# Image Reconstruction Theory: methods of non-integral geometry

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# Reconstruction Problem

**Task:** The image reconstruction is the restoration of the internal structure of a given object without having cuts and/or cracks of this object.

**Application:** Computerized (reconstructive) tomography methods used in the different domains including medicine, astrophysics and seismology.

# Motivations I

Nowadays, the present actual methods and algorithms of CT-reconstruction possess a series of problems associated with

- the computational restrictions (the minimization of X-ray radiation which acts on patients);
- the definite lacks which lead to the appearance of blurred zones together with the "black" zones in the digital images.

The quality of the internal structure image has been directly defined by the efficiency of the mathematical methods used for the reconstruction procedure.

However, the modern modifications relates mainly to the technical methods that are being implemented, generally speaking, on the stage:

$$\text{"digital image} \Longleftrightarrow \text{visualization"}.$$

The main lack is to use the "symmetric" math. for the reconstruction of "non-symmetric" object.

The consequences of the "symmetric" math. are

- the use of Courand-Hilbert's identities traced from Green's formulae leading to the substantially different forms of inversion for the even and odd position space:

$$\boxed{f_R \sim \mathcal{R}^{-1}_{even}, \quad f_R \sim \mathcal{R}^{-1}_{odd} \quad \text{and} \quad \mathcal{R}^{-1}_{even} \neq \mathcal{R}^{-1}_{odd}}$$

- the corresponding singularities are excluded by construction, not are being treated.

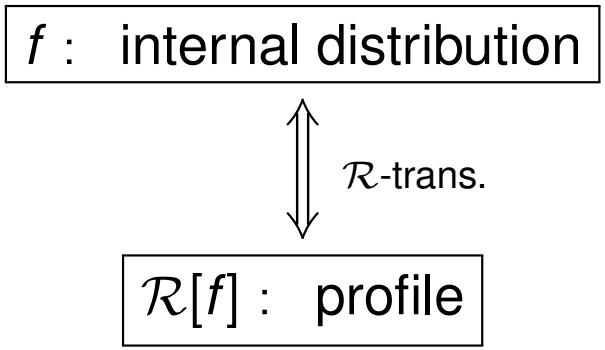
We study the effects of symmetry breaking within “non-symmetric” math. for the reconstruction of “non-symmetric” object.

The consequences of the “non-symmetric” math. are

- excluding Courant-Hilbert’s identities (and Green’s formulae). It leads to the universal forms of inversion which are valid for both the even and odd position space:

$$\boxed{f_R \sim (i)^{n-2}\ \mathcal{R}_S^{-1} + (i)^{n-1}\ \mathcal{R}_A^{-1}}$$

- the corresponding singularities are being treated by consideration.




DRTs are given by ($\vec{\mathbf{n}}_{\varphi,\theta_i}$ is the $n$-dimensional unit vector pointing along the radial Radon coordinate)

$$G \equiv \mathcal{R}[f](\tau,\varphi,\theta_i) \quad \text{with} \quad \mathcal{R}[f](\tau,\varphi,\theta_i) = \int_{\mathbb{R}^n} d\mu_n(\vec{\mathbf{x}})\, f(\vec{\mathbf{x}})\, \delta\big(\tau - \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle\big).$$

While IRTs are defined as

$$f = \mathcal{R}^{-1} G \quad \text{or} \quad f(\vec{\mathbf{x}}) = \int_{\Omega} d\mu_n(\eta,\varphi,\theta_i)\, \mathcal{R}[f]\big(\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i\big),$$

where $d\mu_n$ depends on the radial $\eta$ and the angular $\varphi, \theta_i$ Radon coordinates.

# Inverse Radon transforms: notice I

Notice that IRTs are more difficult math task compared to DRTs.

Indeed, in order to fully-reconstruct the internal distribution:

- we have to implement the infinite number of $\mathcal{R}$-transforms,
- it is necessary to invert $\mathcal{R}$-transforms using the regularizations.

"Infinite" is impossible from the practical point of view, then "infinite number" $\Rightarrow$ "reasonable number".

# Integral Geometry: standard methods of $\mathcal{R}$-inversion

The standard inversion methods inspired by Integral Geometry underlie the mathematical tools for CT-medicine, astrophysics and seismology.

The main items of the standard inversion of $\mathcal{R}$-inversion:

- it is based on the methods within Integral Geometry. Integral Geometry as a theory studies the symmetry properties of the corresponding measures on the geometrical spaces;
- using Courand-Hilbert's identities (based on Green's formulae), the inverse operators have different representations depending on the space dimension, even or odd dimension.

To invert $\mathcal{R}$-transform in both even and/or odd dimension of space, using the standard method, we have to integrate with the symmetric angular measure:

$$\mathcal{R}^{-1} \sim \int_{|\xi|=1} d\xi(\varphi, \theta_i)\{....\},$$

where $\xi(\varphi, \theta_i)$ is defined by the corresponding angles $\varphi$ and $\theta_i$.

Focusing on $\mathbb{R}^2$, the symmetric integration measure leads to the full region of integration:

$$\boxed{\varphi \in [0,\, 2\pi] \quad \text{within integral geometry}}$$

A new direction in Math, called *Non-integral Geometry*, is devoted to the study of the non-invariant (non-symmetric) integral measures. It investigates the different effects breaking the symmetry of integration measures.

In particular, dwelling on $\mathbb{R}^2$, the angular measure of integration for $\mathcal{R}^{-1}$ receives the angular constraints, for example, as

$$\varphi \in \left\{\Omega_{res.} : -\pi/2 < \varphi < \pi/2\right\} \quad \text{within non-integral geometry}$$

# Non-integral Geometry: universal form of $\mathcal{R}$-inversion

The important feature of the use of non-integral geometry is the universal form of $\mathcal{R}$-inversion.

In contrast to the standard methods of integral geometry, the proposed methods in non-integral geometry are independent on the parity of space. That is, it is valid for both the even and odd dimension of space.

- In non-integral geometry, the universality of $\mathcal{R}$-inversion can be traced from the Fourier slice theorem and, as a consequence, from the necessity of regularization applied to the inversion $\mathcal{R}$- operator

As a result, we deal with the complexity of $\mathcal{R}$-inversion and with the additional, new term $f_A$ contributing the inverse $\mathcal{R}$-transforms.

The Standard Methods of Integral Geometry:

- The basis is formed by Courand-Hilbert's identities traced from Green's formulae. It gives the different repres. of inversion for the spaces with even and odd dimensions;
- We deal with the invariant (symmetric) integration measure that leads to the full integration region: $\varphi \in (0, 2\pi)$.

The Proposed Methods of Non-Integral Geometry (non-sym. int. geometry):

- Do Not use Courand-Hilbert's identities (Green's formulae). Instead, we use the Fourier slice theorem that demands to the regularization of inversion. As a result, we derive the universal repres. of inversion for the spaces independently on even and odd dimensions;
- We deal with the non-invariant (non-symmetric) integration measure that leads to the restricted integration region, for example, $\varphi \in (-\pi/2, \pi/2)$.

After the regularization procedure, we obtain that

$$f(\vec{\mathbf{x}}) \quad \Longrightarrow \quad f_\epsilon(\vec{\mathbf{x}}) = \mathcal{R}_\epsilon^{-1} G \quad \text{and} \quad f_\epsilon(\vec{\mathbf{x}}) = f_S(\vec{\mathbf{x}}) + f_A(\vec{\mathbf{x}}),$$

where

$$f_S(\vec{\mathbf{x}}) = i^{n-2}(-)^{n-1}(n-1)! \int_\Omega d^{n-1}\Theta(\varphi, \theta_i) \times \int_{-\infty}^{+\infty} (d\eta)\, \frac{\mathcal{P}}{\eta^n}\, \mathcal{R}[f](\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i)$$

and

$$f_A(\vec{\mathbf{x}}) = (-)^{n-1} i^{n-1}\, \pi \int_\Omega d^{n-1}\Theta(\varphi, \theta_i) \times \int_{-\infty}^{+\infty} (d\eta)\, \delta(\eta)\, \frac{\partial^{n-1}}{\partial \eta^{n-1}}\, \mathcal{R}[f](\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i).$$

P.S. $\Omega$ defines the restricted region of integration with respect to the angular Radon variables.

As dictated by CT-technology, the different algorithms have been designed for the two-dimensional reconstructions which work with the corresponding transverse projections of the three-dimensional object.

If the object is described by $f(x_1, x_2, x_3)$, we perform the finite number of transverse sections regarding $x_3$-axis:

$$f(x_1, x_2, x_3) \Longrightarrow \left\{ f(x_1, x_2, x_{3_1}),\ f(x_1, x_2, x_{3_2}),\ ....,\ f(x_1, x_2, x_{3_n}) \right\},$$

where the number of sections should be defined by the experiment.

This is *a discrete slice method* of CT-technology. In this case, we always deal with the direct and inverse Radon transforms determined on the two-dimensional space formed by $(x_1, x_2)$ while the third discrete coordinate plays a role of the external parameter.

We introduce the hybrid (Wigner-like) function associated with the outset function:

$$F(x_1, x_2; k) = \int_{-\infty}^{+\infty} (dx_3)\, e^{-ikx_3}\, f(x_1, x_2, x_3), \quad F(x_1, x_2; k) \in \mathbb{C}.$$

Hence, in the discrete slice method, we have the following

$$\widetilde{F}(x_1, x_2; k) = \sum_n e^{-ikx_{3_n}}\, f(x_1, x_2, x_{3_n}), \quad \widetilde{F}(x_1, x_2; k) \in \mathbb{C},$$

which is nothing but the Fourier series expansion.

For the sake of simplicity, we begin with the outset function $f(\vec{\mathbf{x}})$, where $\vec{\mathbf{x}} \in \mathbb{R}^2$, defined as

$$f(\vec{\mathbf{x}}) = m_I \, \Theta\big(\vec{\mathbf{x}} \in \Omega_I\big) + m_{IV} \, \Theta\big(\vec{\mathbf{x}} \in \Omega_{IV}\big).$$

Here, $\Theta$ are associated with the supports given by the following regions:

$$\{\Omega_I : x_1 > 0, x_2 > 0, |\vec{\mathbf{x}}| \leq 1\} \quad \text{and}$$
$$\{\Omega_{IV} : x_1 > 0, x_2 < 0, |\vec{\mathbf{x}}| \leq 1\}.$$

By the direct calculation, we obtain that DRT is given by

$$\boxed{\mathcal{R}[f](\tau,\varphi) = M\sqrt{1-\tau^2}\,\Theta_1(\tau,\,\varphi) + M\tau\,\cot\varphi\,\Theta_2(\tau,\,\varphi) - M\tau\,\tan\varphi\,\Theta_3(\tau,\,\varphi),}$$

where

$$\Theta_1(\tau,\,\varphi) = \Theta(\tau\in[\sin\varphi,\,1]) + \Theta(\tau\in[0,\,1]) + \Theta(\tau\in[\cos\varphi,\,1]),$$
$$\Theta_2(\tau,\,\varphi) = \Theta(\tau\in[0,\,\sin\varphi]),\quad \Theta_3(\tau,\,\varphi) = \Theta(\tau\in[\cos\varphi,\,1]),$$

and $M = m_I + m_{IV}$.

Notice that $\mathcal{R}[f](\tau,\varphi)$ contains the nontrivial $\varphi$-angular dependence. This is a result of the non-symmetric outset function.

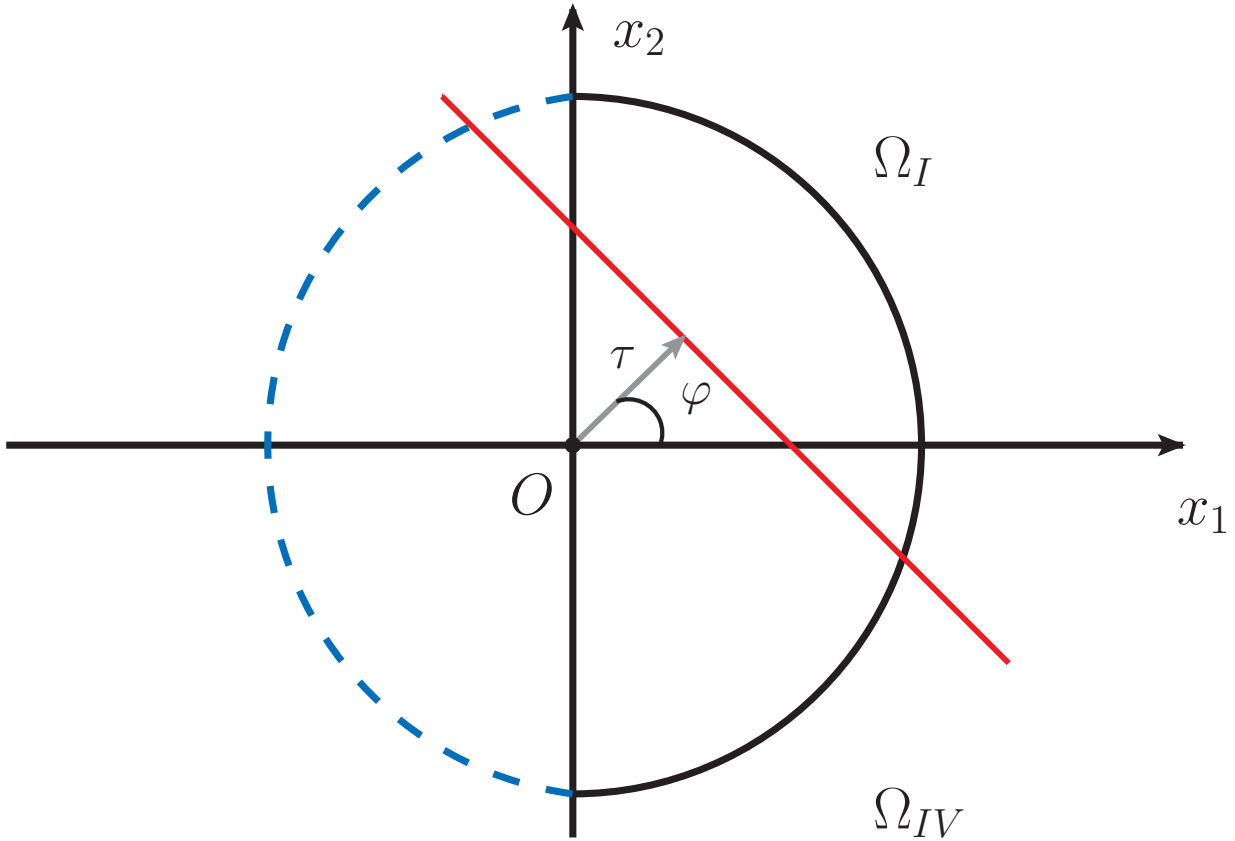



Figure: The restricted region in $\vec{\mathbf{x}}$-plane: $\Omega_I$ and $\Omega_{IV}$. Notations: $\tau$ is the Radon radial variable, $\varphi$ is the Radon angular variable. The line is parametrized as $\tau - \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle = 0$ with $\vec{\mathbf{n}}_\varphi = (\cos\varphi, \sin\varphi)$. For the sake of convenience, the starting point $O$ has been chosen to keep the maximal possible symmetry.

Using DRT, we get the regularized representations:

$$f_S(\vec{\mathbf{x}}) = f_S^{(\tau)}(\vec{\mathbf{x}}) + f_S^{(\varphi)}(\vec{\mathbf{x}}),$$
$$f_A(\vec{\mathbf{x}}) = f_A^{(\tau)}(\vec{\mathbf{x}}) + f_A^{(\varphi)}(\vec{\mathbf{x}})$$

where

$$f_S^{(\tau)}(\vec{\mathbf{x}}) = -M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta) \frac{\mathcal{P}}{\eta^2} \sqrt{1 - \left(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle\right)^2}\, \Theta_1\left(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\right),$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = -M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta) \frac{\mathcal{P}}{\eta^2} \left(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle\right)$$
$$\times \left\{ \cot\varphi\, \Theta_2\left(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\right) - \tan\varphi\, \Theta_3\left(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\right) \right\}$$

and

$$f_A^{(\tau)}(\vec{\mathbf{x}}) = (-i\pi)M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta)\delta(\eta) \frac{\partial}{\partial \eta} \sqrt{1 - \big(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle\big)^2}$$
$$\times \Theta_1\big(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\big),$$

together with

$$f_A^{(\varphi)}(\vec{\mathbf{x}}) = (-i\pi)M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta)\delta(\eta) \frac{\partial}{\partial \eta} \Big(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle\Big)$$
$$\times \Big\{ \cot\varphi\, \Theta_2\big(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\big) - \tan\varphi\, \Theta_3\big(\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle,\, \varphi\big) \Big\}.$$

Despite $\varphi$-angle varies within the interval $(0,\, \pi/2)$, the expressions involve both $\Omega_I$ and $\Omega_{IV}$ owing to the implemented replacements of variables.

The standard contributions with the quadratic $\tau$-dependence of DRT are

$$f_S^{(\tau)}(\vec{\mathbf{x}}) = -M\int_0^{\pi/2}(d\varphi)\Big\{\int_{-A_1(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2}$$
$$+\int_{A_2(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2}+\int_{A_2^\star(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2}\Big\}$$

where the shortened notations have been introduced as (here and below, $\vec{\mathbf{x}}$-dependence is hidden in all $A$-functions)

$$A_1(\varphi)\equiv\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle = x_2\,\sin\varphi + x_1\,\cos\varphi,$$
$$A_2(\varphi) = (1-x_2)\,\sin\varphi - x_1\,\cos\varphi,$$
$$A_2^\star(\varphi) = (1-x_1)\,\cos\varphi - x_2\,\sin\varphi,\quad A_4(\varphi) = 1-A_1(\varphi).$$

The standard contribution with the linear $\tau$-dependence:

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}$$

where

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} = -M\int_0^{\pi/2}(d\varphi)\,\cot\varphi\int_{-A_1(\varphi)}^{A_2(\varphi)}(d\eta)\frac{\eta + A_1(\varphi)}{\eta^2}$$

and

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = M\int_0^{\pi/2}(d\varphi)\,\tan\varphi\int_{A_2^\star(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\eta + A_1(\varphi)}{\eta^2}.$$

For the sake of convenience, we are splitting the contribution $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}$ into two terms where one of them resembles the contributions $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}$. We have

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2}$$

with

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1} = M \int_0^{\pi/2} (d\varphi)\, \tan\varphi \int_{-A_1(\varphi)}^{A_4(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2} = -M \int_0^{\pi/2} (d\varphi)\, \tan\varphi \int_{-A_1(\varphi)}^{A_2^{\star}(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2}.$$

After $\eta$-integration, we get the following expressions:

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} = -M\int_0^{\pi/2}(d\varphi)\,\cot\varphi$$
$$\times\Big[\log\frac{x_1\,\cos\varphi-(1-x_2)\,\sin\varphi}{x_1\,\cos\varphi+x_2\,\sin\varphi}+\frac{x_1\,\cos\varphi+x_2\,\sin\varphi}{x_1\,\cos\varphi-(1-x_2)\,\sin\varphi}-1\Big],$$

and

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1} = M\int_0^{\pi/2}(d\varphi)\,\tan\varphi$$
$$\times\Big[\log\frac{x_1\,\cos\varphi+x_2\,\sin\varphi-1}{x_1\,\cos\varphi+x_2\,\sin\varphi}+\frac{x_1\,\cos\varphi+x_2\,\sin\varphi}{x_1\,\cos\varphi+x_2\,\sin\varphi-1}-1\Big].$$

The $\eta$-integrated contribution of $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2}$ can be derived from $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}$. We have

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2} = -M\int_0^{\pi/2}(d\varphi)\,\tan\varphi$$
$$\times\Big[\log\frac{x_2\,\sin\varphi-(1-x_1)\,\cos\varphi}{x_1\,\cos\varphi+x_2\,\sin\varphi}+\frac{x_1\,\cos\varphi+x_2\,\sin\varphi}{x_2\,\sin\varphi-(1-x_1)\,\cos\varphi}-1\Big].$$

For our analysis, we have

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}^{(a)} = -M\int_0^{\pi/2}(d\varphi)\,\cot\varphi\,\log\frac{x_1\,\cos\varphi-(1-x_2)\,\sin\varphi}{x_1\,\cos\varphi+x_2\,\sin\varphi},$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2,(a)} = -M\int_0^{\pi/2}(d\varphi)\,\tan\varphi\,\log\frac{x_2\,\sin\varphi-(1-x_1)\,\cos\varphi}{x_1\,\cos\varphi+x_2\,\sin\varphi}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}^{(b)} = -M\int_0^{\pi/2}(d\varphi)\,\cot\varphi\Big[\frac{x_1\,\cos\varphi+x_2\,\sin\varphi}{x_1\,\cos\varphi-(1-x_2)\,\sin\varphi}-1\Big],$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2,\,(b)} = -M\int_0^{\pi/2}(d\varphi)\,\tan\varphi\Big[\frac{x_1\,\cos\varphi+x_2\,\sin\varphi}{x_2\,\sin\varphi-(1-x_1)\,\cos\varphi}-1\Big].$$

To restore the origin outset function, as the first stage, we propose to minimize the norm as

$$\min_{\epsilon} \left\{ \| f(\vec{\mathbf{x}}) - f_{\epsilon}(\vec{\mathbf{x}}) \|^2 \right\}.$$

provided the function $f(\vec{\mathbf{x}})$ is known. Then, the optimization procedure can be applied to

$$\min_{\epsilon,\Lambda} \left\{ \| \mathcal{R}[f_{\epsilon}] - \mathcal{R}[f] \|^2 + \Lambda \| f(\vec{\mathbf{x}}) - f_{\epsilon}(\vec{\mathbf{x}}) \|^2 \right\}.$$

Here, we deal with the set of regularizing parameters: $\epsilon = \left\{ \varepsilon, \lambda, ... \right\}$.

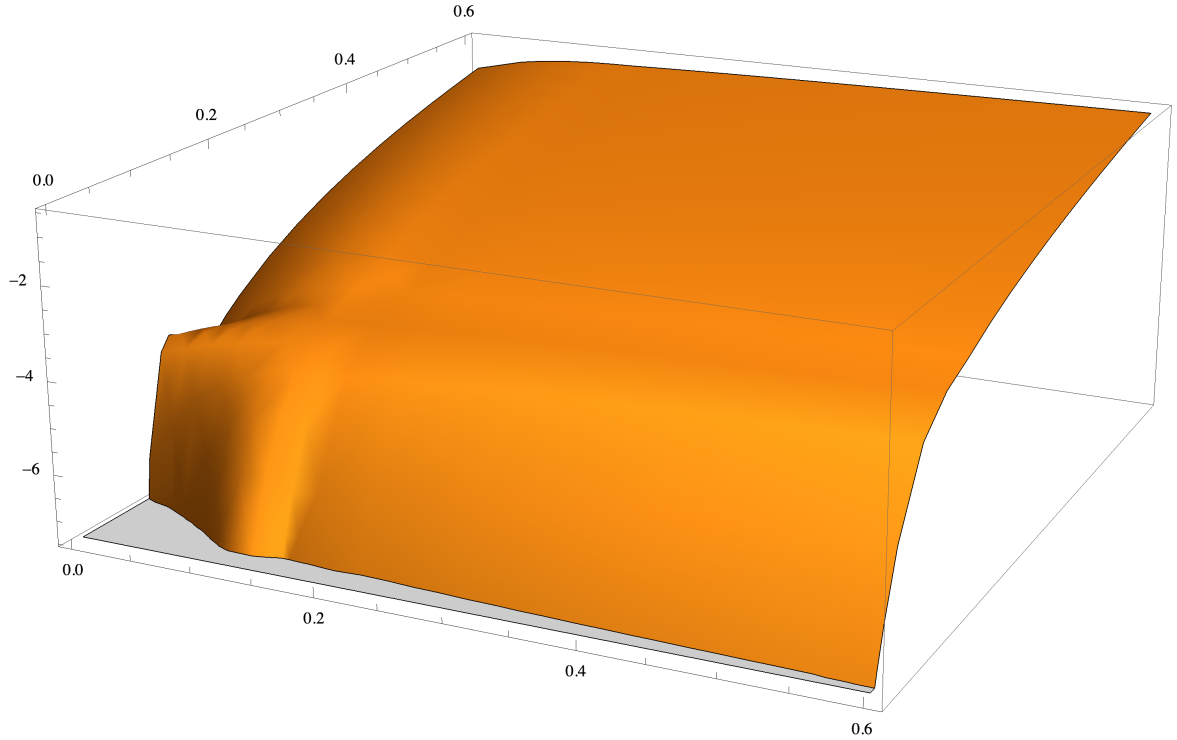



Figure: The restoration of the flat origin outset function $f(\vec{\mathbf{x}})$ defined on the support $\Omega_I$ and $\Omega_{IV}$.

# Summary

- We have presented the basis of non-integral geometry which can be treated as a new field of functional analysis.
- We have studied the effects of symmetry breaking related to the integration measure. As demonstrated, the non-invariant (non-symmetric) measure results directly in the complexity of the universal inversion with the additional contributions.
- We have advocated a unique role of the additional term $f_A$. Namely, the term $f_A$ is very important to regularize the universal IRT of the outset function located in some domain. Despite we have considered the simplest form of the outset function, the discovered role of the additional term as the regularizing contribution is the very general property even for the practical outset function.
- We have emphasized that the mentioned regularization is a very important for the image reconstruction procedure