Equation of state of rotating gluon plasma from lattice simulations

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in collaboration with

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DSPIN-26



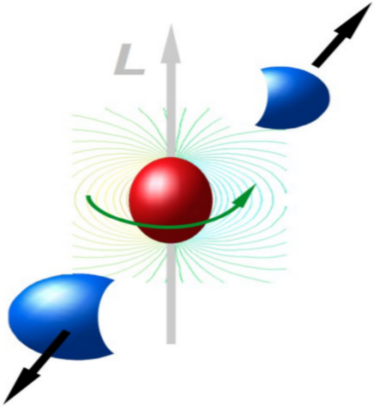
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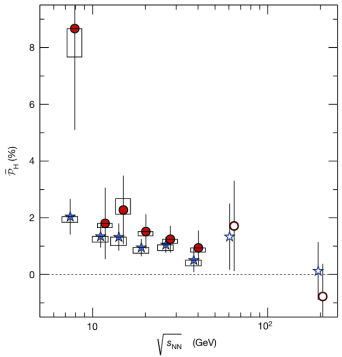
ИФВЭ IHEP

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- In non-cetral heavy ion collisions, the droplets of QGP with angular momentum are crated.
- The rotation occurs with relativistic velocities.



$\omega = 10$ MeV ~ 0.05 fm^{-1} $\quad v \sim c$ at $r \sim 20$ fm



[L. Adamczyk et al. (STAR), Nature **548**, 62–65 (2017), arXiv:1701.06657 [nucl-ex]]

$\langle\omega\rangle \sim 7$ MeV $\quad (\sqrt{s_{NN}}$-averaged)

- How does the rotation affect QCD properties?

Equation of State, Angular Momentum and Moment of Inertia

A mechanical response of a thermodynamic ensemble to rigid rotation $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$ is described in terms of the total angular momentum $\boldsymbol{J} = J\boldsymbol{e}_z$. The energy in co-rotating reference frame is

$$E = E^{(lab)} - \boldsymbol{J}\cdot\boldsymbol{\Omega}\,, \qquad F = E - TS\,, \qquad dF = -SdT - \boldsymbol{J}\cdot d\boldsymbol{\Omega} + \dots\,,$$

The angular momentum and moment of inertia are given by

$$\boldsymbol{J} = -\left(\frac{\partial F}{\partial \boldsymbol{\Omega}}\right)_T\,, \qquad I(T,\Omega) = \left(\frac{\partial J(T,\Omega)}{\partial \Omega}\right)_T = -\left(\frac{\partial^2 F}{\partial \Omega^2}\right)_T\,. \tag{1}$$

The free energy of a slowly rotating system can be represented as a series in the angular velocity Ω:

$$F(T,R,\Omega) = F_0(T,R) - \frac{F_2(T,R)}{2}\Omega^2 - \frac{F_4(T,R)}{4}\Omega^4 - \dots\,. \tag{2}$$

It is convenient to use specific quantities $f = \dfrac{F}{V}$, $f_n = \dfrac{F_n}{V}$, and $\mathsf{i}_n = \dfrac{F_n}{VR^n} = \dfrac{f_n}{R^n}$,

$$f(T,\Omega R) = f_0(T) - \frac{\mathsf{i}_2(T)}{2}(\Omega R)^2 - \frac{\mathsf{i}_4(T)}{4}(\Omega R)^4 - \dots\,. \tag{3}$$

The (bluk-averaged) total angular momentum density $j = J/V$ may be expressed as follows

$$\frac{j}{R} \equiv \frac{J}{VR} = \mathsf{i}_2(\Omega R) + \mathsf{i}_4(\Omega R)^3 + \mathsf{i}_6(\Omega R)^5 + \dots \,. \tag{4}$$

And the moment of inertia is

$$I = VR^2\Big(\mathsf{i}_2 + 3\mathsf{i}_4(\Omega R)^2 + 5\mathsf{i}_6(\Omega R)^4 + \dots\Big) \tag{5}$$

where i_2 is specific moment of inertia, $\mathsf{i}_4, \mathsf{i}_6$ describe rotational deformation.

Analogy in classical mechanics

The moment of inertia of a body with mass density $\rho(\boldsymbol{r}) = \rho_0 + \rho_2(\Omega r)^2 + \rho_4(\Omega r)^4 + \dots$, is

$$I_{\text{class}} = \int_V d^3x\, r^2\, \rho(\boldsymbol{r}) = VR^2\left(\alpha_2\rho_0 + \alpha_4\rho_2 + \alpha_6\rho_4 + \dots\right) , \tag{6}$$

or

$$\mathsf{i}_n = \frac{\alpha_n}{2n-1}\rho_{n-2} , \qquad \text{where} \quad \alpha_n = \frac{1}{VR^n}\int_V d^3x\, r^n . \tag{7}$$

The coefficients α_n depends on the system geometry. Classical expectations: $\rho_0 > 0$, $\rho_2 > 0$, i.e. $\mathsf{i}_2 > 0$, $\mathsf{i}_4 > 0$.

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- For a large rotating system, approximation of local thermalization may be applied, and free energy density at specific radius r is

$$\tilde{f}(T,r,\Omega) = \tilde{f}(T,u) = f_0(T) - \frac{\tilde{f}_2(T)}{2}u^2 - \frac{\tilde{f}_4(T)}{4}u^4 - \frac{\tilde{f}_6(T)}{6}u^6 + \dots , \tag{8}$$

where $u = \Omega r$. The free energy of the whole system is an integral quantity,

$$F(T,R,\Omega) = \int_V d^3r\, \tilde{f}(T,\Omega r)\,, \tag{9}$$

and $\mathsf{i}_n(T) = \alpha_n \tilde{f}_n(T)$. Therefore, $\tilde{f}_n(T) \sim \rho_{n-2}(T)$.

- The angular momentum density is also a function of coordinate. Indeed,

$$J(T,R,\Omega) = \int_V d^3r\, \tilde{j}(T,r,\Omega)\,, \tag{10}$$

and the local quantity, $\tilde{j}$ is

$$\tilde{j}(T,r,\Omega) = \left(\tilde{f}_2(T) + \tilde{f}_4(T)u^2 + \tilde{f}_6(T)u^4 + \dots\right) u\, r\,. \tag{11}$$

Our aim is to calculate i_n and $\tilde{f}_n$ for rotating gluodynamics.

It is convenient to describe the system in the co-rotating reference frame, $x^\mu = (t, x, y, z)$,

$$\varphi = \left[\bar{\varphi} - \Omega t\right]_{2\pi}, \quad t = \bar{t}, \quad z = \bar{z}, \quad r = \bar{r}, \tag{12}$$

where $\bar{x}^\mu \equiv x^\mu_{\text{lab}}$, with the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 - r^2\Omega^2 & y\Omega & -x\Omega & 0 \\ y\Omega & -1 & 0 & 0 \\ -x\Omega & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

The action of Yang-Mills theory in the Minkowski curved spacetime is

$$S_G^{(\text{Mink})} = -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{-g}\, g^{\mu\alpha} g^{\nu\beta} F^a_{\mu\nu} F^a_{\alpha\beta} = \int d^4x \left(\mathcal{L}_0^{(\text{Mink})} + \Omega \mathcal{L}_1^{(\text{Mink})} + \Omega^2 \mathcal{L}_2^{(\text{Mink})}\right) \tag{14}$$

where $\Omega = \partial_t \varphi_{\text{lab}}$.

The causality restriction: $\boxed{\Omega r < 1}$.

- In the laboratory reference frame, the angular momentum operator can be defined via the stress-energy tensor:

$$J_{i,\,\mathrm{lab}} = \frac{1}{2}\epsilon_{ijk}\int d^3\bar{x}\left(\bar{x}^j T^{0k}_{\mathrm{lab}} - \bar{x}^k T^{0j}_{\mathrm{lab}}\right), \qquad \text{where} \quad T^{\mu\nu}_{\mathrm{lab}} = -\frac{1}{g^2_{YM}}\left(\bar{F}^a_{\mu\alpha}\bar{F}^{a,\alpha}_{\ \ \nu} - \frac{1}{4}g_{\mu\nu}\bar{F}^a_{\alpha\beta}\bar{F}^{a,\alpha\beta}\right). \tag{15}$$

The z-component of total angular momentum reads as follows:

$$J_{\mathrm{lab}} \equiv J_{z,\,\mathrm{lab}} = -\frac{1}{g^2_{YM}}\int d^3\bar{x}\big(-\bar{x}\bar{F}^a_{\mathrm{yx}}\bar{F}^a_{\mathrm{xt}} - \bar{x}\bar{F}^a_{\mathrm{yz}}\bar{F}^a_{\mathrm{zt}} + \bar{y}\bar{F}^a_{\mathrm{xy}}\bar{F}^a_{\mathrm{yt}} + \bar{y}\bar{F}^a_{\mathrm{xz}}\bar{F}^a_{\mathrm{zt}}\big), \tag{16}$$

where $\bar{F}^a_{\mu\nu}$ is a gluon tensor in the laboratory frame. It is $\int d^3x\mathcal{L}_1^{(\mathrm{Mink})}$ up to $F^a_{\mu\nu} \to \bar{F}^a_{\mu\nu}$.

- Using the tensor transformation, $\bar{F}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}F_{\alpha\beta}$, one gets the angular momentum in terms of rotating fields

$$J = \int d^3x\left(\mathcal{L}_1^{(\mathrm{Mink})} + 2\Omega\mathcal{L}_2^{(\mathrm{Mink})}\right), \tag{17}$$

with $\mathcal{L}_2^{(\mathrm{Mink})} \propto (F^a_{xy})^2$, $(F^a_{yz})^2$, $(F^a_{xz})^2$.

The rotating system at thermal equilibrium is studied on the lattice. The partition function is

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\hat{H}/T_0}\right] = \int \mathcal{D}[U]\; e^{-S_G[U,\Omega]}\,, \tag{18}$$

where the Euclidean action, S_G, is formulated in curved space ($t \to -i\tau$), $x^\mu = (x, y, z, \tau)$,

$$g^E_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & -y\Omega_I \\ 0 & 1 & 0 & x\Omega_I \\ 0 & 0 & 1 & 0 \\ -y\Omega_I & x\Omega_I & 0 & 1 + r^2\Omega_I^2 \end{pmatrix}, \tag{19}$$

and the angular velocity is imaginary, $\Omega_I = \partial_\tau \varphi_{\rm lab} = -i\partial_t \varphi_{\rm lab} = -i\Omega$, to avoid the sign problem.

There is no causality restriction in Euclidean space.

The gluon action is a quadratic function in angular velocity:

$$S_G = \frac{1}{4g_{YM}^2}\int d^4x\,\sqrt{g_E}\,g_E^{\mu\nu}g_E^{\alpha\beta}F^a_{\mu\alpha}F^a_{\nu\beta} \equiv S_0 + S_1\Omega_I + S_2\frac{\Omega_I^2}{2} =$$

$$= \frac{1}{g_{YM}^2}\int d^4x\left(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \Omega_I\left[-yF^a_{xy}F^a_{y\tau} - yF^a_{xz}F^a_{z\tau} + xF^a_{yx}F^a_{x\tau} + xF^a_{yz}F^a_{z\tau}\right] + \right.$$

$$\left. + \Omega_I^2\left[r^2(F^a_{xy})^2 + y^2(F^a_{xz})^2 + x^2(F^a_{yz})^2 + 2xyF^a_{xz}F^a_{zy}\right]\right) \quad (20)$$

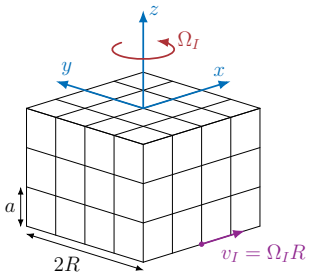
so, that $S_1 = \int d^4x\, rO_1$ and $S_2 = \int d^4x\, r^2O_2$.

- For the thermodynamic ensemble (18), an (imaginary) angular momentum reads

$$J_I = -\left(\frac{\partial F}{\partial\Omega_I}\right)_T = T\left(\frac{\partial \ln\mathcal{Z}}{\partial\Omega_I}\right)_T = -T\langle S_1 + 2\Omega_I S_2\rangle\,. \quad (21)$$

Note that $J_I = -\partial F/\partial\Omega_I = -i\partial F/\partial\Omega = iJ$.

Lattice setup and observables



- Lattice size: $N_t \times N_z \times N_s^2 \quad (N_x = N_y = N_s)$
- Temperature: $T = 1/(N_t a)$
- "Radius" of the system: $R = a(N_s - 1)/2$
- Boundary velocity: $v_I^2 = (\Omega_I R)^2 < 1/2$
- Angular velocity: $\frac{\Omega_I}{T} = \frac{v_I}{RT} \ll 1$
- periodic b.c. in directions τ, z.
- different types of b.c. in directions x,y:
  open / periodic / Dirichlet / ...

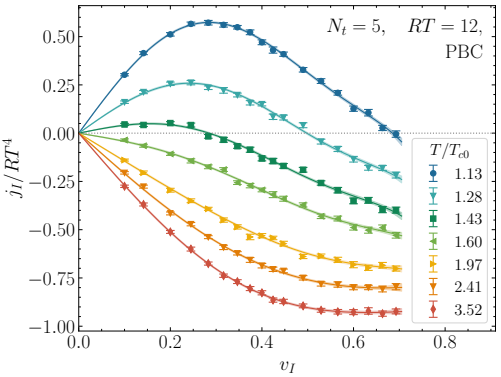
We used large lattices with $RT = 8$, $\underline{12}$, 18 and $N_t = 4$, 5, 6, 7,

Causality restriction

- Analytic continuation is allowed only for bounded system with $\Omega r < 1$
- Boundary conditions are important! (they influence the result in all approaches)

- Total angular momentum calculations
- Free energy calculations (Integral method)
- Angular momentum density analysis and local thermalization
- Derivatives at $\Omega = 0$

Total angular momentum calculations



Imaginary (bulk-averaged) angular momentum density is calculated,

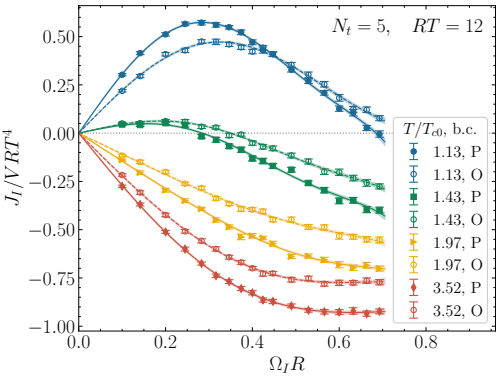
$$\frac{j_I}{RT^4} = -\frac{2N_t^4}{(N_s-1)}\Big[\langle s_1 + 2\Omega_I s_2\rangle\Big]^{\rm sub} \qquad (22)$$

where $[\langle s_i\rangle]^{\rm sub} = \langle s_i\rangle_T - \langle s_i\rangle_{T=0} = \langle s_i\rangle_{N_t} - \langle s_i\rangle_{N_t^\infty}$.

The data fitted by the function

$$\frac{j_I(T)}{RT^4} = -\frac{\mathsf{i}_2(T)}{T^4}v_I + \frac{\mathsf{i}_4(T)}{T^4}v_I^3 - \frac{\mathsf{i}_6(T)}{T^4}v_I^5 + \dots . \qquad (23)$$

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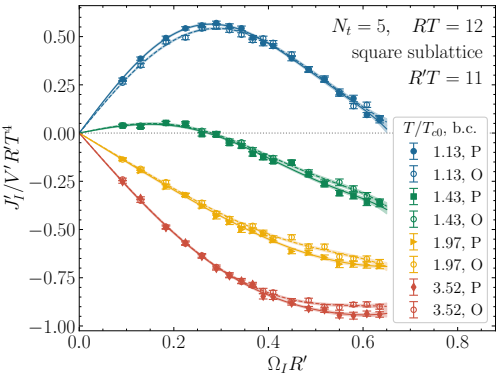
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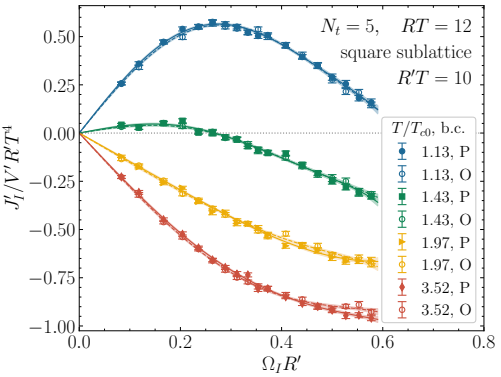
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- When a few layers adjusted to the boundary are ignored, the results for OBC/PBC coincides.

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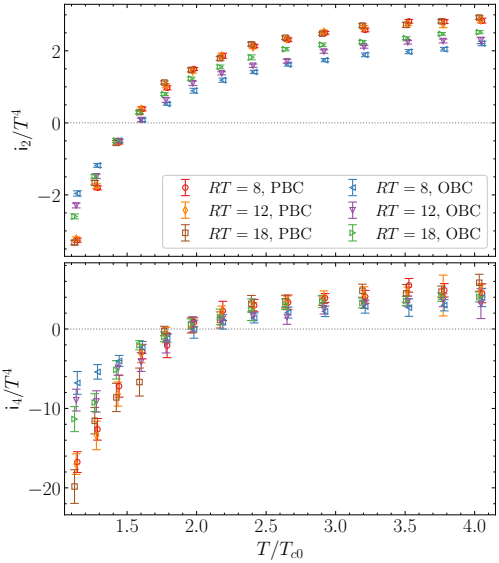
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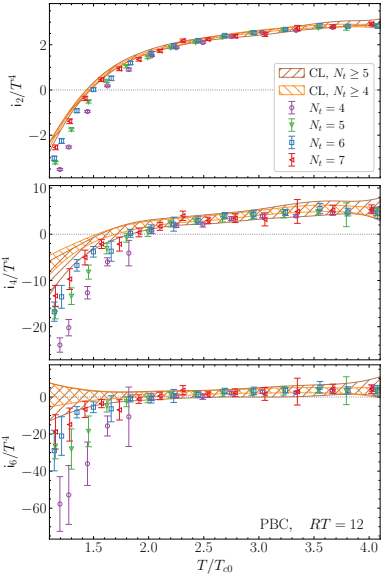
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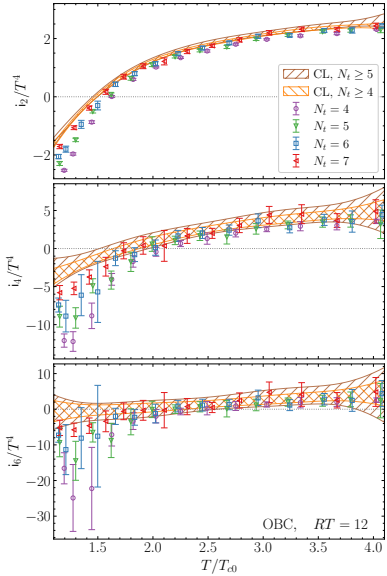
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- When a few layers adjusted to the boundary are ignored, the results for OBC/PBC coincides.
- As the lattice size increases, the results for OBC/PBC become closer to each other.

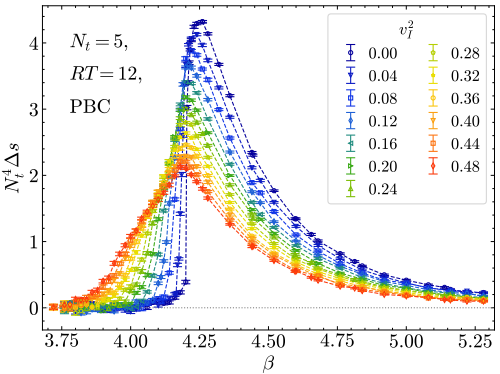
Total angular momentum calculations





- $i_2 < 0$ at $T < T_s \approx 1.5T_{c0}$
- $i_4 < 0$ at $T \lesssim 1.6 - 1.9T_{c0}$
- $i_6 \approx 0$ with large uncertainty
- The results for PBC/OBC are in qualitative agreement

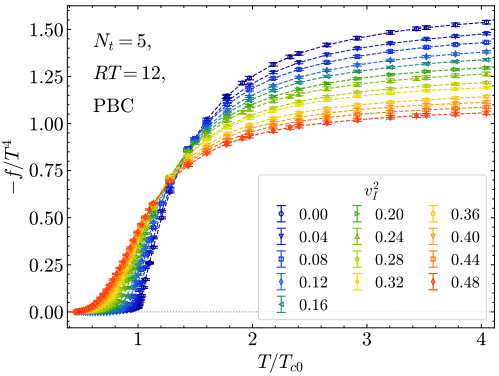
- Total angular momentum calculations
- Free energy calculations (Integral method)
- Angular momentum density analysis and local thermalization
- Derivatives at $\Omega = 0$



The (bulk-averaged) free energy density is calculated as β-integral:

$$\left.\frac{f}{T^4}\right|_{\beta_0}^{\beta} = -N_t^4 \int_{\beta_0}^{\beta} d\beta' \Delta s(\beta')\,, \tag{24}$$

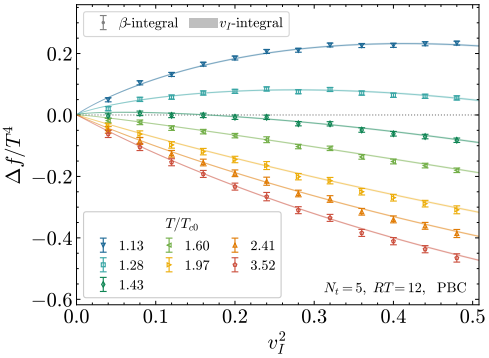
where $\Delta s = [\langle s(\beta') \rangle]^{\text{sub}}$, and $\frac{\langle T^\mu_\mu \rangle}{T^4} = -N_t^4 \frac{d\beta}{d\ln a} \Delta s$.



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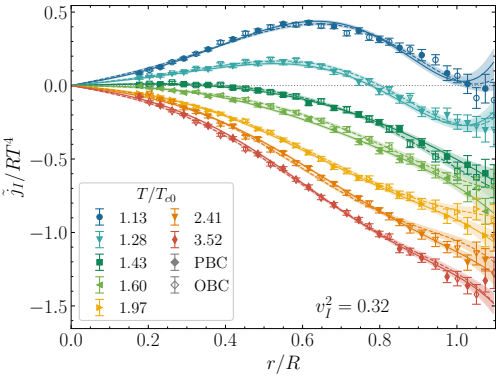
An excess of the free energy, $\Delta f(v_I) = f(v_I) - f(0)$, also may be computed as v_I-integral,

$$\frac{\Delta f(v_I, T)}{T^4} = -\int_0^{v_I} dv_I' \frac{j_I(T, v_I')}{RT^4}\,. \tag{25}$$

- The results of two integral methods for all lattices are in agreement.
- Uncertainties of β-integral method are larger.

- Total angular momentum calculations
- Free energy calculations (Integral method)
- Angular momentum density analysis and local thermalization
- Derivatives at $\Omega = 0$

Angular momentum density

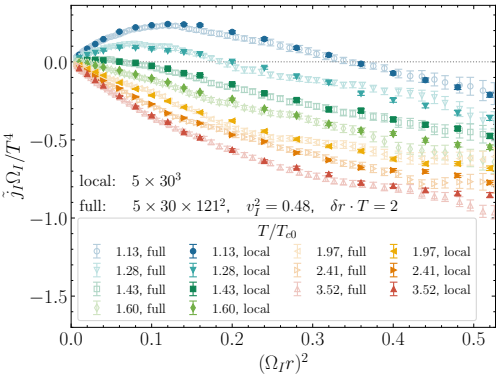


The angular momentum density is given by

$$\frac{\tilde{j}_I(x,y)}{RT^4} = -\frac{N_t^4}{v_I}\Big[\langle \Omega_I \tilde{s}_1(x,y) + 2\Omega_I^2 \tilde{s}_2(x,y)\rangle\Big]^{\text{sub}}, \quad (26)$$

and

$$\frac{\tilde{j}_I(T,r,\Omega_I)}{RT^4} = \left(-\frac{\tilde{f}_2(T)}{T^4} + \frac{\tilde{f}_4(T)}{T^4}u_I^2 - \frac{\tilde{f}_6(T)}{T^4}u_I^4 + \dots\right) u_I \frac{r}{R},$$



The angular momentum density is given by

$$\frac{\tilde{j}_I(x,y)}{RT^4} = -\frac{N_t^4}{v_I}\Big[\langle \Omega_I \tilde{s}_1(x,y) + 2\Omega_I^2 \tilde{s}_2(x,y)\rangle\Big]^{\rm sub}, \quad (26)$$

and

$$\frac{\tilde{j}_I(T,r,\Omega_I)}{RT^4} = \left(-\frac{\tilde{f}_2(T)}{T^4} + \frac{\tilde{f}_4(T)}{T^4}u_I^2 - \frac{\tilde{f}_6(T)}{T^4}u_I^4 + \dots\right) u_I \frac{r}{R},$$

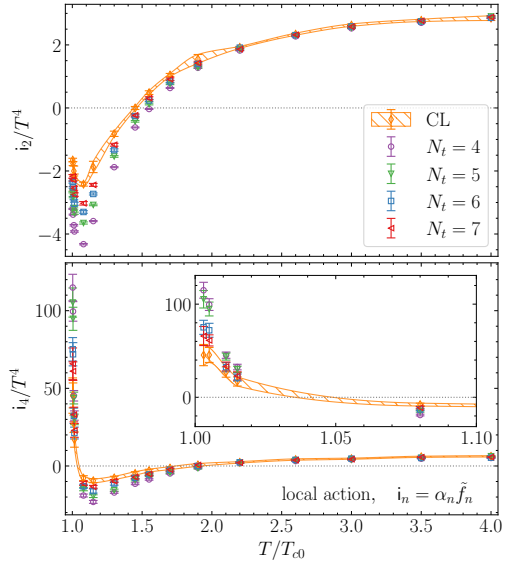
At the point r_0 with local velocity $u_I = \Omega_I r_0$,

$$\frac{j_I^{(loc)}}{r_0 T^4} = -N_t^4\Big[\langle s_1 + 2u_I s_2\rangle\Big]^{\rm sub}. \quad (27)$$

Data are fitted with the following ansatz,

$$\frac{j_I^{(loc)}(T,u_I)\Omega_I}{T^4} = -\frac{\tilde{f}_2(T)}{T^4}u_I^2 + \frac{\tilde{f}_4(T)}{T^4}u_I^4 - \frac{\tilde{f}_6(T)}{T^4}u_I^6 + \dots.$$

Angular momentum density



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- Total angular momentum calculations
- Free energy calculations (Integral method)
- Angular momentum density analysis and local thermalization
- Derivatives at $\Omega = 0$

One may directly calculate the specific moments of inertia by taking the derivatives at vanishing rotation,

$$\mathsf{i}_{2k} = -\frac{1}{(2k-1)!\,VR^{2k}}\,\frac{\partial^{2k}F}{\partial\Omega^{2k}}\bigg|_{\Omega=0} = \frac{(-1)^k T}{(2k-1)!\,VR^{2k}}\,\frac{\partial^{2k}\ln\mathcal{Z}}{\partial\Omega_I^{2k}}\bigg|_{\Omega_I=0}. \tag{28}$$

For the two leading coefficients one gets,

$$\mathsf{i}_2 = \mathsf{i}_2^{\mathrm{mech}} + \mathsf{i}_2^{\mathrm{magn}}, \tag{29}$$

$$\mathsf{i}_4 = \mathsf{i}_4^{\mathrm{mech}} + \mathsf{i}_4^{\mathrm{mix}} + \mathsf{i}_4^{\mathrm{magn}}, \tag{30}$$

where

$$\mathsf{i}_2^{\mathrm{mech}} = -\frac{T}{VR^2}\langle S_1^2\rangle\,, \qquad \mathsf{i}_2^{\mathrm{magn}} = \frac{T}{VR^2}\langle 2S_2\rangle\,,$$

$$\mathsf{i}_4^{\mathrm{mech}} = \frac{T}{6VR^4}\left(\langle S_1^4\rangle - 3\langle S_1^2\rangle^2\right), \qquad \mathsf{i}_4^{\mathrm{mix}} = \frac{T}{VR^4}\left(2\langle S_1^2\rangle\langle S_2\rangle - 2\langle S_2 S_1^2\rangle\right), \qquad \mathsf{i}_4^{\mathrm{magn}} = \frac{T}{VR^4}\left(2\langle S_2^2\rangle - 2\langle S_2\rangle^2\right).$$

Note that $J_I = iJ = -T\langle S_1\rangle = 0$ at $\Omega = 0$ and $T\langle S_2\rangle = \dfrac{\alpha_2}{3}\langle(\boldsymbol{H}^a)^2\rangle$.

Similar decomposition takes place for $\tilde{f}_n$.

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Similar decomposition takes place for $\tilde{f}_n$.

The operators S_i has the following structure: $S_1 = \int d^4x\, r O_1$ and $S_2 = \int d^4x\, r^2 O_2$.
Let's consider $\mathsf{i}_n^{\text{magn}}$,

$$\mathsf{i}_2^{\text{magn}} = \frac{2T}{VR^2}\int d^4x r^2 \langle O_2(\tau,\boldsymbol{r})\rangle = \frac{2}{VR^2}\int d^3r r^2 \langle O_2(0,0)\rangle = 2\alpha_2\langle O_2\rangle = \alpha_2 \tilde{f}_2^{\text{magn}}, \tag{31}$$

$$\mathsf{i}_4^{\text{magn}} = \frac{2T}{VR^4}\int d\tau_1 d\tau_2 d^3r_1 d^3r_2 r_1^2 r_2^2 \times \left(\langle \boldsymbol{T} O_2(\tau_2,\boldsymbol{r}_2) O_2(\tau_1,\boldsymbol{r}_1)\rangle - \langle O_2(0,0)\rangle^2\right) \tag{32}$$

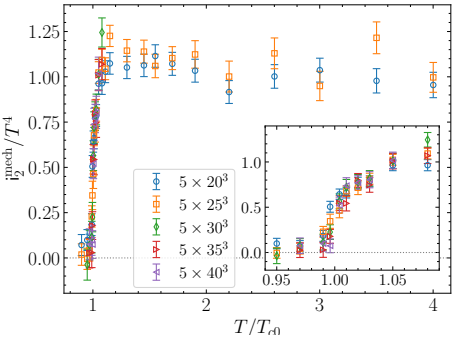
$$= \frac{2}{VR^4}\int d\tau d^3r d^3\xi ((\xi^2 + r^2/4)^2 - (\boldsymbol{r}\boldsymbol{\xi})^2) \times \left(\langle O_1(\tau,\boldsymbol{r}) O_1(0,\boldsymbol{0})\rangle - \langle O_1(0,\boldsymbol{0})\rangle^2\right) \tag{33}$$

$$\approx 2\alpha_4 \int d\tau d^3r \left(\langle O_2(\tau,\boldsymbol{r}) O_2(0,\boldsymbol{0})\rangle - \langle O_2(0,\boldsymbol{0})\rangle^2\right) = \alpha_4 \tilde{f}_4^{\text{magn}}, \tag{34}$$

where $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{\xi} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$, $\tau = \tau_2 - \tau_1$, and approximation $r_\xi \ll R$ is used.

- Local approximation follows from structure of S_i.
- $\alpha_n = \langle r^n\rangle / VR^n \sim 1$.
- Other and high order terms has the same scaling with volume, so $F_n\Omega^n = \alpha_n \tilde{f}_n V(\Omega R)^n$ for large system $R \gg r_\xi$.

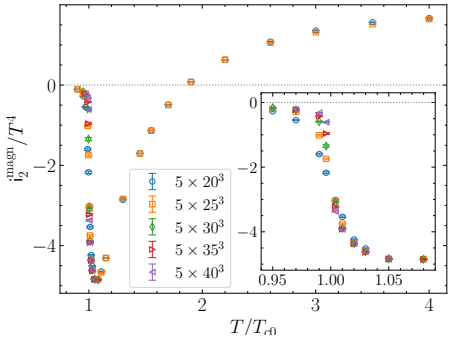
Derivatives at $\Omega = 0$



$$\frac{\mathrm{i}_2^{\mathrm{mech}}}{T^4} = -\frac{N_t^4}{(N_s/2)^2}\Big[(N_\sigma^3 N_t)\langle s_1^2\rangle\Big]^{\mathrm{sub}} \propto \Big[\langle J^2\rangle\Big]^{\mathrm{sub}}\,, \quad (35)$$

Note $\langle J^2\rangle > 0$.

Derivatives at $\Omega = 0$



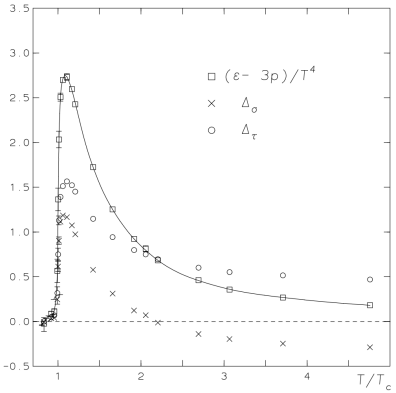
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- Magnetic contribution behaves similar to the magnetic part of thermal gluon condensate $\langle\!\langle G_{\mathrm{magn}}\rangle\!\rangle_T \propto -\Delta_\sigma$.

Derivatives at $\Omega = 0$



[G. Boyd et al., Nucl. Phys. B **469**, 419–444 (1996), arXiv:hep-lat/9602007]

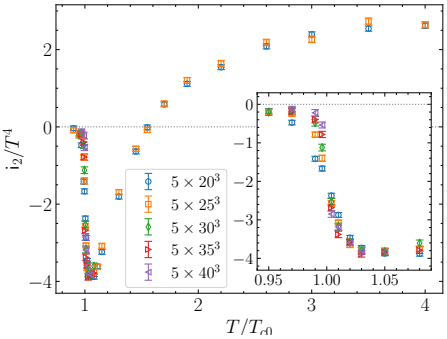
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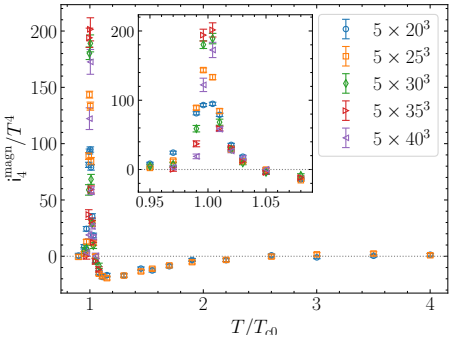
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- Magnetic contribution behaves similar to the magnetic part of thermal gluon condensate $\langle\!\langle G_{\rm magn}\rangle\!\rangle_T \propto -\Delta_\sigma$.
- It is respectful for $\mathsf{i}_2 < 0$.

Derivatives at $\Omega = 0$



$$\frac{i_4^{\rm magn}}{T^4} = \frac{N_t^4}{(N_s/2)^4}\Big[(N_t N_\sigma^3)\left(2\langle s_2^2\rangle - 2\langle s_2\rangle^2\right)\Big]^{\rm sub}, \quad (37)$$

It may be written as

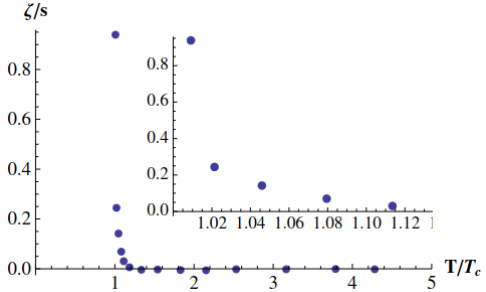
$$\tilde{f}_4^{\rm magn} = -2\beta^2\,\frac{\partial\left\langle\left((H_x^a)^2+(H_z^a)^2\right)\right\rangle}{\partial\tilde{\beta}}\bigg|_{\tilde{\beta}=\beta}. \quad (38)$$

with $S_G = \int d^4x\left[\beta\left((\boldsymbol{E}^a)^2+(H_y^a)^2\right)+\tilde{\beta}\left((H_x^a)^2+(H_z^a)^2\right)\right]$.
It is similar to the expression for the bulk viscosity ζ,

$$\zeta = \frac{1}{9\omega_0}\left[-\left(\frac{\beta(g)}{4g}\right)^2\frac{\partial\langle(\boldsymbol{E}^a)^2+(\boldsymbol{H}^a)^2\rangle}{\partial\beta}+16|\epsilon_v|\right]. \quad (39)$$

- $i_4^{\rm magn}$ can be considered as a magnetic component of the bulk viscosity.

Derivatives at $\Omega = 0$



[D. Kharzeev and K. Tuchin, JHEP **09**, 093 (2008), arXiv:0705.4280 [hep-ph]]

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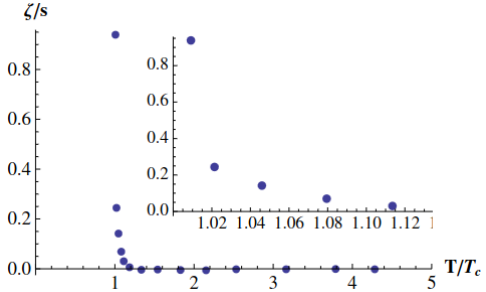
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- $\mathrm{i}_4^{\mathrm{magn}}$ can be considered as a magnetic component of the bulk viscosity.
- Other contributions are quite noisy

Conclusions

- Using lattice simulations with *imaginary* angular velocity, we calculated angular momentum, free energy and angular momentum density for rotating gluodynamics.
- We found that in the bulk of the system results are not affected by boundary conditions.
- From this data, we compute the specific moment of inertia i_2 and deformation coefficient i_4 for OBC/PBC and $\tilde{f}_2$, $\tilde{f}_4$ from approximation of local thermalization.
- We found that $\mathsf{i}_2 < 0$ in temperature range $T_c < T < T_s \approx 1.5T_c$ and $\mathsf{i}_4 < 0$ at $T_c \lesssim T < 1.6 - 1.9T_c$.
  The gluon plasma is easily deformable in the vicinity of the phase transition.
  Negative i_2 may be a manifestation of Negative Barnett Effect.
- At high temperatures, $\mathsf{i}_2 > 0$, $\mathsf{i}_4 > 0$.
- We calculated separate contributions to i_2, i_4 at $\Omega = 0$ and found that $\mathsf{i}_2^{\mathrm{magn}}$, $\mathsf{i}_4^{\mathrm{magn}}$ are respectful for the non-trivial behaviour above T_c, where inhomogeneous mixed phase is expected for *real* rotation.

Thank you for your attention!

QCD on the lattice

Lattice simulation is a powerful method to study strong-interacting systems:

- Based on the *path-integral* representation of the partition function:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[A]\mathcal{D}[\psi,\bar{\psi}]\, e^{-S_E(\psi,\bar{\psi},A)}\, \mathcal{O}(\psi,\bar{\psi},A)\,, \qquad \mathcal{Z} = \int \mathcal{D}[A]\mathcal{D}[\psi,\bar{\psi}]\, e^{-S_E(\psi,\bar{\psi},A)}\,,$$

$$\langle \mathcal{O} \rangle \approx \frac{1}{N_{conf}} \sum_i O(\{U_i\})\,, \qquad \text{with} \quad p(\{U_i\}) \sim e^{-S_E}$$

- The spacetime is discretized on the hypercubic Euclidean lattice with finite spacing a.
- In continuum, the discretized action S_E exactly reproduces S_{QCD}.
- The integrals are computed using Monte-Carlo algorithms. HPC is required.
- The continuum limit $a \to 0$ should be taken.
- Statistical and systematic uncertainties are under control.

Interpretation of the results: negative Barnett effect

Total angular momentum $\boldsymbol{J} = I\boldsymbol{\Omega}$ is a sum of the orbital and spin parts:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}\,, \tag{40}$$

and $I < 0$. The possible physical picture: instability, or *negative* Barnett effect for gluon.

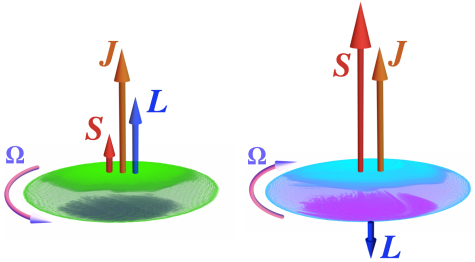
In the temperature range $T_c \lesssim T < T_s \simeq 1.5T_c$:

(i) a sizable fraction of the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is accumulated in the spin of gluons $\boldsymbol{S}$;

(ii) therefore, $\boldsymbol{S} \uparrow\uparrow \boldsymbol{J}$ and $\boldsymbol{S} \uparrow\downarrow \boldsymbol{L}$.

Let's introduce $\boldsymbol{L} = I_L\boldsymbol{\Omega}$, $\boldsymbol{S} = I_S\boldsymbol{\Omega}$, therefore

$$I_L > 0\,, \quad I_S < 0\,, \quad I = I_L + I_S < 0\,.$$



(left) usual Barnett effect
(right) negative Barnett effect

[V. V. Braguta et al., Phys. Rev. D **110**, 014511 (2024), arXiv:2310.16036 [hep-ph]]