electro-weak theory and quantum chromodynamics: [Gauge $U(2)$(spont. broken) $\otimes SU(3)_c$(unbroken)] $\otimes$ [Global Flavor $SU(n)$(broken)];
- String theory: the diffeomorphisms of the world-sheet $(z\,,\bar{z})$;
- Supergravity, Superstrings, Superbranes: Supersymmetry, local, global, conformal , ...

Group: some manifold $G = \{g_n\}\,, \quad n = 1, 2, \ldots,$ such that the following axioms are valid:

1. Closedness under the appropriate product:

$$g_1 \cdot g_2 = g_3 \in G$$

2. The existence of the unit element $I \in G$:

$$g \cdot I = I \cdot g = g$$

3. The existence of the inverse element for any $g_n \in G$:

$$g \cdot g^{-1} = g^{-1} \cdot g = I$$

4. Associativity of the product:

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3).$$

Simplest examples: 1)$(1, -1)$ w.r.t. the standard multiplication; 2)integer numbers, w.r.t. the summation, with 0 as the unit element, ...

Groups: 1) finite groups; 2) infinite countable groups; 3) continuous or topological groups (Lie groups). We will be interested in the third type.

- Lie groups:

$$G = \{g(x)\} \quad x := (x^1, x^2, \ldots, x^r), \quad r(\text{rank}) = \text{Dim}\, G,$$
$$g(x) \cdot g(y) = g(z(x,y)) \in G\,, \quad g(0) = I\,, \quad z(0,y) = y\,,\ z(x,0) = x\,.$$

- For Lie groups one can always parametrize an element as

$$g(x) = \exp\{x^i T_i\}\,, \quad [T_i, T_k] = c_{ik}^l T_l, \quad c_{ik}^l = -c_{ki}^l\,,$$

where T_i are generators and c_{ik}^l are structure constants.

- The generators T_i span an algebra called Lie algebra. The Lie algebra is specified by its structure constants which, in virtue of the Jacobi identity

$$[T_l, [T_k, T_i]] + [T_i, [T_l, T_k]] + [T_k, [T_i, T_l]] = 0\,,$$

satisfy the fundamental relation

$$c_{ki}^m c_{lm}^p + c_{lk}^m c_{im}^p + c_{il}^m c_{km}^p = 0\,.$$

- **Example**: the group $SU(2)$:

$$g = \exp\{i\lambda_a T_a\}\,,\ (T_a)^\dagger = T_a\,, \quad [T_a, T_b] = i\varepsilon_{abc} T_c\,, \quad a,b,c = 1,2,3\,,$$
$$\varepsilon_{abc}\varepsilon_{dce} + \varepsilon_{eac}\varepsilon_{bcd} + \varepsilon_{dec}\varepsilon_{acb} = 0\,.$$

The general definition of **Lie algebra**:

This is a vector space $(X, Y, ...)$ on which the operation called commutation is defined: for any two elements there exists the third element, the commutator $[X, Y]$ satisfying the following axioms:

- Linearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$, where α, β are some real or complex numbers commuting with any element;
- Anticomutativity: $[X, Y]$ = -$[Y, X]$;
- Jacoby: $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

Obviously, the generators T_i defined earlier satisfy all these axioms and so constitute a **Lie algebra**.

There are two vast classes of symmetries in the Nature

- **I. Internal symmetries**: isotopic $SU(2)$,flavour $SU(n)$, etc. Their main feature: they are realized as transformations of fields without affecting the space-time coordinates. The generators are matrices acting on some external indices of fields, no any x-derivatives are present.
  Example: realization of $SU(2)$ on the doublet of fields $\psi_i(x)$ ("neutron - proton")

$$\delta\psi_i(x) = i\lambda_a \frac{1}{2}(\sigma_a)_i^k \psi_k(x)\,, \quad [\frac{1}{2}\sigma_a, \frac{1}{2}\sigma_b] = i\varepsilon_{abc}\frac{1}{2}\sigma_c\,,$$
$$\sigma_a\sigma_b = \delta_{ab}\mathbf{I} + i\varepsilon_{abc}\sigma_c\,,$$

  σ_a are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- **II. Space-time symmetries**: Lorentz, Poincaré and conformal groups. Generators in the realization on fields involve x-derivatives.
  Example: transformation of the scalar field $\varphi(x)$ in the Poincaré group:

$$\delta\varphi(x) := -ic^m P_m - i\omega^{[mn]} L_{mn}\varphi(x) = -c^m\partial_m\varphi(x) - \omega^{[mn]}\frac{1}{2}(x_m\partial_n - x_n\partial_m)\varphi$$

$$P_m = \frac{1}{i}\partial_m\,, \quad L_{mn} = \frac{1}{2i}(x_m\partial_n - x_n\partial_m)\,, \quad m,n = 0,1,2,3\,.$$

- The primary fundamental symmetry principle is the invariance of the action

$$S = \int d^4x \mathcal{L}(\phi_A, \partial\phi_A, \psi_\alpha, ...)\,, \quad \delta S = \frac{\delta S}{\delta \phi_A}\delta\phi_A = 0 \leftrightarrow \delta\mathcal{L} = \partial_m \mathcal{A}^m\,.$$

- **Example**: free Lagrangian of the scalar field

$$\mathcal{L}^{(1)}_{free} = \frac{1}{2}\partial^m\phi(x)\partial_m\phi(x)$$

transform under the Poincare group as

$$\delta_\omega \mathcal{L}^{(1)}_{free} = -\frac{1}{2}\partial_n(\omega^{mn}x_m\partial^s\phi\partial_s\phi)\,, \quad \delta_c\mathcal{L}^{(1)}_{free} = -\frac{1}{2}c^m\partial_m(\partial^n\phi\partial_n\phi),$$

whence the invariance of the relevant action follows.

- In the systems with few scalar fields one can realize internal symmetries. The free Lagrangian of one complex field

$$\mathcal{L}^{(2)}_{free} = \partial^m\phi(x)\partial_m\bar{\phi}(x)$$

is invariant under $U(1)$ symmetry

$$\delta\phi = i\lambda\phi\,, \quad \delta\bar{\phi} = -i\lambda\bar{\phi},$$

three real scalar fields can be joined into a triplet of the group $SU(2)$

$$\mathcal{L}^{(3)}_{free} = \frac{1}{2}\partial^m\phi_a(x)\partial_m\phi_a(x)\,, \quad \delta\phi_a = \varepsilon_{abc}\lambda_b\phi_c \;\Rightarrow\; \delta\mathcal{L}^{(3)}_{free} = 0\,.$$

- One more possibility to construct $SU(2)$ invariant Lagrangian is to join two complex scalar fields into $SU(2)$ doublet

$$\mathcal{L}^{(4)}_{free} = \partial^m \phi_\alpha(x) \partial_m \bar{\phi}^\alpha(x) \,,$$
$$\delta\phi_\alpha = \frac{i}{2}\lambda_a(\sigma_a)^\beta_\alpha \phi_\beta \,, \quad \delta\bar{\phi}^\alpha = -\frac{i}{2}\lambda_a(\sigma_a)^\alpha_\beta \bar{\phi}^\beta \,, \Rightarrow \; \delta\mathcal{L}^{(4)}_{free} = 0 \,.$$

- Extending the sets of fields (and adding interaction terms), we can further enlarge internal symmetries.

- The characteristic feature of all these symmetries is that the group parameters are ordinary commuting numbers, and so do not mix bosonic fields (Bose-Einstein statistics, integer spins $0, 1, \ldots$) with fermionic fields (Fermi-Dirac statistics, half-integer spins $1/2, 3/2, \ldots$)
- The bosonic and fermionic parts of the Lagrangian are invariant separately.

- Let us now consider a sum of the free Lagrangians of the massless complex scalar field $\varphi(x)$ and the Weyl fermionic field $\psi^\alpha(x)$

$$\mathcal{L}_{\phi+\psi} = \partial^m \varphi \partial_m \bar{\varphi} - \frac{i}{4}\left[\psi^\alpha (\sigma^m)_{\alpha\dot{\alpha}} \partial_m \bar{\psi}^{\dot{\alpha}} - \partial_m \psi^\alpha (\sigma^m)_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}\right],$$

  where $(\sigma^m)_{\alpha\dot{\alpha}} = (\delta_{\alpha\dot{\alpha}}, (\sigma^a)_{\alpha\dot{\alpha}})$ are the so called sigma matrices, the basic object of the spinor two-component formalism of the Lorentz group (they are invariant under simultaneous Lorentz transformation of the vector $m, m = 0, 1, 2, 3$, and spinor $\alpha, \dot{\alpha} = 1, 2$ indices).
- The evident symmetries of this Lagrangian are Poincaré and phase $U(1)$ symmetries which separately act on $\varphi(x)$ and $\psi^\alpha(x)$.
- However, there is a new much less obvious symmetry. Namely, this Lagrangian transform by a total derivative under the following transformations mixing bosonic and fermionic fields

$$\delta\varphi = -\epsilon^\alpha \psi_\alpha\,, \quad \delta\bar{\varphi} = -\bar{\psi}_{\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}}\,, \quad \delta\psi_\alpha = -2i(\sigma^m)_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} \partial_m \varphi\,.$$

- One sees that the transformation parameters $\epsilon^\alpha, \bar{\epsilon}^{\dot{\alpha}}$ have the dimension $cm^{1/2}$, so these transformations do not define an internal symmetry (the relevant group parameters are dimensionless). Moreover, for the action to be invariant, these parameters should anticommute among themselves and with the fermionic fields, $\{\epsilon, \epsilon\} = \{\epsilon, \bar{\epsilon}\} = \{\epsilon(\bar{\epsilon}), \psi\} = 0$, and commute with the scalar field, $[\epsilon(\bar{\epsilon}), \varphi] = 0$.

- To see which kind of algebraic structure is behind this invariance one needs to consider the Lie bracket of two successive transformations on the scalar $\varphi(x)$

$$(\delta_1\delta_2 - \delta_2\delta_1)\varphi = -(\epsilon_2^\alpha \delta_1\psi_\alpha) - (\epsilon_1^\alpha \delta_2\psi_\alpha)$$
$$= 2\left(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1\right)\left(\frac{1}{i}\partial_m\varphi\right).$$

Thus the result is an infinitesimal 4-translation with the composite parameter $i\left(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1\right)$.

- Rewriting the ϵ variation in the form

$$\delta\varphi = i\left(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\varphi\,,$$

and taking into account that the spinor parameters anticommute with $Q_\alpha, \bar{Q}^{\dot{\alpha}}$, we find that the above Lie bracket structure is equivalent to the following anticommutation relations for the supergenerators

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\left(\sigma^m\right)_{\alpha\dot{\beta}}P_m\,, \quad P_m = \frac{1}{i}\frac{\partial}{\partial x^m}\,,$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,,$$
$$[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}}] = 0\,.$$

This is what is called $\mathcal{N} = 1$ Poincaré superalgebra.

Consequences of Poincare supersymmetry

The full set of the (anti)commutation relations of the $\mathcal{N}=1$ Poincaré superalgebra reads

$$
\begin{aligned}
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,(\sigma^m)_{\alpha\dot\beta} P_m\,,\\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,,\\
&[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot\alpha}] = 0\,,\\
&[J_{mn}, Q_\alpha] = -\frac{1}{2}\,(\sigma_{mn})_\alpha^{\ \beta} Q_\beta\,, \quad [J_{mn}, \bar{Q}_{\dot\alpha}] = \frac{1}{2}\,(\tilde\sigma_{mn})_{\dot\alpha}^{\ \dot\beta} \bar{Q}_{\dot\beta}\,,\\
&[J_{mn}, P_s] = i\,(\eta_{ns} P_m - \eta_{ms} P_n)\,,\\
&[J_{mn}, J_{sq}] = i\,(\eta_{ns} J_{mq} - \eta_{ms} J_{nq} + \eta_{nq} J_{sm} - \eta_{mq} J_{sn})\,,\\
&[R, Q_\alpha] = Q_\alpha\,, \quad [R, \bar{Q}_{\dot\alpha}] = -\bar{Q}_{\dot\alpha} \quad [R, P_m] = [R, J_{mn}] = 0\,.
\end{aligned}
\tag{1}
$$

Here J_{mn} are the full Lorentz group generators and R is a generator of an extra internal $U(1)$ transformations (the so-called R symmetry). Also,

$$
\eta_{mn} = \mathrm{diag}(1,-1,-1,-1)\,, \quad (\sigma_{mn})_\alpha^{\ \beta} = \frac{i}{2}\left(\sigma^m\tilde\sigma^n - \sigma^n\tilde\sigma^m\right)_\alpha^{\ \beta}\,,
$$

$$
(\tilde\sigma^{mn})_{\dot\alpha}^{\ \dot\beta} = \frac{i}{2}\left(\tilde\sigma^m\sigma^n - \tilde\sigma^n\sigma^m\right)_{\dot\alpha}^{\ \dot\beta}\,, \quad \tilde\sigma^{m\dot\alpha\alpha} = (\delta^{\dot\alpha\alpha}, -\sigma^{a\dot\alpha\alpha})
$$

Some important common features and consequences of supersymmetry can be figured out just from these (anti)commutation relations.

- The Poincaré superalgebra is an example of Z_2-graded algebra: one ascribes parities ± 1 to its all elements and the structure relations should respect these parities

$$[\text{odd}, \text{odd}] \sim \text{even}\,, \;\; [\text{even}, \text{odd}] \sim \text{odd}\,, \;\; [\text{even}, \text{even}] \sim \text{even}\,.$$

  The spinor generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ have the parity -1 and so are odd; all bosonic generators have the parity +1 and so are even.

- Lie superalgebras satisfy the same axioms as the Lie algebras, the difference is that the relevant generators satisfy the graded Jacobi identities. E.g.,

$$\{[B_1, F_2], F_3\} - \{[F_3, B_1], F_2\} + [\{F_2, F_3\}, B_1] = 0\,,$$
$$[\{F_1, F_2\}, F_3] + [\{F_3, F_1\}, F_2] + [\{F_2, F_3\}, F_1] = 0\,,$$

  where B_1 is a bosonic generator and F_1, F_2 are fermionic ones.

- Since the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ are fermionic, irreducible SUSY multiplets should unify bosons with fermions. Action of the spinor generators on the bosonic state yields a fermionic state and vice versa.

- Since the translation operator P_m is non-vanishing on any field given on the Minkowski space, the same should be true for the spinor generators as well. So any field should belong to a non-trivial supermultiplet.
- It follows from the relations $[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}}] = 0$ that $[P^2, Q_\alpha] = [P^2, \bar{Q}_{\dot{\alpha}}] = 0$. The operator P^2 is a Casimir of the Poincaré group, $P^2 = m^2$. So it is also a Casimir of the Poincaré supergroup. Hence all components of the irreducible supermultiplet should have the same mass. No mass degeneracy between bosons and fermions is observed, so supersymmetry should be broken in one or another way.
- In any representation of supersymmetry, such that the operator P_m is invertible, there should be equal numbers of bosons and fermions.
- In any supersymmetric theory the energy P_0 should be non-negative. Indeed, from the basic anticommutator it follows

$$\sum_{\alpha=1,2} \left(|Q_\alpha|^2 + |\bar{Q}_{\dot{\alpha}}|^2 \right) = 4P_0 \geq 0 \,.$$

- Rigid supersymmetry, with constant parameters, implies the translation invariance. Gauge supersymmetry, with the parameters being arbitrary functions of the space-time point, implies the invariance under arbitrary diffeomorphisms of the Minkowski space. Hence the theory of gauged supersymmetry necessarily contains gravity. This is supergravity. Its basic gauge fields are graviton (spin 2) and gravitino (spin 3/2).

Supersymmetry allows one to evade the famous Coleman-Mandula theorem about impossibility of non-trivial unification of the space-time symmetries with the internal ones. It states that any symmetry of such type (in dimensions ≥ 3), under the standard assumptions about the spin-statistic relation, is inevitably reduced to the direct product of the Poincaré group and the internal symmetry group.

The arguments of this theorem do not apply to superalgebras, when one deals with both commutation and anticommutation relations. Haag, Lopushanski, Sohnius showed that the most general superextension of the Poincaré group algebra is given by the following relations

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}k}\} = 2\delta^i_k (\sigma^m)_{\alpha\dot{\beta}} P_m \,,$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \,, \quad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij} \,,$$

$$[T^i_j, Q^k_\alpha] = -i\left(\delta^k_j Q^i_\alpha - \frac{1}{\mathcal{N}}\delta^i_j Q^k_\alpha\right), \quad [T^i_j, \bar{Q}_{\dot{\alpha}k}] = i\left(\delta^i_k \bar{Q}_{\dot{\alpha}j} - \frac{1}{\mathcal{N}}\delta^i_j \bar{Q}_{\dot{\alpha}k}\right),$$

$$[T^i_j, T^k_l] = i\left(\delta^i_l T^k_j - \delta^k_j T^i_l\right),$$

where T^i_j ($(T^i_j)^\dagger = -T^j_i$, $T^i_i = 0$) are generators of the group $SU(\mathcal{N})$ The generators $Z^{ij} = -Z^{ji}, \bar{Z}_{ij} = -\bar{Z}_{ji}$ are central charges, they commute with all generators except the $SU(\mathcal{N})$ ones

$$[Z, Z] = [Z, \bar{Z}] = [Z, P] = [Z, J] = [Z, Q] = [Z, \bar{Q}] = 0 \,.$$

The relevant supergroup is called $\mathcal{N}$-extended Poincaré supergroup.

Due to the property that the spinor generators Q^i_α, $\bar{Q}_{\dot{\beta}k}$ carry the internal symmetry indices, the supermultiplets of extended supersymmetries join fields having not only different statistics and spins, but also belonging to different representations of the internal symmetry group $U(\mathcal{N})$. In other words, in the framework of extended supersymmetry the actual unification of the space-time and internal symmetries comes about. The relevant supergravities involve, as a subsector, gauge theories of internal symmetries, i.e. they yield non-trivial unification of the Einstein gravity with Yang-Mills theories.

Lecture II

An important ingredient of supersymmetric theories is the auxiliary fields.
Let us come back to the realization of $\mathcal{N}=1$ supersymmetry on the fields $\varphi(x), \psi_\alpha(x)$ and calculate Lie bracket of the odd transformations on $\psi_\alpha(x)$:

$$(\delta_1\delta_2-\delta_2\delta_1)\psi_\alpha = -2i\left(\epsilon_1\sigma^m\bar{\epsilon}_2-\epsilon_2\sigma^m\bar{\epsilon}_1\right)\partial_m\psi_\alpha + 2i\left[\epsilon_{1\alpha}\bar{\epsilon}_{2\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\beta}\partial_m\psi_\beta-(1\leftrightarrow 2)\right]$$

The first term in the r.h.s. is the translation one, as for $\varphi(x)$. However, there is one extra term. It is clear that the Lie bracket on all members of the supermultiplet should have the same form, i.e. reduce to translations. The condition of vanishing of the 2-nd term is

$$\tilde{\sigma}^m\partial_m\psi = \sigma^m\partial_m\bar{\psi} = 0\,.$$

But this is just the free equation of motion for $\psi_\alpha(x)$. Thus $\mathcal{N}=1$ SUSY is closed only on-shell, i.e. modulo equations of motion.
How to achieve the off-shell closure? The solution is to introduce a new field $F(x)$ of non-canonical dimension cm^{-2} and to extend the free action of φ, ψ_α:

$$\mathcal{L}_{\phi+\psi+F} = \partial^m\varphi\partial_m\bar{\varphi} - \frac{i}{4}\left[\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} - \partial_m\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\right] + F\bar{F}\,.$$

It is invariant, up to a total derivative, under the modified transformations having the correct closure for all fields:

$$\delta\phi = -\epsilon^\alpha\psi_\alpha\,,\ \delta\psi_\alpha = -2i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_m\phi - 2\epsilon_\alpha F\,,\ \delta F = -i\bar{\epsilon}^{\dot{\alpha}}(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\psi^\alpha\,.$$

The auxiliary fields satisfy the algebraic equations of motion

$$F = \bar{F} = 0\,.$$

After substitution of this solution back in the Lagrangian and SUSY transformations, we reproduce the previous on-shell realization. They do not propagate also in the quantum case, possessing delta-function propagators.

The only (but very important!) role of the auxiliary fields is just to ensure the correct off-shell realization of supersymmetry, such that it does not depend on the precise choice of the invariant Lagrangian, like in the cases of ordinary symmetries.

The simplest non-trivial choice is

$$\mathcal{L}_{\rm WZ} = \partial^m\phi\partial_m\bar{\phi} - \frac{i}{4}\left[\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} - \partial_m\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\right] + F\bar{F}$$
$$+ \left[m\left(\phi F - \frac{1}{4}\psi\psi\right) + g\left(\phi^2 F - \frac{1}{2}\phi\psi\psi\right) + c.c.\right].$$

This model was the first example of renormalizable supersymmetric quantum field theory and it is called the Wess-Zumino model, after names of its discoverers. The Lagrangian $\mathcal{L}_{\rm WZ}$ is invariant under the same transformations as the free Lagrangian we have considered before.

The Wess-Zumino model Lagrangian was originally found by the "trying and error" method. The systematic way of constructing invariant off-shell Lagrangians is the superfield method we will discuss in the next Lectures.

Using this systematic method, one can equally construct more general Lgarangians of the fields (ϕ, ψ_α, F), invariant under the same linear off-shell $\mathcal{N} = 1$ supersymmetry transformations. After eliminating the auxiliary fields from these Lagrangians by their equations of motion, we will obtain the Lagrangians in terms of the physical fields (ϕ, ψ_α) only. These physical Lagrangians are invariant under the nonlinear on-shell $\mathcal{N} = 1$ supersymmetry transformations the precise form of which depends on the form of the on-shell Lagrangian, though it is uniquely specified by the off-shell Lagrangian.

To summarize, the fields (ϕ, ψ_α, F) form the set closed under the off-shell $\mathcal{N} = 1$ supersymmetry transformations, and it is impossible to select any lesser closed set of fields in it. Thus these fields constitute the simplest irreducible multiplet of $\mathcal{N} = 1$ supersymmetry. It is called scalar $\mathcal{N} = 1$ supermultiplet.

Irreps of $\mathcal{N} = 1$ supersymmetry

The fields on Minkowski space are distributed over the irreducible multiplets of the Poincaré group according to the eigenvalues of two Casimirs of this group: square of P_m (which is m^2) and square of the Pauli-Lubanski vector (square of spin). For the case of zero mass the diverse Poincaré group multiplets are characterized by the helicity, the projection of spin on the direction of motion. What about irreps of supersymmetry? Once again, the contents of the supermultiplets are different for massive and massless cases.

- **Massive case**. Choose the rest frame

$$P_m = (m, 0, 0, 0)\,.$$

In this frame

$$\text{(a) } \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,; \quad \text{(b) } \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}}\,,$$

i.e. $\mathcal{N} = 1$ superalgebra becomes the Clifford algebra of two mutually conjugated fermionic creation and destruction operators. $\bar{Q}_{\dot{\alpha}}$ and Q_α. Define the "Clifford vacuum" as the irrep of the Poincaré group with mass m and spin s:

$$Q_\alpha |s>= 0\,.$$

An irrep of the full supersymmetry can be then produced by the successive action of $\bar{Q}_{\dot{\alpha}}$ on the vacuum

State	Spin	# of components
$\lvert s\rangle$	s	$2s+1$
$\bar{Q}_{\dot{\alpha}}\lvert s\rangle$	$s \pm 1/2$	$4s+2$
$(\bar{Q})^2\lvert s\rangle$	s	$2s+1$

Here $(\bar{Q})^2 \equiv \bar{Q}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}$. Further action by $\bar{Q}_{\dot{\alpha}}$ yields zero. Thus the full number of states is $2^2(2s+1)$, one half being fermions and the second one bosons. The dimensionality of the Clifford vacuum (the number of independent states in it) is just $d_{|s>} = 2s+1$.

Since off shell $P^2 \neq 0$, this spin contents characterizes any off-shell supermultiplet. E.g., the scalar multiplet corresponds to $s = 0$: in this case $s + 1/2 = 1/2$ and we are left just with two complex scalars and one Weyl fermion.

Thus massive $\mathcal{N} = 1$ supermultiplets are entirely specified by the spin s of their Clifford vacua. This spin is called **superspin** Y of the given $\mathcal{N} = 1$ supermultiplet. Each multiplet with $P^2 \neq 0$ and superspin Y involves the following set of spins

$$Y\,,\ Y+\frac{1}{2},\ Y-\frac{1}{2}\,,\ Y\,.$$

The scalar supermultiplet ($Y = 0$) contains spins $1/2, (0)^2$ and describes $\mathcal{N} = 1$ matter. The supermultiplet with $Y = 1/2$ involves states with spins $1, (1/2)^2, 0$ and stands for the gauge supermultiplet. The supermultiplet with $Y = 3/2$ has the spin content $(3/2)^2, 2, 1$. It is the so-called $\mathcal{N} = 1$ Weyl supermultiplet. It corresponds to conformal $\mathcal{N} = 1$ supergravity.

- **Massless case**. We can choose the frame

$$P_m = (p, 0, 0, p)\,, \quad P^m P_m = 0\,.$$

The only non-zero anticommutator in this system is

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,p\,(I + \sigma^3)_{\alpha\dot\alpha}\,,$$

The full set of the antcommutation relations is

$$\{Q_1, \bar{Q}_{\dot 1}\} = 4p\,, \quad \{Q_1, \bar{Q}_{\dot 2}\} = \{Q_2, \bar{Q}_{\dot 2}\} = \{Q_2, \bar{Q}_{\dot 1}\} = 0\,,$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,.$$

Then one can define the Clifford vacuum $|\lambda>$ with the helicity λ by the conditions

$$Q_1|\lambda>= Q_2|\lambda>= \bar{Q}_{\dot 2}|\lambda>= 0\,.$$

The only creation operator is $\bar{Q}_{\dot{1}}$. Due to its nilpotency, $(\bar{Q}_{\dot{1}})^2 = 0$, the procedure of constructing the irreducible set of states terminates at the 1st step:

State	Helicity	# of components
$\|\lambda\rangle$	λ	1
$\bar{Q}_{\dot{1}}\|\lambda\rangle$	$\lambda - 1/2$	1

Thus in $\mathcal{N} = 1$ supersymmetry the massless supermultiplets are formed by pairs of states with the adjacent helicities, $|\lambda\rangle$, $|\lambda - 1/2\rangle$. In particular, massless particle with zero helicity should be accompanied by a particle with the helicity $-1/2$, a particle with $\lambda = 1/2$ should be paired with a particle having $\lambda = 0$, helicities ± 1 can be embedded either into the multiplets $(1, 1/2)$, $(-1/2, -1)$, or $(-1, -3/2)$, $(3/2, 1)$, the minimal embeddings for the helicities ± 2 are into the multiplets $(2, 3/2)$ and $(-3/2, -2)$, etc. The multiplets with the opposite helicities are related through CPT conjugation.

- **Massless multiplets of $\mathcal{N}$ extended supersymmetry**. In this case the only non-vanishing anticommutator is

$$\{Q_1^i, \bar{Q}_{1j}\} = 4\delta_j^i p\,, \tag{2}$$

The Clifford vacuum $|\lambda>$ is defined by

$$Q_1^i|\lambda\rangle = Q_2^i|\lambda\rangle = \bar{Q}_{2i}|\lambda\rangle = 0\,, \tag{3}$$

and the irreducible tower of states is constructed by acting on the vacuum by $\mathcal{N}$ independent creation operators $\bar{Q}_{1i}$:

State	Helicity	# of components
$\|\lambda\rangle$	λ	1
$\bar{Q}_{1i}\|\lambda\rangle$	$\lambda - 1/2$	$\mathcal{N}$
$\bar{Q}_{1i}\bar{Q}_{1j}\|\lambda\rangle$	$\lambda - 1$	$\mathcal{N}(\mathcal{N}-1)/2$
$\vdots$	$\vdots$	$\vdots$
$(\bar{Q})^{\mathcal{N}}\|\lambda\rangle$	$\lambda - \mathcal{N}/2$	1

For $\mathcal{N} = 2$ supersymmetry, irreps are formed by the states $|\lambda\rangle$, $|\lambda - 1/2\rangle^2$, $|\lambda - 1\rangle$, etc.

The multiplets with opposite helicities can be obtained via CPT conjugation. Of special interest are the so called "self-conjugated" multiplets which, from the very beginning, involve the full spectrum of helicities from λ to $-\lambda$. Equating

$$\lambda - \mathcal{N}/2 = -\lambda \;\Rightarrow \lambda = \mathcal{N}/4\,, \tag{4}$$

find that, until $\mathcal{N} = 8$, there exist the following self-conjugated massless supermultiplets

$$\begin{aligned}
&\mathcal{N} = 2 \text{ matter multiplet:} && 1/2, (0)^2, -1/2;\\
&\mathcal{N} = 4 \text{ gauge multiplet:} && 1, (1/2)^4, (0)^6, (-1/2)^4, -1;\\
&\mathcal{N} = 8 \text{ supergravity multiplet:} && 2, (3/2)^8, (1)^{28}, (1/2)^{56}, (0)^{70},\\
& && (-1/2)^{56}, (-1)^{28}, (-3/2)^8, -2.
\end{aligned}$$

Note that for $\mathcal{N} > 8$ the massless supermultiplets would include helicities > 2. The relevant theories are called "higher-spin theories" and, for self-consistency at the full interaction level, should include the whole infinite set of such spins (helicities). Such complicated theories are under intensive study at present, but their consideration is out the scope of my lectures.

Lecture III

- When considering one or another symmetry and constructing physical models invariant w.r.t. it, it is very important to find out the proper space and/or the fundamental multiplet on which this symmetry is realized in the most natural and simplest way.
- The Poincaré group has a natural realization in the Minkowski space x^m, $m = 0, 1, 2, 3$ as the group of linear rotations and shifts of x^m preserving the flat invariant interval $ds^2 = \eta_{mn} dx^m dx^n$. Analogously, supersymmetry has a natural realization in the Minkowski superspace.
- The translation generators P_m can be realized as shifts of x^m, $x^{m\prime} = x^m + c^m$. In the case of $\mathcal{N} = 1$ supersymmetry we have additional spinor generators Q_α, $\bar{Q}_{\dot\alpha}$ and anticommuting parameters ϵ^α, $\bar\epsilon^{\dot\alpha}$. Then it is natural to introduce new spinor coordinates θ^α, $\bar\theta^{\dot\alpha}$ having the same dimension $cm^{1/2}$ as the spinor parameters and to realize the spinorial generators as shifts of these new coordinates

$$\theta^{\alpha\prime} = \theta^\alpha + \epsilon^\alpha\,, \quad \bar\theta^{\dot\alpha\prime} = \bar\theta^{\dot\alpha} + \bar\epsilon^{\dot\alpha}\,.$$

- The extended manifold

$$\mathcal{M}^{(4|4)} = \left(x^m\,,\ \theta^\alpha\,,\ \bar\theta^{\dot\alpha}\right),$$

is called $\mathcal{N} = 1$ **Minkowski superspace**.

- Its natural generalization is

$$\mathcal{M}^{(4|4\mathcal{N})} = \left(x^m\,,\ \theta_i^\alpha\,,\ \bar{\theta}^{\dot{\alpha}\,i}\right),$$

and it is called $\mathcal{N}$ **extended Minkowski superspace**.

- The spinor coordinates are called odd or Grassmann coordinates and have the Grassmann parity -1, while x^m are even coordinates having the Grassmann parity $+1$

$$[\theta_i^\alpha\,, x^m] = [\bar{\theta}^{\dot{\alpha}\,i}\,, x^m] = 0\,, \quad \{\theta_i^\alpha\,, \theta_k^\beta\} = \{\theta_i^\alpha\,, \bar{\theta}^{\dot{\beta}\,k}\} = 0\,.$$

The spinor coordinates also anticommute with the parameters $\epsilon^\alpha\,, \bar{\epsilon}^{\dot{\alpha}}$.

- Since two superstranslations yield a shift of x^m, they should be non-trivially realized on x^m. In the $\mathcal{N} = 1$ case:

$$x^{m\prime} = x^m - i(\epsilon\sigma^m\bar{\theta} - \theta\sigma^m\bar{\epsilon})\,, \quad (\delta_1\delta_2 - \delta_2\delta_1)x^m = 2i(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1).$$

(an analogous transformation takes place in the general case of $\mathcal{N}$ extended supersymmetry).

- **Superfields** are functions on superspace, such that they have definite transformation properties under supersymmetry. The general scalar $\mathcal{N} = 1$ superfield is $\Phi(x, \theta, \bar{\theta})$ with the following transformation law

$$\Phi'(x', \theta', \bar{\theta}') = \Phi(x, \theta, \bar{\theta}) .$$

- The most important property of superfield is that its series expansion in Grassmann coordinates terminates at the finite step. The reason is that these coordinates are nilpotent, because they anticommute. E.g., $\{\theta_\alpha, \theta_\beta\} = 0 \Rightarrow \theta_1\theta_1 = \theta_2\theta_2 = 0$. Then

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) \quad = \quad & \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ & + \theta\sigma^m\bar{\theta} A_m(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 D(x) ,\end{aligned}$$

  where $\theta^2 := \theta^\alpha\theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$, $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}}$, $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = 1$.

- Here one deals with the set of 8 bosonic and 8 fermionic independent complex component fields. The reality condition

$$\overline{(\Phi)} = \Phi$$

  implies the following reality conditions for the component fields

$$\begin{aligned}& \phi(x) = \overline{\phi(x)} , \quad \bar{\chi}_{\dot{\alpha}}(x) = \overline{\psi_\alpha(x)} , \quad M(x) = \overline{N(x)} , \quad A_m(x) = \overline{A_m(x)} , \\ & \bar{\lambda}^{\dot{\alpha}}(x) = \overline{\rho^\alpha(x)} , \quad D(x) = \overline{D(x)} .\end{aligned}$$

  They leave in Φ just $(8 + 8)$ independent real components.

- The transformation law $\Phi'(x,\theta,\bar\theta) = \Phi(x-\delta x, \theta-\epsilon, \bar\theta-\bar\epsilon)$ implies

$$\delta\Phi = -\epsilon^\alpha \frac{\partial\Phi}{\partial\theta^\alpha} - \bar\epsilon_{\dot\alpha}\frac{\partial\Phi}{\partial\bar\theta_{\dot\alpha}} - \delta x^m \frac{\partial\Phi}{\partial x^m} \equiv i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\Phi\,,$$

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \bar\theta^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}\,, \quad \bar Q_{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}\,,$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2P_m\,,\ \{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0\,, \quad P_m = \frac{1}{i}\frac{\partial}{\partial x^m}\,.$$

- The relevant component transformations are read off from the formula $\delta\Phi = \delta\phi + \theta^\alpha\delta\psi_\alpha + \ldots + \theta^2\bar\theta^2\delta D$. They are

$$\delta\phi = -\epsilon\psi - \bar\epsilon\bar\chi\,, \quad \delta\psi_\alpha = -i(\sigma^m\bar\epsilon)_\alpha\partial_m\phi - 2\epsilon_\alpha M - (\sigma^m\bar\epsilon)_\alpha A_m\,,\ldots\,,$$

$$\delta D = \frac{i}{2}\partial_m\rho\sigma^m\bar\epsilon - \frac{i}{2}\epsilon\sigma^m\partial_m\bar\lambda\,.$$

- These transformations uniformly close on x^m translations without use of any dynamical equations. However, the supermultiplet of fields encompassed by $\Phi(x,\theta,\bar\theta)$ is reducible: it contains in fact both the scalar and gauge $\mathcal{N}=1$ supermultiplets (superspins $Y=0$ and $Y=1/2$). How to describe irreducible supermultiplets in the superfield language?

- An important element of the superspace formalism are spinor covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}, \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m},$$
$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^m)_{\alpha\dot\alpha}\partial_m, \quad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0.$$

- The covariant derivatives anticommute with supercharges, $\{D, Q\} = \{D, \bar Q\} = 0$, so $D_\alpha\Phi$ and $\bar D_{\dot\alpha}\Phi$ are again superfields, e.g., $\delta D_\alpha\Phi = D_\alpha\delta\Phi = D_\alpha i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\Phi = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)D_\alpha\Phi$.
- Now, it becomes possible to define the "irreducible" superfields. **Analogy**: in Minkowski space the vector field A_m is known to carry two Poincaré spins 1 and 0. The irreducible components are distinguished by imposing on A_m the supplementary differential conditions

$$\partial^m A_m = 0 \leftrightarrow \text{spin } 1, \quad \partial_m A_n - \partial_n A_m = 0 \leftrightarrow \text{spin } 0.$$

- Analogous conditions can be imposed on the superfield Φ in order to single out the irreducible multiplets with the superspins 0 and $1/2$. These conditions are defined with the help of the covariant spinor derivatives.

- The simplest condition of this type is the chirality or anti-chirality conditions

$$(a)\ \bar{D}_{\dot{\alpha}}\Phi_L(x,\theta,\bar{\theta}) = 0\,, \quad \text{or} \quad (b)\ D_\alpha\Phi_R(x,\theta,\bar{\theta}) = 0\,.$$

- Eq. (a), e.g., implies

$$\Phi_L(x,\theta,\bar{\theta}) = \varphi_L(x_L,\theta) = \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) + \theta\theta F(x_L)\,,$$
$$x_L^m = x^m + i\theta\sigma^m\bar{\theta}\,.$$

  i.e. we are left with the independent fields ϕ, ψ_α, F.

- From the general transformation laws of the component fields it follows that this set is closed under $\mathcal{N}=1$ supersymmetry:

$$\delta\phi = -\epsilon\psi\,, \quad \delta\psi_\alpha = -2i(\sigma^m\bar{\epsilon})_\alpha\partial_m\phi - 2\epsilon_\alpha F\,, \quad \delta F = -i\bar{\epsilon}\tilde{\sigma}^m\partial_m\psi\,.$$

  This is just the transformation law of the scalar $\mathcal{N}=1$ supermultiplet.

- The geometric interpretation: the coordinate set (x_L^m, θ^α) is closed under $\mathcal{N}=1$ supersymmetry:

$$\delta x_L^m = 2i\theta\sigma^m\bar{\epsilon}\,, \quad \delta\theta^\alpha = \epsilon^\alpha\,. \tag{5}$$

  It is called left-chiral $\mathcal{N}=1$ superspace.

- In the basis $(x_L^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ the chirality condition (a) is reduced to the Grassmann Cauchy-Riemann conditions:

$$\bar{D}_{\dot{\alpha}}\Phi_L(x_L,\theta,\bar{\theta}) = 0 \;\Rightarrow\; \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\Phi_L = 0 \;\Rightarrow\; \Phi_L = \varphi_L(x_L,\theta)\,. \tag{6}$$

Lecture IV

- Having superfields, one can cosntruct out of them, as well as out of their vector and covariant spinor derivatives, scalar superfield Lagrangians. Any local product of superfields is again a superfield:

$$\mathcal{L} = \mathcal{L}(\Phi, D_\alpha\Phi, \bar{D}_{\dot\alpha}\Phi, \partial_m\Phi, \ldots)\,, \quad \delta\mathcal{L} = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)\mathcal{L}\,.$$

- It is easy to realize that the variation of the higher component of any expansion of any superfield is a total derivative. Then one takes the higher component field in the θ expansion of the superfield Lagrangian and integrates it over Minkowski space. It will be just an action invariant under $\mathcal{N} = 1$ supersymmetry!
- A manifestly covariant way to write supersymmmetric actions is to use the Berezin integral. It is equivalent to differentiation in Grassmann coordinates. In the considered case of $\mathcal{N} = 1$ superspace it is defined by the rules

$$\int d^2\theta\,(\theta)^2 = 1\,, \quad \int d^2\bar\theta\,(\bar\theta)^2 = 1\,, \quad \int d^2\theta d^2\bar\theta\,(\theta)^4 = 1\,, \quad (\theta)^4 \equiv (\theta)^2(\bar\theta)^2$$

Hence the Berezin integration yields an efficient and manifestly supersymmetric way of singling out the coefficients of the highest-order θ monomials in the superfield Lagrangians.

- The simplest invariant action of chiral superfields producing the kinetic terms of the scalar multiplet is as follows

$$S_{kin} = \int d^4x d^4\theta\, \varphi(x_L, \theta)\bar{\varphi}(x_R, \bar{\theta})\,, \quad x_R^m = \overline{(x_L^m)} = x^m - i\theta\sigma^m\bar{\theta}\,.$$

After performing integration over Grassmann coordinates, one obtains

$$S \sim \int d^4x \left(\partial^m\bar{\phi}\partial_m\phi - \frac{i}{2}\psi\sigma^m\partial_m\bar{\psi} + F\bar{F}\right).$$

- The total Wess-Zumino model action is reproduced by adding, to this kinetic term, also potential superfield term

$$S_{pot} = \int d^4x_L d^2\theta \left(\frac{g}{3}\varphi^3 + \frac{m}{2}\varphi^2\right) + \text{c.c.}\,.$$

- This action is the only renormalizble action of the scalar $\mathcal{N} = 1$ multiplet. In principle, one can construct more general actions, e.g., the action of Kähler sigma model and generalized potential term

$$\tilde{S}_{kin} = \int d^4x d^4\theta\, K\left[\varphi(x_L, \theta), \bar{\varphi}(x_R, \bar{\theta})\right], \quad \tilde{S}_{pot} = \int d^4x_L d^2\theta\, P(\varphi) + \text{c.c.}\,.$$

- The multiplet with the superspin $Y = 1/2$ is described by the gauge superfield $V(x, \theta, \bar{\theta})$ possessing the gauge freedom

$$\delta V(x, \theta, \bar{\theta}) = i[\bar{\lambda}(x^m - i\theta\sigma^m\bar{\theta}, \bar{\theta}) - \lambda(x^m + i\theta\sigma^m\bar{\theta}, \theta)],$$

  where $\lambda(x_L, \theta)$ is an arbitatry chiral superfild parameter.

- Using this freedom, one can choose the so called Wess-Zumino gauge

$$V_{WZ}(x, \theta, \bar{\theta}) = 2\,\theta\sigma^m\bar{\theta}\,A_m(x) + 2i\bar{\theta}^2\theta^\alpha\,\psi_\alpha(x) - 2i\theta^2\bar{\theta}_{\dot{\alpha}}\,\bar{\psi}^{\dot{\alpha}}(x) + \theta^2\bar{\theta}^2\,D(x)\,.$$

- Thus in the WZ gauge we are left with the irreducible set of fields forming the gauge (or vector) off-shell supermultiplet: the gauge field $A_m(x)$, $A'_m(x) = A_m + \partial_m\lambda(x)$, the fermionic field of gaugino $\psi_\alpha(x)$, $\bar{\psi}_{\dot{\alpha}}(x)$ and the auxiliary field $D(x)$.

- The invariant action is written as an integral over the chiral superspace

$$S^{N=1}_{gauge} = \frac{1}{16}\int d\zeta_L\,(W^\alpha W_\alpha) + \text{c.c.}\,, \quad W_\alpha = -\frac{1}{2}\bar{D}^2 D_\alpha V\,,\ \bar{D}_{\dot{\alpha}} W_\alpha = 0\,.$$

- Everything is easily generalized to the non-abelian case. The corresponding component off-shell action reads

$$S = \int d^4x\,\mathrm{Tr}\left[-\frac{1}{4}F^{mn}F_{mn} - i\psi\sigma^m\mathcal{D}_m\bar{\psi} + \frac{1}{2}D^2\right].$$

- What about superfield approach to higher $\mathcal{N}$ supersymmetries? The difficulties arise because the relevant superspaces contain too many θ coordinates and it is a very complicated problem to define the superfields which would correctly describe the relevant irreps.
- For $\mathcal{N}=2$, the off-shell gauge multiplet contains the vector gauge field $A_m(x)$, the complex scalar physical field $\varphi(x)$, the $SU(2)$ doublet of Weyl fermions $\psi^i_\alpha(x), \bar{\psi}_{\dot{\alpha}\, i}(x)$ and the auxiliary real $SU(2)$ triplet $D^{(ik)}(x)$.
- There is no simple way to define $\mathcal{N}=2$ analog of the $\mathcal{N}=1$ gauge prepotential V. However, one can define the appropriate covariant superfield strength W. In the abelian case, it is defined by the constraints

$$(a)\ \bar{D}^i_{\dot{\alpha}} W = 0\,, \quad (b)\ D^{\alpha i} D^k_\alpha W = \bar{D}^i_{\dot{\alpha}} \bar{D}^{\dot{\alpha} k} \bar{W}\,.$$

The invariant action is an integral over chiral $\mathcal{N}=2$ superspace

$$S \sim \int d^4x_L d^4\theta W^2 + \text{c.c.}\,.$$

- What about maximaly extended $\mathcal{N}=4$ super Yang-Mills? It has no superfield formulation with all $\mathcal{N}=4$ supersymmetries being manifest and off-shell. There is $\mathcal{N}=1$ superfield formulation with one gauge superfield and three chiral superfields; $\mathcal{N}=2$ formulation in terms of $\mathcal{N}=2$ gauge superfield and one hypermultiplet. The latter possesses an off-shell formulation only in the $\mathcal{N}=2$ harmonic superspace.

$d = 1$ Superspace

- $\mathcal{N} = 2, d = 1$ superspace:

$$\mathcal{M}^{(1|2)} = (t, \theta, \bar{\theta}) \,, \quad \delta\theta = \epsilon \,, \quad \delta\bar{\theta} = \bar{\epsilon} \,, \quad \delta t = i(\epsilon\bar{\theta} + \bar{\epsilon}\theta) \,.$$

- $\mathcal{N}=2$ covariant spinor derivatives:

$$D = \partial_\theta - i\bar{\theta}\partial_t \,, \qquad \bar{D} = -\partial_{\bar{\theta}} + i\theta\partial_t \,, \qquad \{D, \bar{D}\} = 2i\partial_t \,.$$

- The simplest superfield is the real one, $\Phi(t, \theta, \bar{\theta})$,

$$\Phi'(t', \theta', \bar{\theta}') = \Phi(t, \theta, \bar{\theta}) \;\Rightarrow\; \delta\Phi = -\delta t\partial_t\Phi - \epsilon\partial_\theta\Phi - \bar{\epsilon}\partial_{\bar{\theta}}\Phi \,.$$

- On the component fields appearing in the θ expansion of Φ,

$$\Phi(t, \theta, \bar{\theta}) = x(t) + \theta\psi(t) - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}y(t) \,,$$

$\mathcal{N}=2$ supersymmetry acts as

$$\delta x = \bar{\epsilon}\bar{\psi} - \epsilon\psi \,, \; \delta\psi = \bar{\epsilon}(i\dot{x} - y) \,, \; \delta\bar{\psi} = -\epsilon(i\dot{x} + y) \,, \; \delta y = i(\epsilon\dot{\psi} + \bar{\epsilon}\dot{\bar{\psi}}) \,.$$

- The superfield $\Phi(t, \theta, \bar{\theta})$ encompasses the irreducible $\mathcal{N} = 2, d = 1$ multiplet $(\mathbf{1}, \mathbf{2}, \mathbf{1})$. Other $\mathcal{N} = 2, d = 1$ multiplets exist as well, e.g., $(\mathbf{2}, \mathbf{2}, \mathbf{0})$.

- The simplest invariant superfield action containing interaction reads

$$S^{(N=2)} = \int dt d^2\theta \left[\bar{D}\Phi\, D\Phi + W(\Phi) \right].$$

Here $W(\Phi)$ is the superpotential. After integrating over Grassmann coordinates, we obtain

$$S^{(N=2)} = \int dt \left[\dot{x}^2 - i\left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}\right) + y^2 + y\partial_x W(x) + (\psi\bar{\psi})\partial_x^2 W(x) \right].$$

- The next step is to eliminate the auxiliary field y by its algebraic equation of motion

$$y = -\frac{1}{2}\partial_x W\,.$$

The on-shell action is

$$S^{(N=2)} = \int dt \left[\dot{x}^2 - i\left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}\right) - \frac{1}{4}(\partial_x W)^2 + (\psi\bar{\psi})\partial_x^2 W(x) \right].$$

- The action is invariant under the transformations

$$\delta x = \bar{\epsilon}\bar{\psi} - \epsilon\psi\,,\ \delta\psi = \bar{\epsilon}(i\dot{x} + \frac{1}{2}\partial_x W)\,,\ \delta\bar{\psi} = -\epsilon(i\dot{x} - \frac{1}{2}\partial_x W)\,.$$

Hamiltonian formalism and quantization

- The quantum Hamiltonian obtained in a standard way from the canonical one reads

$$H = \frac{1}{4}\left[\hat{p}^2 + \left(\frac{dW}{d\hat{x}}\right)^2\right] - \frac{1}{2}\frac{d^2W}{d\hat{x}^2}\left(\hat{\psi}\hat{\bar{\psi}} - \hat{\bar{\psi}}\hat{\psi}\right),$$

  where we have Weyl-ordered the fermionic term. The supercharges calculated by the Noether procedure and then brought into the quantum form through passing to the operators are

$$Q = \hat{\psi}\left(\hat{p} + i\frac{dW}{d\hat{x}}\right), \qquad \bar{Q} = \hat{\bar{\psi}}\left(\hat{p} - i\frac{dW}{d\hat{x}}\right).$$

- The algebra of the basic quantum operators is

$$[\hat{x}, \hat{p}] = i\,, \qquad \{\hat{\psi}, \hat{\bar{\psi}}\} = \tfrac{1}{2}\,.$$

  Using it, we can calculate the anticommutators of the quantum supercharges and check that they form $\mathcal{N} = 2, d = 1$ superalgebra

$$\{Q, \bar{Q}\} = 2H\,, \qquad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0\,. \tag{7}$$

- By the graded Jacobi identities:

$$[Q, H] = [\bar{Q}, H] = 0\,.$$

Spectral problem: double degeneracy

- We use the standard realization for $\hat{p}$, $\hat{p} = \frac{1}{i}\frac{\partial}{\partial x}$, and the Pauli-matrix realization for the fermionic operators

$$\hat{\psi} = \frac{1}{2\sqrt{2}}\left(\sigma_1 + i\sigma_2\right), \quad \hat{\bar{\psi}} = \frac{1}{2\sqrt{2}}\left(\sigma_1 - i\sigma_2\right), \quad \hat{\psi}\hat{\bar{\psi}} - \hat{\bar{\psi}}\hat{\psi} = \tfrac{1}{2}\,\sigma_3\,.$$

Then the Hamiltonian and supercharges become 2×2 matrices

$$H = \frac{1}{4}\left[-\partial_x{}^2 + (W_x)^2\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{4}\,W_{xx}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(\partial_x - W_x)\,, \quad \bar{Q} = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}(\partial_x + W_x)\,.$$

- Thus the wave functions form a doublet and, taking into account the conditions $[Q, H] = [\bar{Q}, H] = 0$, the relevant matrix spectral problem is

$$H\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lambda\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

- It is equivalent to the two ordinary problems

$$H_\pm = \lambda\,\psi_\pm\,, \qquad H_\pm = -\frac{1}{4}(\partial_x \mp W_x)(\partial_x \pm W_x).$$

- Using the intertwining property

$$H_-(\partial_x + W_x) = (\partial_x + W_x)H_+\,, \quad H_+(\partial_x - W_x) = (\partial_x - W_x)H_-\,,$$

now it easy to show that the states

$$Q\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \begin{pmatrix}-i(\partial_x - W_x)\psi_-\\ 0\end{pmatrix}, \quad \bar{Q}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \begin{pmatrix}0\\ -i(\partial_x + W_x)\psi_+\end{pmatrix}$$

are the eigenfunctions of H_+ and H_- with the same eigenvalue λ. Thus we observe the double degeneracy of the spectrum. This double degeneracy is the most chracteristic feature of the $\mathcal{N} = 2$ supersymmetry in $d = 1$ (and of any higher $\mathcal{N}$ supersymmetry).

- In general, the Hilbert space of quantum states of $\mathcal{N} = 2$ SQM is divided into the following three sectors

$$(a)\ \text{Ground state}: \ Q\Psi_0 = \bar{Q}\Psi_0 = H\Psi_0 = 0\,,$$
$$(b)\ H\Psi_1 = E\Psi_1\,,\ Q\Psi_1 \neq 0\,,\ \bar{Q}\Psi_1 = 0\,,$$
$$(c)\ H\Psi_2 = E\Psi_2\,,\ \bar{Q}\Psi_2 \neq 0\,,\ Q\Psi_2 = 0\,.$$

- One can argue that many QM models with the double degeneracy of the energy spectrum can be identified with some $\mathcal{N} = 2$ SQM model.

Summary

- Supersymmetry between fermions and bosons is a new uinusual concept in the mathematical physics which allowed to construct a lot of new theories with remarkable and surprising features: supergravities, superstrings, superbranes, $N = 4$ super Yang-Mills theory (the first example of the ultraviolet-finite quanrtum field theory), etc. It also allowed to establish unexpected relations between these theories, e.g., the AdS/CFT (or “gravity/gauge”) correspondence, AGT correspondence, etc.
- It predicts new particles (superpartners) which still await their experimental discovery.
- The natural approach to supersymmetric theories is the superfield methods.

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THANK YOU FOR YOUR ATTENTION!