Mikhail Klimenko
Sector of a New Source and Complex of Moderators,
Group No. 1 New Neutron Source

- I. M. Frank Scholarship for the Development of Basic Facilities
- AYSS Grant Competition

Scientific advisor: **Dr. Evgeny Perepelkin, FLNP JINR**

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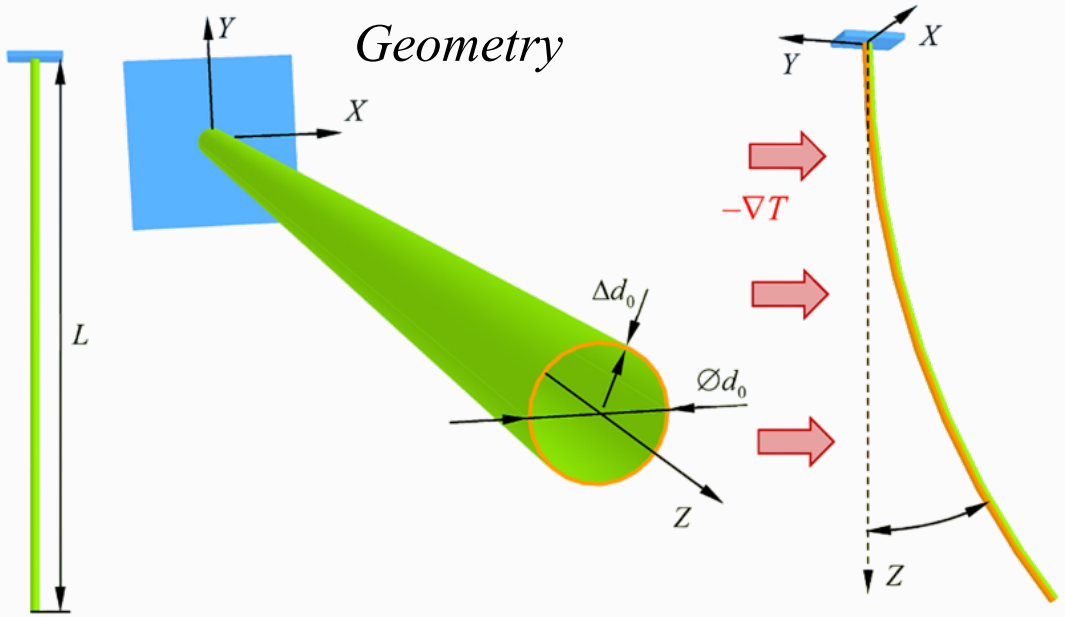
I. M. Frank Scholarship

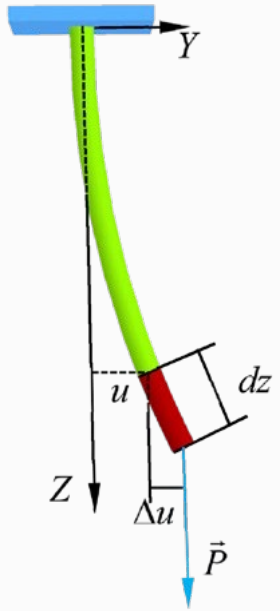
1. Oscillations of fuel rods under thermal load.
2. Eigenfrequencies of a four-component fuel rod.
3. System of oscillating fuel rods.

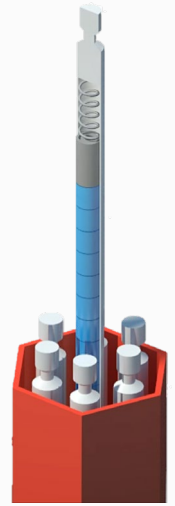
AYSS Grant

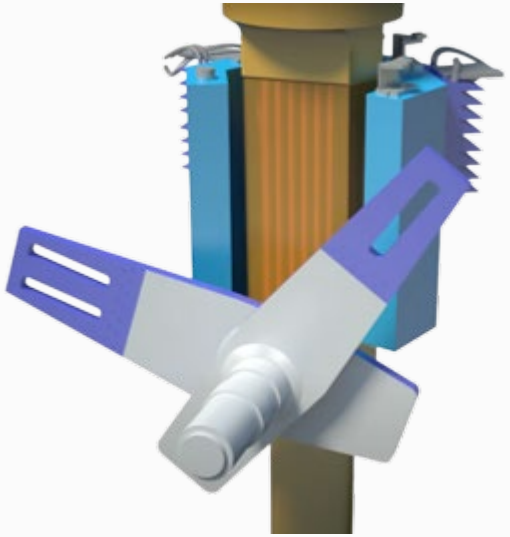
4. Thermal-hydraulic analysis of the NEPTUN reactor modulator cooling.

1. Oscillations of fuel rods under thermal load







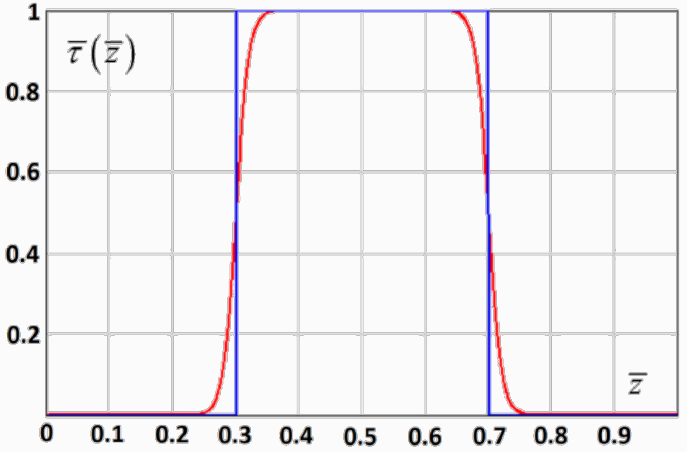


$$\begin{cases} a_1\dfrac{\partial^2 u}{\partial t^2}+a_2\dfrac{\partial u}{\partial t}+\dfrac{\partial^4 u}{\partial z^4}+b\dfrac{\partial}{\partial z}\left(P_z\dfrac{\partial u}{\partial z}\right)=\bar{f}(z,t), \\ u(z,0)=u_0(z),\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=u_1(z),\ u(0,t)=0, \\ \dfrac{\partial^2 u}{\partial z^2}\Big|_{z=L}=\dfrac{\partial^3 u}{\partial z^3}\Big|_{z=L}=0,\ \dfrac{\partial u}{\partial t}\Big|_{z=0}=\dfrac{\partial u}{\partial z}\Big|_{z=0}=0, \end{cases}$$

$$P_z(z)=M_0 g\left(1-\frac{z}{L}\right),\ a_1=\frac{\rho S}{EJ_x},\ a_2=\frac{d_0\eta}{EJ_x},\ b=\frac{1}{EJ_x}.$$

$$\bar{f}(z,t)=-\frac{\alpha_T \iota}{d_0}\beta(t)\frac{\partial^2\tau(z)}{\partial z^2}$$

IBR – 2 and the fuel rod assembly



Results

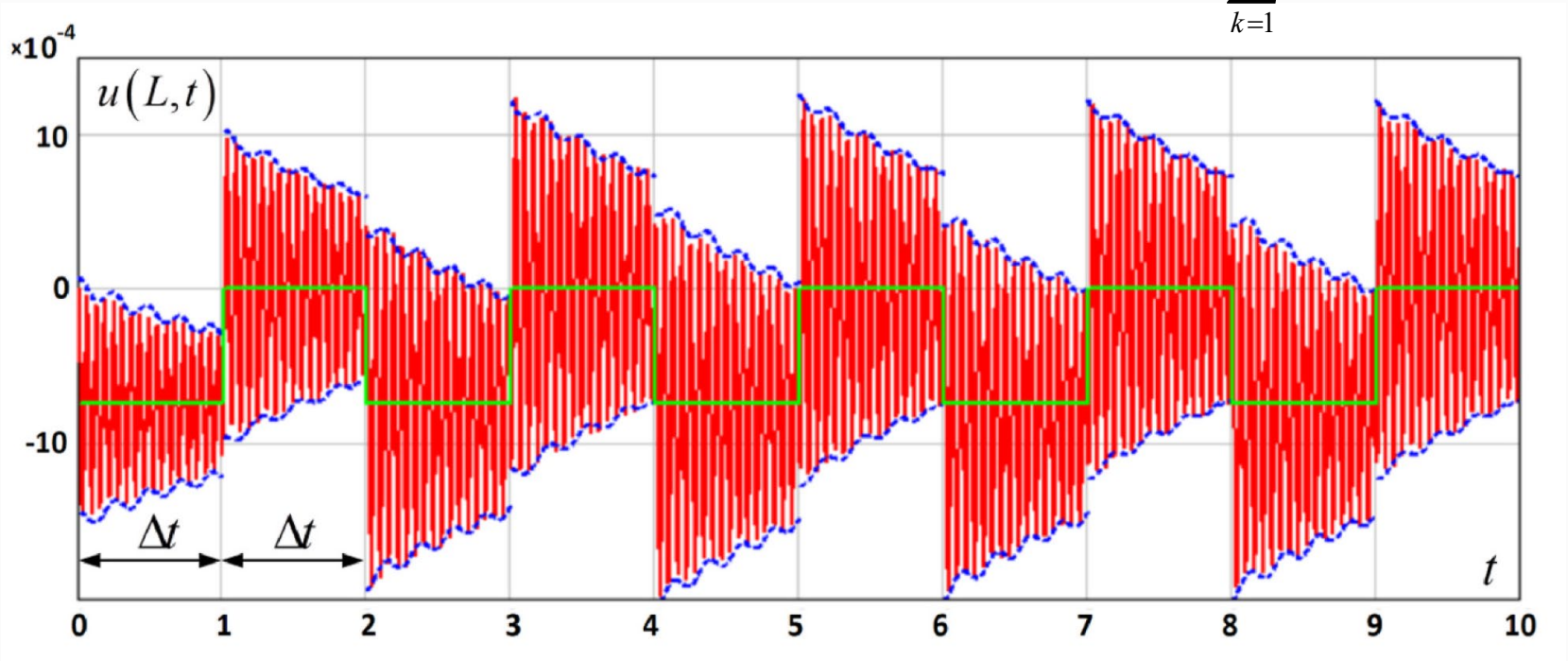
$$u_{p.i.}(z,t)=\sum_{n=1}^{\infty}\bar{w}_n^{p.i.}(t)W_n(\bar{z}),\quad \bar{w}_n(t)=\frac{1}{a_1\beta_n}\bar{I}_n(t)+e^{-\gamma t}\left[\bar{w}_n^{(0)}\cos(\beta_n t)+\frac{1}{\beta_n}(\bar{w}_n^{(1)}+\gamma\bar{w}_n^{(0)})\sin(\beta_n t)\right],$$

$$\bar{I}_n(t)=\int_0^t \bar{f}_n(t)e^{-\gamma(t-\bar{t})}\sin\left[\beta_n(t-\bar{t})\right]d\bar{t},\ \bar{f}_n(t)=-\frac{\alpha_T\imath\tau_n''}{d_0L^2}\Theta(t,v),\ \gamma=\frac{a_2}{2a_1},\beta_n^2+\gamma^2=\frac{\lambda_n}{a_1L^4}.$$

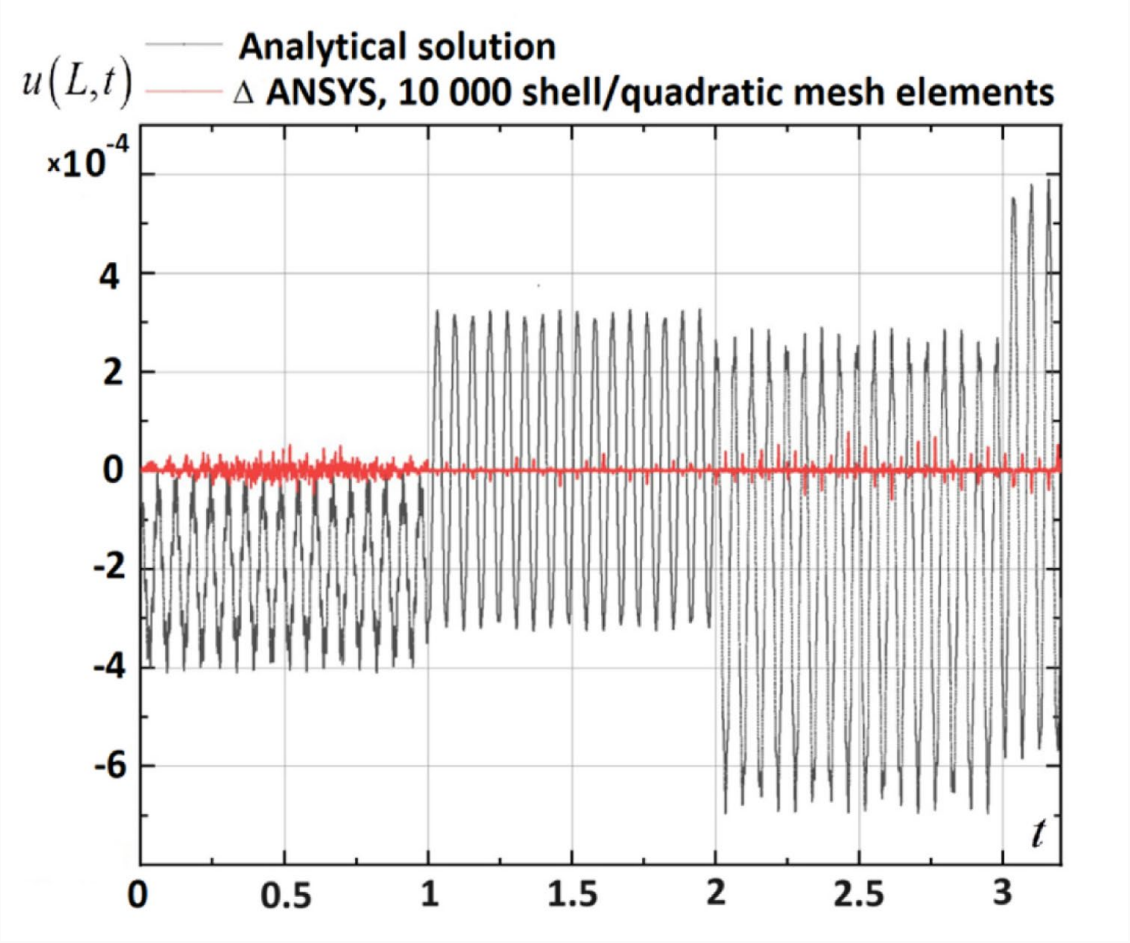
Function of periodic heat pulses

$$\Theta(t,v)=\sum_{k=0}^{N}\left[\theta(k(v+1)\Delta t-t+\Delta t)+\theta(t-k(v+1)\Delta t)\right]-N-1,$$

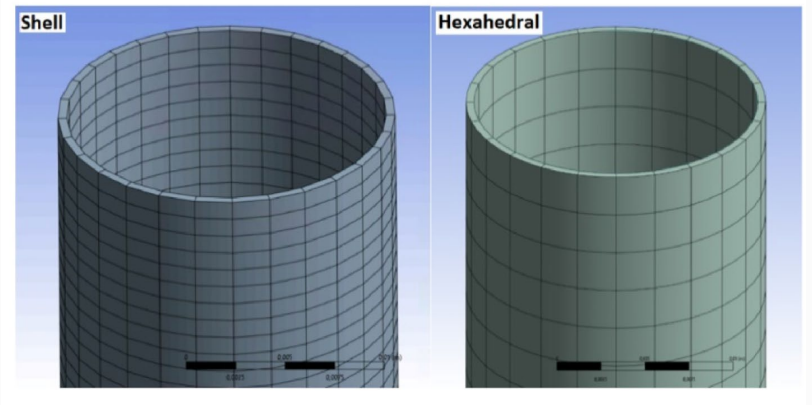
Oscillations of the rod's end point $u(L,t)=\sum_{k=1}^{K}W_k(1)\bar{w}_k(t).$



Results



Comparison with numerical computation

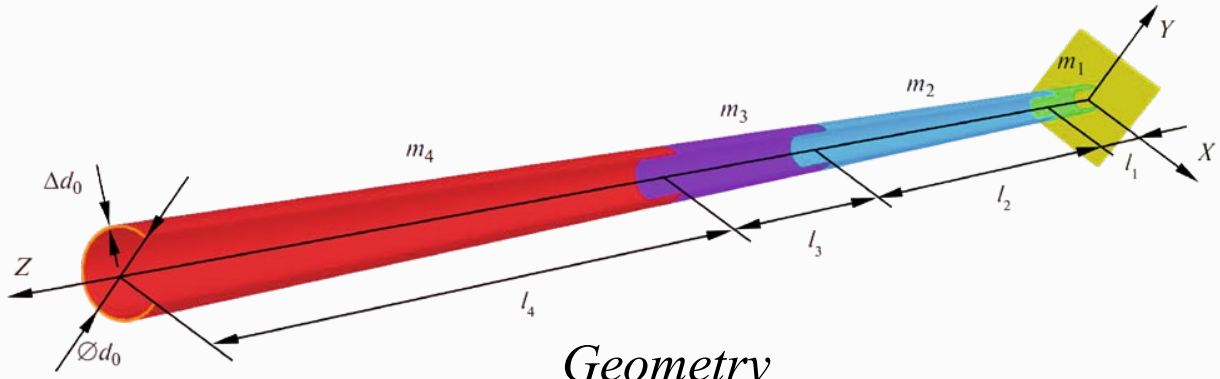


Element type	*shell*	*hexahedral*
Number of elements	10000	30000
Number of steps	8171	8133
Computation time	2*h* 20*m*	7*h* 15*m*

Results were published in a Q1 journal

1. E.E. Perepelkin, B.I. Sadovnikov, N.G. Inozemtseva, M.V. Klimenko, Investigation of the dynamics of transverse oscillations of a vertical rod under gravity, friction, and thermal expansion, Progress in Nuclear Energy, 2024

2. Four-component fuel rod



Geometry

$$u(z,t)=u^j(z,t), z\in\Gamma_j, \Gamma_j=\left\{z: z_{j-1}\le z\le z_j\right\},$$

$$u^j(z,t)\sim Z^j(z)T^j(t),\ \overline{z}=\frac{z}{L},$$

$$a_j u_{tt}^j+u_{zzzz}^j=0,\ a_j=\frac{\rho_j S_j}{E_j J_j},\ j=1...4,$$

$$\begin{cases} T_{tt}^j+\omega^2 T^j=0, \\ \overline{Z}_{zzzz}^j-\lambda_j\overline{Z}^j=0, \end{cases}$$

$$u(z,0)=u_0(z),\ u_t(z,0)=u_1(z),$$

$$\lambda_j=\alpha_j^4=\overline{\omega}^2 a_j L^4.$$

Boundary conditions

$$u^i(z_i,t)=u^{i+1}(z_i,t),\ i=\overline{1,3},$$

$$E_j\frac{\partial^2 u(z_j,t)}{\partial z^2}=E_{j+1}\frac{\partial^2 u(z_j,t)}{\partial z^2},\ j=\overline{1,3},$$

$$u_t^i(z_i,t)=u_t^{i+1}(z_i,t),\ i=\overline{1,3},$$

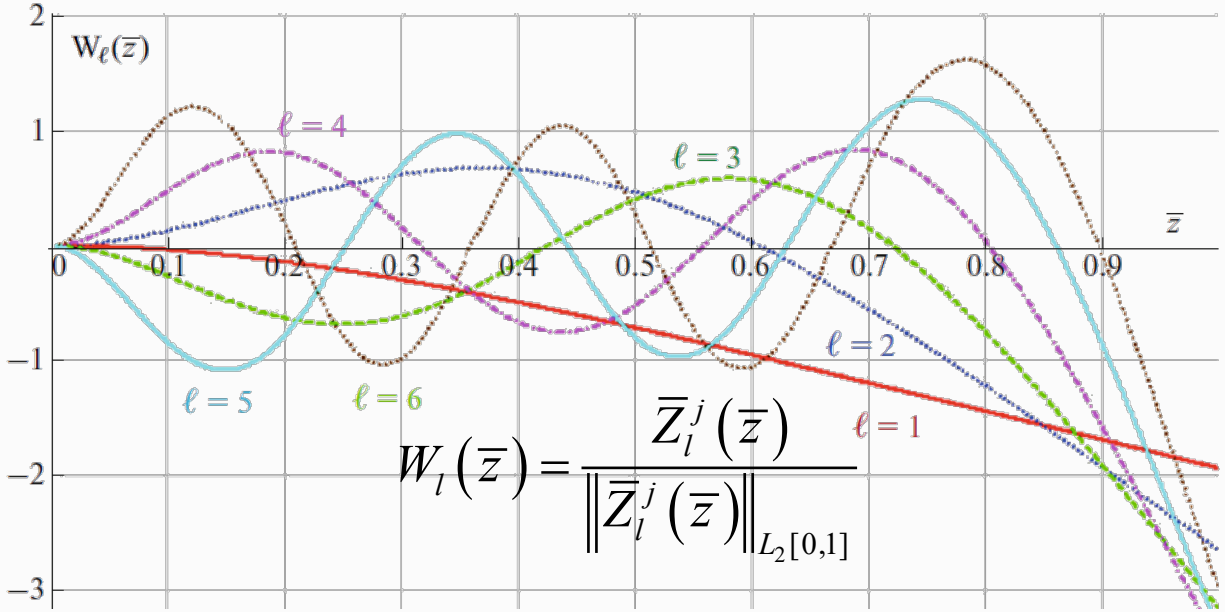
$$E_j\frac{\partial^3 u(z_j,t)}{\partial z^3}=E_{j+1}\frac{\partial^3 u(z_j,t)}{\partial z^3},$$

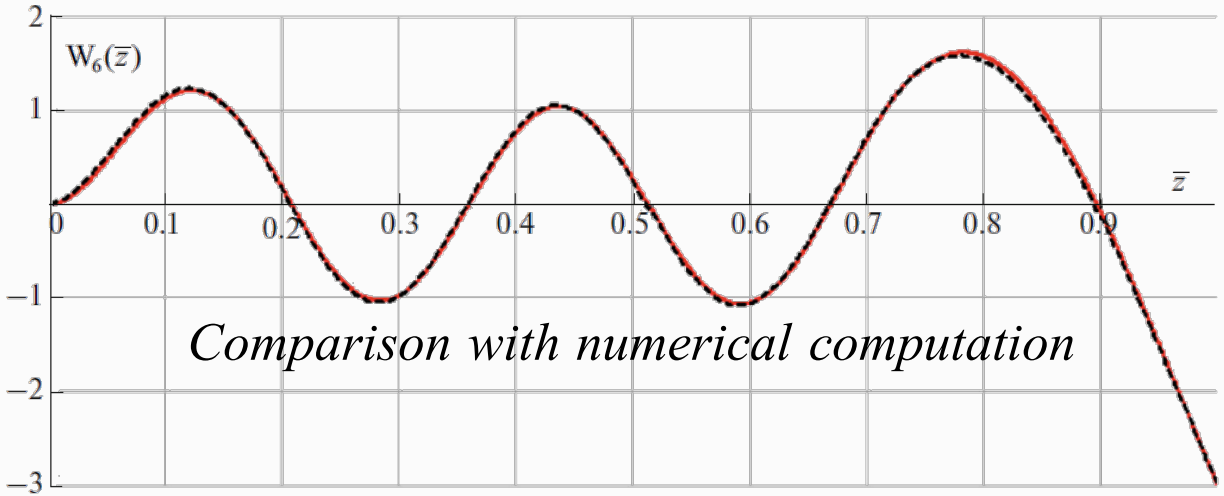
$$u_z^i(z_i,t)=u_z^{i+1}(z_i,t),$$

$$u_{zz}^4(L,t)=u_{zzz}^4(L,t)=0,$$

$$u^1(0,t)=u_z^1(0,t)=0,$$

Results





Mode	ϑ_l, Hz
1	7.971
2	43.184
3	108.809
4	201.141
5	332.333
6	514.099
7	733.499
8	964.722
9	1209.222

Results were published in the paper:

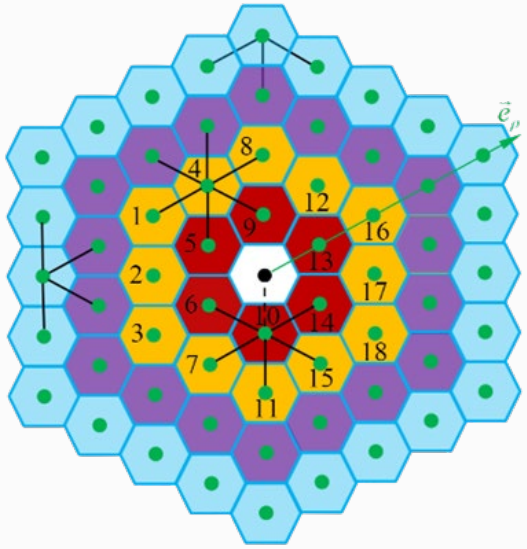
2 E.E. Perepelkin, N.G. Inozemtseva, M.V. Klimenko et al. Numerical-Analytical Method for Finding the Natural Frequencies and Eigenfunctions of a Four-Component Rod System. Phys. Part. Nuclei 56, 88–103 (2025)

3. System of oscillating fuel rods

Generalised coordinates $\vec{r}_n = \vec{r}_n\left(q_1,\ldots,q_{2N}\right),\ n=1\ldots N,$

$$\left(q_1,\ldots,q_{2N}\right)=\left(x_1,\ldots,x_N,y_{N+1},\ldots,y_{2N}\right)\mapsto\left(r_1,\ldots,r_N,\varphi_{N+1},\ldots,\varphi_{2N}\right),$$

$$\left(\dot{q}_1,\ldots,\dot{q}_{2N}\right)=\left(\dot{x}_1,\ldots,\dot{x}_N,\dot{y}_{N+1},\ldots,\dot{y}_{2N}\right)\mapsto\left(\dot{r}_1,\ldots,\dot{r}_N,\dot{\varphi}_{N+1},\ldots,\dot{\varphi}_{2N}\right),$$



Hamiltonian

$$H\left(q,p\right)=\sum_{n=1}^{N}\frac{1}{2m_n}\left(\left[p_n^{(r)}\right]^2+\frac{1}{r_n^2}\left[p_n^{(\varphi)}\right]^2\right)+\sum_{n=1}^{N}\sum_{k\in\Omega_n}\Phi\left(\sqrt{r_n^2+r_k^2-2r_nr_k\cos\left(\varphi_n-\varphi_k\right)}\right),$$

where $p_n^{(r)}=m_n\sum_{j=1}^{N}g_{n,j}\dot{q}_j,\ p_n^{(\varphi)}=m_n\sum_{j=N+1}^{2N}g_{n+N,j}\dot{q}_j,\ p_n^{(r)}=m_n\dot{r}_n,\ p_n^{(\varphi)}=m_nr_n^2\dot{\varphi}_n,$

$$g_{sk}\mapsto\begin{pmatrix}\mathrm{I}_N & 0\\ 0 & G_N\end{pmatrix},\ G_N=\mathrm{diag}\left(r_1^2,\ldots,r_N^2\right),\ \mathrm{I}_N-\textit{identity matrix.}$$

Canonical variables $\vec{z}=\left\{q_1,\ldots,q_{2N},p_1,\ldots,p_{2N}\right\}^T,$

$$\dot{\vec{z}}=J\partial_z H,\ \textit{where}\ J=\begin{pmatrix}0 & \mathrm{I}_{2N}\\ -\mathrm{I}_{2N} & 0\end{pmatrix},\ \partial_z=\left\{\partial_{z_1},\ldots,\partial_{z_{4N}}\right\}^T=\left\{\partial_{q_1},\ldots,\partial_{q_{2N}},\partial_{p_1},\ldots,\partial_{p_{2N}}\right\}^T.$$

Dissipative system

Viscous friction: $$\boxed{L(\vec{r},\vec{v},t)=\left[\frac{mv^2}{2}-U(\vec{r})\right]e^{2\gamma t}} \Rightarrow \quad H(\vec{r},\vec{\mathrm{p}},t)=\frac{\mathrm{p}^2}{2m}e^{-2\gamma t}+U(\vec{r})e^{2\gamma t},$$

where $\vec{\mathrm{p}}\overset{\text{def}}{=}\nabla_v L=m\vec{v}e^{2\gamma t}=\vec{p}e^{2\gamma t}.$ $\vec{\eta}\overset{\text{def}}{=}\left\{u_1^{(r)},...,u_N^{(r)},u_1^{(\varphi)},...,u_N^{(\varphi)},\mathrm{p}_1^{(r)},...,\mathrm{p}_N^{(r)},\tilde{\mathrm{p}}_1^{(\varphi)},...,\tilde{\mathrm{p}}_N^{(\varphi)}\right\}^T.$

Approximation of the system's Hamiltonian

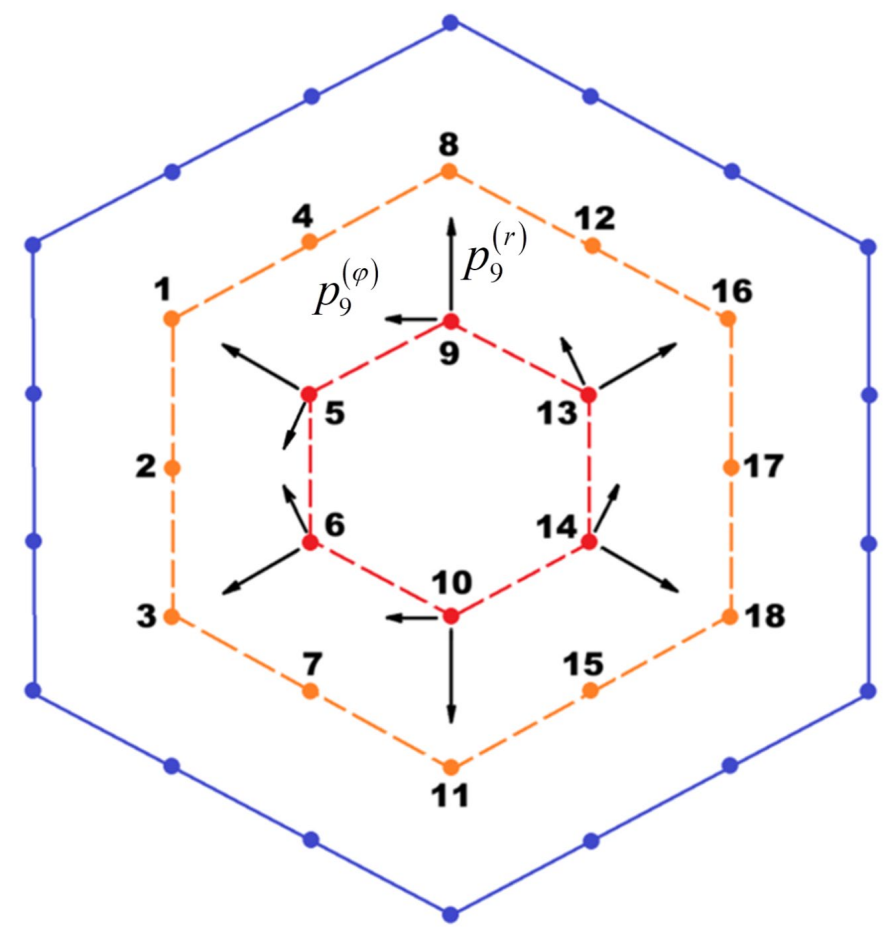
$$H_a(u,p)=\frac{1}{2}u^T Ku+\frac{1}{2}p^T Dp,$$

where $K_{ij}=\left.\frac{\partial^2 U}{\partial u_i \partial u_j}\right|_{(R,\theta)}$, $\mathrm{D}=\begin{pmatrix} d^{(1,1)} & 0 \\ 0 & d^{(2,2)} \end{pmatrix}$, $d^{(1,1)}=d^{(2,2)}=\mathrm{diag}\left(\frac{1}{m_1},...,\frac{1}{m_N}\right)$,

$$\mathrm{G}=\begin{pmatrix} 0 & \mathrm{D} \\ -\mathrm{K} & 0 \end{pmatrix}, \quad \boxed{\dot{\vec{\eta}}=\mathrm{G}_\gamma(t)\vec{\eta},} \quad \mathrm{G}_\gamma(t)=\mathrm{G}\,\mathrm{E}(t), \quad \mathrm{E}(t)\overset{\text{def}}{=}\begin{pmatrix} e^{2\gamma t}\,\mathrm{I}_{2N} & 0 \\ 0 & e^{-2\gamma t}\,\mathrm{I}_{2N} \end{pmatrix}.$$

Solution $$\boxed{\vec{\eta}(t)=e^{-\gamma t}\mathcal{M}\left[\mathcal{E}(t)\vec{C}_+ +\mathcal{E}(-t)\vec{C}_-\right],}$$

Results

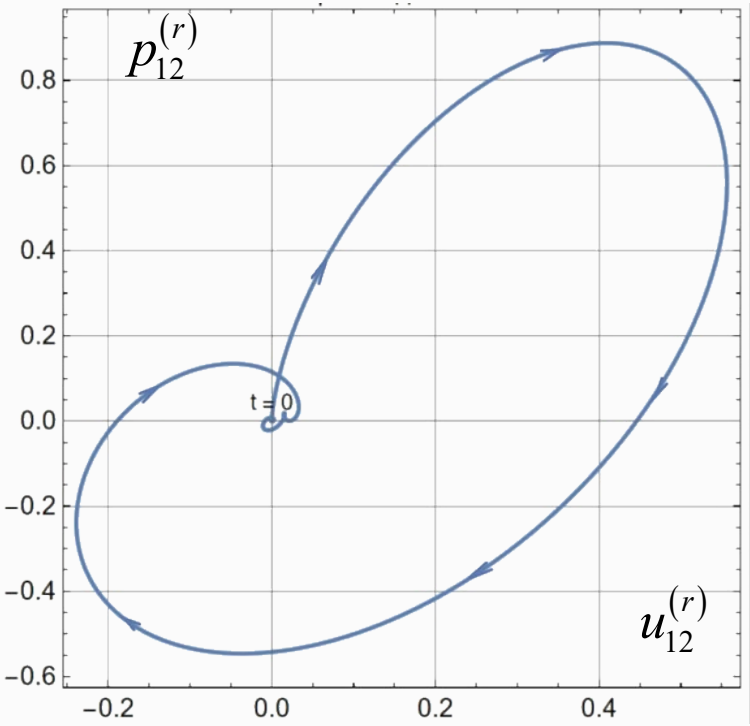


$$u_n^{(\rho)}\Big|_{t=0} = u_n^{(\varphi)}\Big|_{t=0} = 0,\ n \in N,$$

$$p_n^{(\rho)}\Big|_{t=0} = \begin{cases} 2p_0,\ n \in \Omega^1, \\ 0,\ n \in \Omega^2, \end{cases}$$

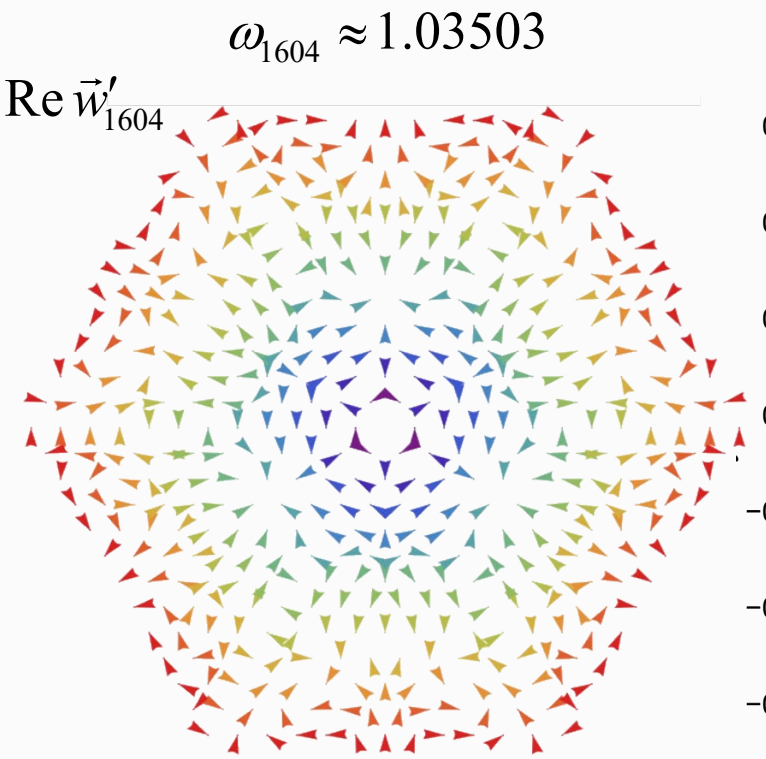
$$p_n^{(\varphi)}\Big|_{t=0} = \begin{cases} \pm p_0,\ n \in \Omega^1, \\ 0,\ n \in \Omega^2. \end{cases}$$

Friction case $\gamma = 0.5$

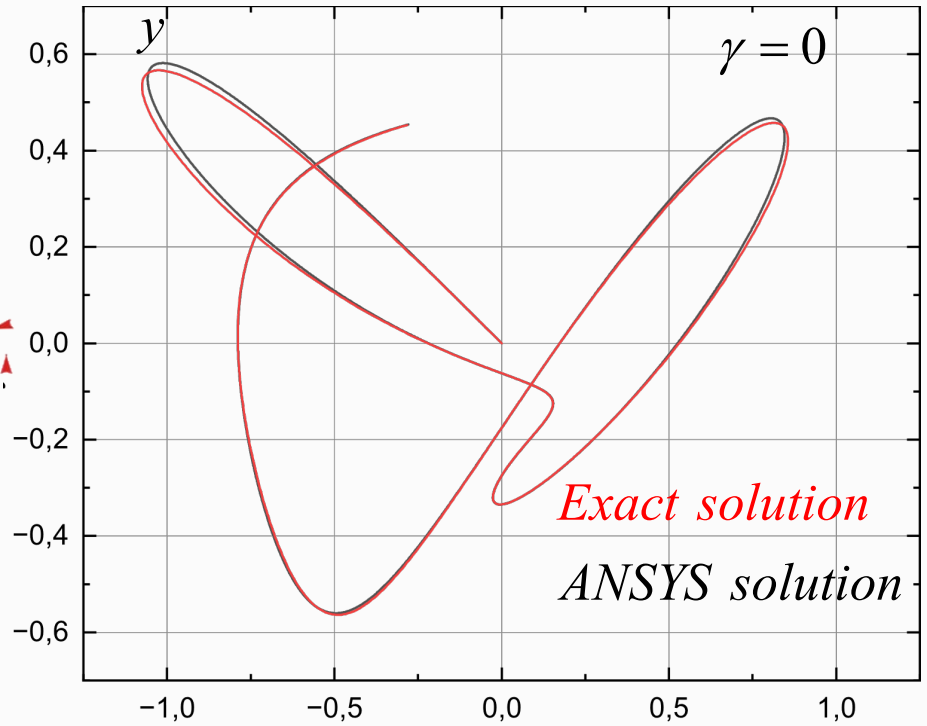


Phase space

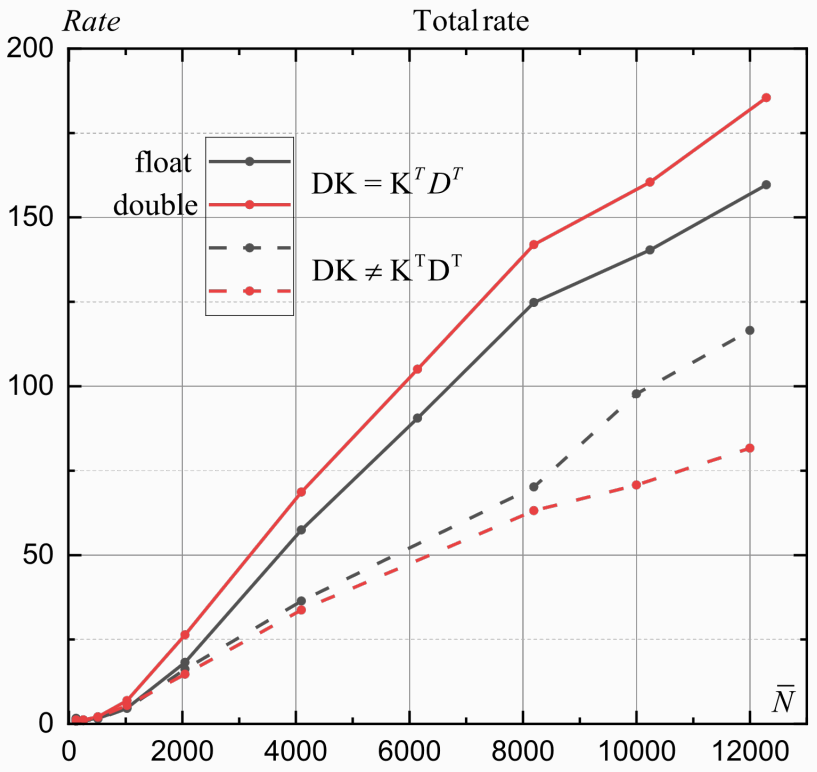
Results



Eigenmodes



Relative error $\varepsilon < 3.5\%$

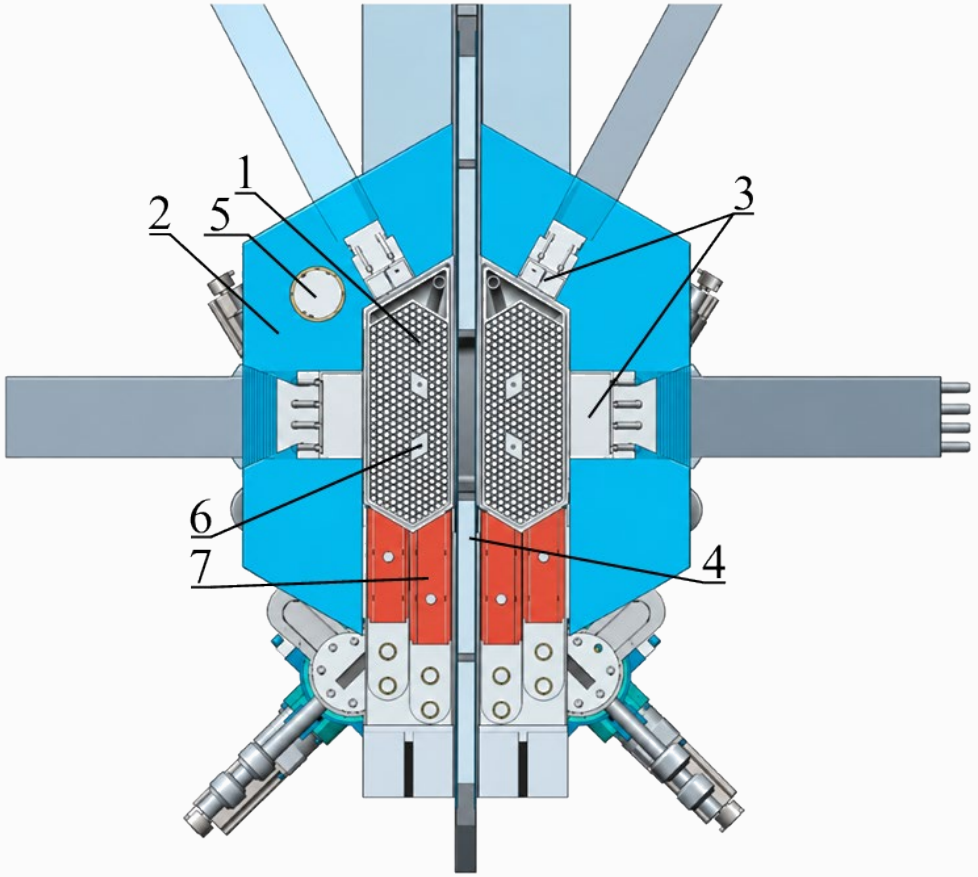


Acceleration rate on GPU architecture

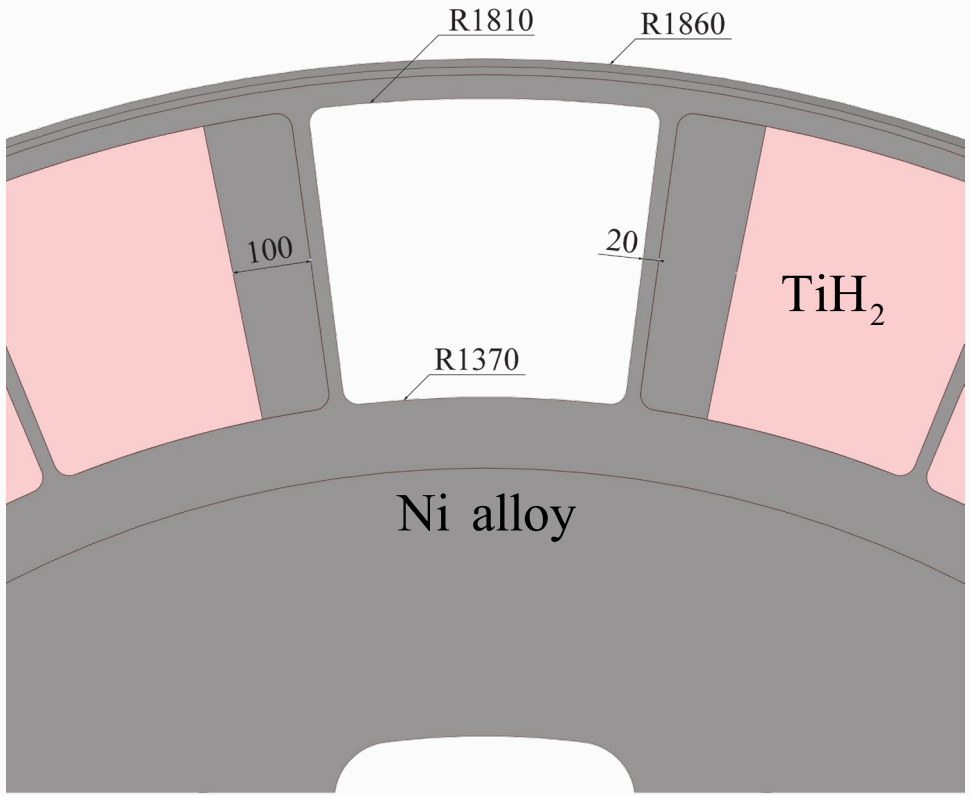
Results were published in a Q1 journal

3. E.E. Perepelkin, B.I. Sadovnikov, N.G. Inozemtseva, M.V. Klimenko, Analysis of dynamics of fuel rods in pulsed fast reactor, Progress in Nuclear Energy, Volume 201, 2026

Current research: NEPTUN reactor modulator



1 — Fuel; 2 — Stationary reflector; 3 — Water moderator; 4 — Reactivity modulator; 5 — Automatic regulator control rod; 6 — Compensating rods; 7 — Emergency protection units.

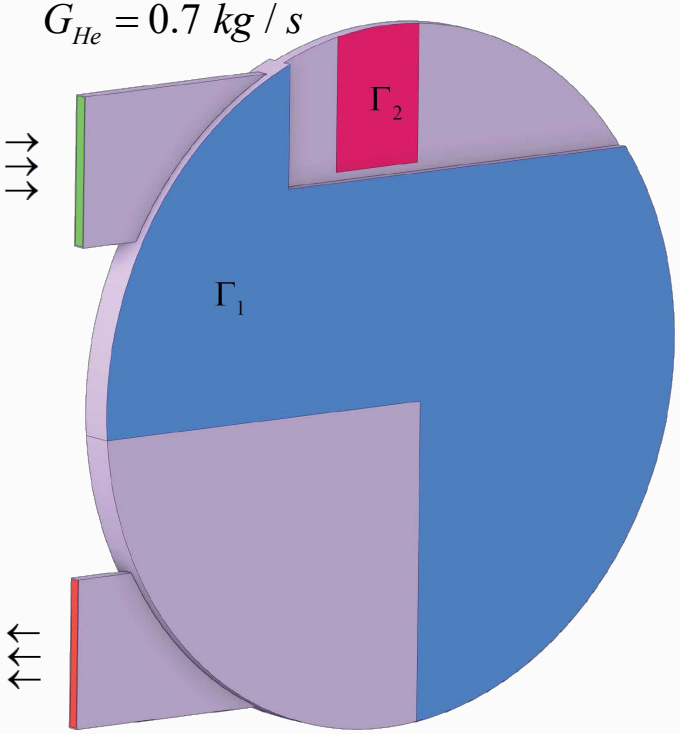


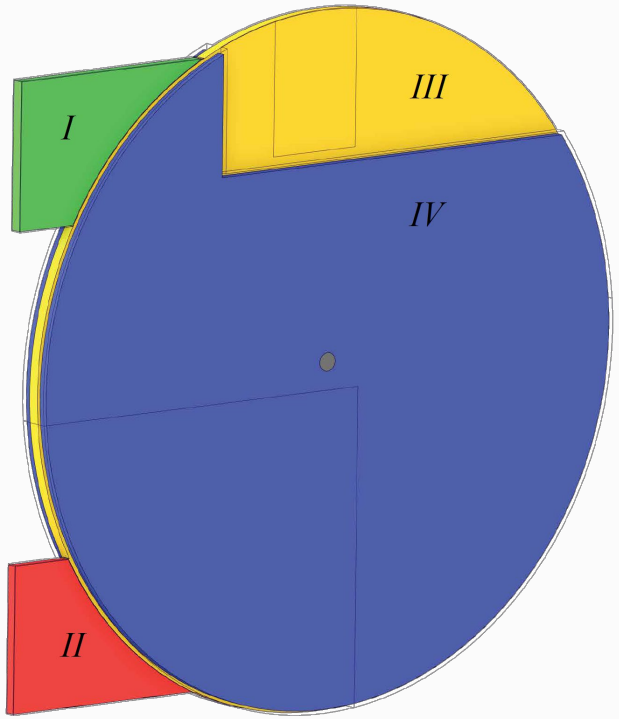
Modulator geometry

Current research: NEPTUN reactor modulator

$G_{air} = 3.6\ kg/s$

$G_{He} = 0.7\ kg/s$





Boundary conditions

$\Gamma_1 : t_c = 30°C$

$\Gamma_2 : \phi_{\Gamma_2} = \varepsilon\sigma\left(T_{az}^4 - T_s^4\right) + h_{\Gamma_2}\left(T_{gap} - T_s\right)$

$T_{in} = 50°C,\ T_{az} = 290°C$

Computational domain

I – Inlet

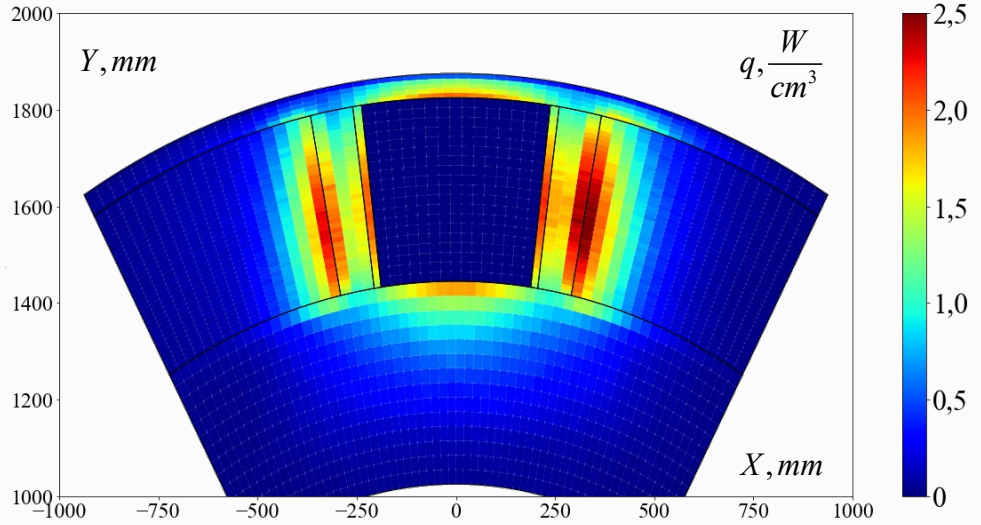
II – Outlet

III – Cooling gas

IV – Water cooling

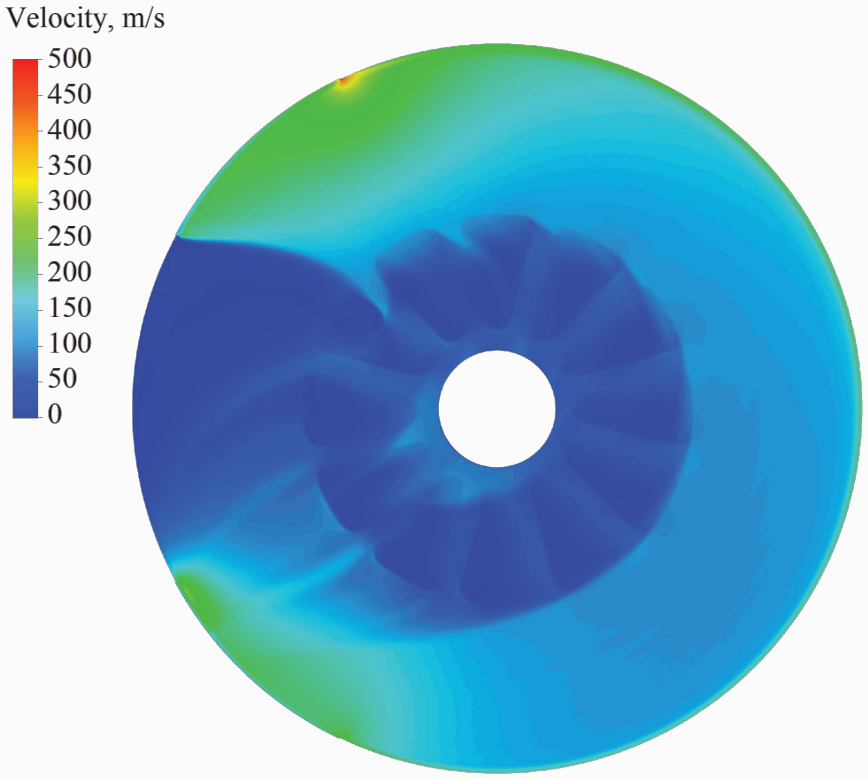
Ansys FLUENT

- Multiple reference frame model
- 12M computational elements
- Convection in the IV water cooling domain
- Volumetric heat generation rate from coupled neutron–photon transport calculations



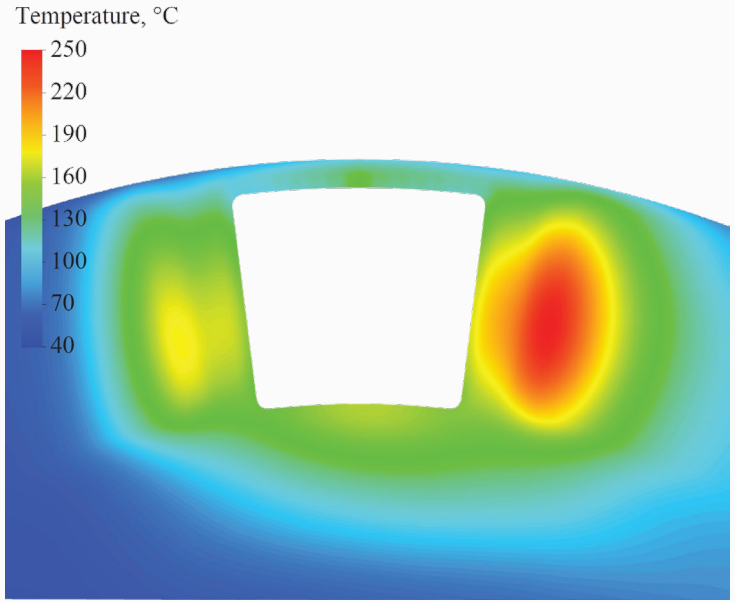
Volumetric heat generation rate

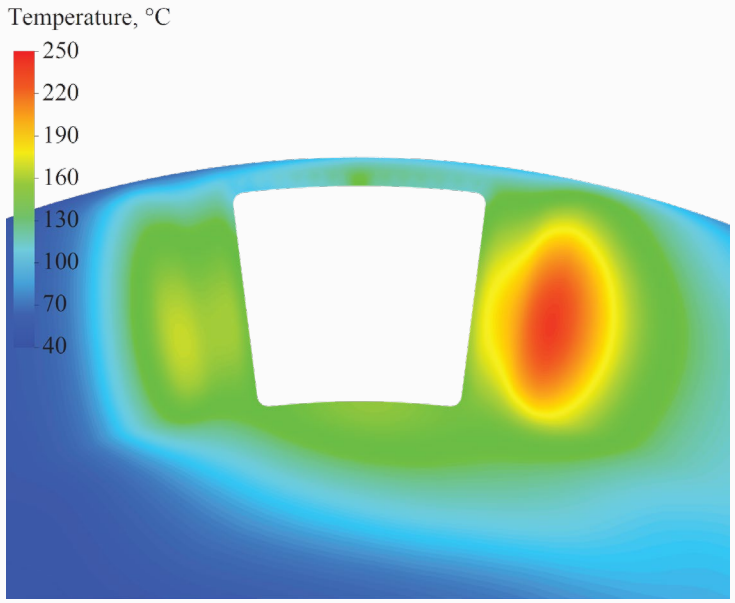
Intermediate results



Relative velocity distribution, air cooling

Temperature field in the middle section





$Air,\ T_{max} = 245°C$

$Helium,\ T_{max} = 237°C$

NIKIET results: $536 / 467°C$ [4].

Maximum allowed temperature for TiH_2: $250°C$ [5], $400°C$ [6].

4. NIKIET, R&D report IBRZ-Ot-5775, 2024
5. TU 95.1149-83. Compact titanium hydride products. Technical specifications, No. 16E-973.
6. Know-how registration certificate No. 20140022: Method for producing a composite material based on modified titanium hydride and Portland cement / V.I. Pavlenko et al., BSTU named after V.G. Shukhov, 2014.

Planned research steps

- **Completing the thermal-hydraulic analysis of the modulator cooling** to determine the maximum temperature in the structure, and publishing the results. A paper on this work is currently at an advanced stage of preparation.
- **Study of the dehydrogenation of the moderator material.** Titanium hydride has been selected as the moderator material. Its stability must be investigated under the neutron flux and in the temperature field obtained from the thermal-hydraulic analysis. Expected result: an estimate of the moderator's service life.
- **Study of the modulator's mechanical properties.** The obtained temperature distribution will allow analysis of the modulator deformation during operation. Expected result: justification of the mechanical stability of the modulator design.

Publications:

1. E.E. Perepelkin, B.I. Sadovnikov, N.G. Inozemtseva, M.V. Klimenko, Analysis of dynamics of fuel rods in pulsed fast reactor, Progress in Nuclear Energy, Volume 201, 106547 (2026), https://doi.org/10.1016/j.pnucene.2026.106547.
2. E.E. Perepelkin, B.I. Sadovnikov, N.G. Inozemtseva, M.V. Klimenko, Construction of Schrödinger, Pauli and Dirac equations from Vlasov equation in case of Lorenz gauge. Eur. Phys. J. Plus 140, 622 (2025), https://doi.org/10.1140/epjp/s13360-025-06547-y.
3. E. E. Perepelkin, N. G. Inozemtseva, M. V. Klimenko, R. V. Polyakova, M. B. Sadovnikova, Numerical-Analytical Method for Finding the Natural Frequencies and Eigenfunctions of a Four-Component Rod System. Phys. Part. Nuclei, Volume 56, pp. 88–103 (2025), https://doi.org/10.1134/S1063779624701247.
4. E.E. Perepelkin, B.I. Sadovnikov, N.G. Inozemtseva, M.V. Klimenko, Investigation of the dynamics of transverse oscillations of a vertical rod under gravity, friction, and thermal expansion. Progress in Nuclear Energy, Volume 177, id.105419 (2024), https://doi.org/10.1016/j.pnucene.2024.105419.
5. E. E. Perepelkin, A. E. Verkhoglyadov, I. V. Kushnir, M. V. Klimenko, Natural Frequencies and Eigenfunctions of a Composite Rod. Physics of Particles and Nuclei Letters, Volume 21, Issue 2, pp. 154-160 (2024), https://doi.org/10.1134/S1547477124020092.

Thank you for your attention!

Appendix: construction of the solution for 1.

$$u(z,t) = u_{g.h.}(z,t) + u_{p.i.}(z,t),$$

$$u(z,t) \sim Z(z)T(t),$$

$$\begin{cases} a_1 T'' + a_2 T' + \lambda T = 0, \\ Z^{(4)} + b\left[P_z(z)Z'\right]' - \lambda Z = 0. \end{cases}$$

$$\bar{z}L = z,\ p = bL^2 M_0 g,\ \bar{\lambda} = \lambda L^4.$$

For $\lambda = 0$, *bifurcation*:

$$\bar{Z}^{(\lambda=0)} = \bar{Z}^{(0)} = \int_0^{\bar{z}} \vartheta(x)dx,$$

$$\vartheta''(\bar{z}) + p(1-\bar{z})\vartheta(\bar{z}) = 0,$$

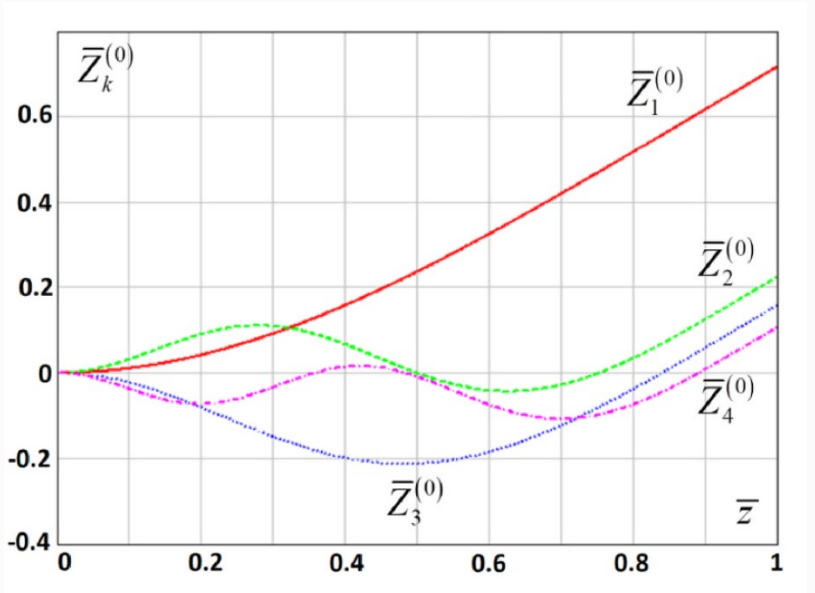
$$\vartheta(0) = \vartheta'(1) = 0.$$

$$\bar{x} = (\bar{z}-1)\sqrt[3]{p},\ \vartheta(\bar{z}) = \psi(\bar{x}),$$

$$\psi''(\bar{x}) - \bar{x}\psi(\bar{x}) = 0.$$

Eigenvalue problem

$$\begin{cases} \dfrac{d^4\bar{Z}}{d\bar{z}^4} + p\dfrac{d}{d\bar{z}}\left[(1-\bar{z})\dfrac{d\bar{Z}}{d\bar{z}}\right] - \bar{\lambda}\bar{Z} = 0, \\ \bar{Z}(0) = \bar{Z}'(0) = 0,\ \bar{Z}''(1) = \bar{Z}'''(1) = 0. \end{cases}$$



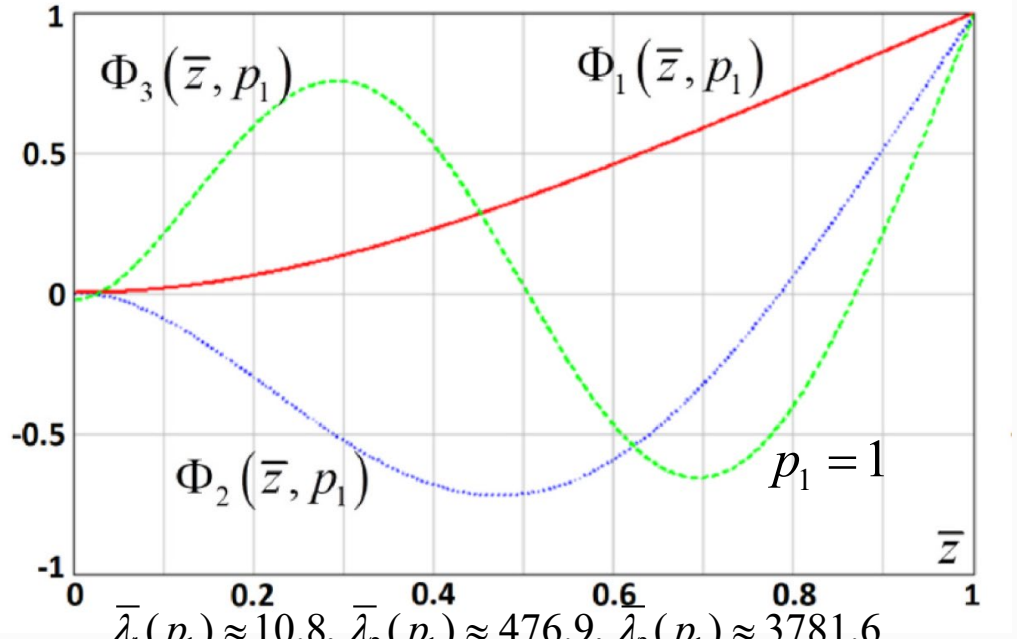
$$p_1 \approx 7.84,\ p_2 \approx 56.0,\ p_3 \approx 148.5,\ p_4 \approx 285.4$$

Appendix: construction of the solution for 1.

$$For\ \lambda \neq 0,$$

$$L_p = \frac{d^4}{d\bar{z}^4} + p\frac{d}{d\bar{z}}\left[(1-\bar{z})\frac{d}{d\bar{z}}\right]$$

$$\begin{cases} L_p\Phi_k(\bar{z},p) = \bar{\lambda}_k(p)\Phi_k(\bar{z},p), \\ \Phi_k(0,p) = \Phi'_k(0,p) = 0, \\ \Phi''_k(1,p) = \Phi'''_k(1,p) = 0. \end{cases}$$



$$\bar{\lambda}_1(p_1) \approx 10.8,\ \bar{\lambda}_2(p_1) \approx 476.9,\ \bar{\lambda}_3(p_1) \approx 3781.6$$

$$\Phi_k(\bar{z},p) = \sum_n \varepsilon_n(p,\bar{\lambda}_k)(1-\bar{z})^n,$$

$$\varepsilon_n(p,\lambda) = \frac{(n-4)!}{n!}\left[\lambda\varepsilon_{n-4} - p\varepsilon_{n-3}(n-3)^2\right], n > 3.$$

$$u(z,t) = \sum_{k=1}^{+\infty} \bar{w}_k(t)\Phi_k(\bar{z}),$$

$$\int_0^1 \Phi_k(\bar{z},p)\Phi_n(\bar{z},p)d\bar{z} = \left\|\Phi_n\right\|^2 \delta_{n,k}.$$

$$\Rightarrow$$

$$a_1\bar{w}''_n + a_2\bar{w}'_n + \frac{\bar{\lambda}_n}{L^4}\bar{w}_n = \bar{f}_n(t,p),$$

$$\bar{f}_n(t,p) = \frac{1}{\left\|\Phi_n\right\|^2}\int_0^1 \bar{f}(z,t)\Phi_n(\bar{z},p)d\bar{z}.$$

Appendix: construction of the solution for 1.

Particular case of $P_z = 0,\ p = 0,$

$$K_1(z) = \frac{1}{2}(\cosh(z) + \cos(z)), K_2(z) = \frac{1}{2}(\sinh(z) + \sin(z)),$$

$$K_3(z) = \frac{1}{2}(\cosh(z) - \cos(z)), K_4(z) = \frac{1}{2}(\sinh(z) - \sin(z)),$$

$$W(\bar{z}) = \sum_{i=1}^{4} C_i K_i(\alpha \bar{z}) \Rightarrow \cos(\alpha_k)\cosh(\alpha_k) = -1,\ \bar{\lambda}_k = \alpha_k^4.$$

Solution using generalised functions

$$\begin{cases} \mathcal{L}_n \bar{w}_n = \bar{w}_n'' + 2\gamma \bar{w}_n' + \left(\beta_n^2 + \gamma^2\right)\bar{w}_n = \dfrac{\bar{f}_n}{a_1}, \\ \bar{w}_n(0) = \bar{w}_n^{(0)}, \bar{w}_n'(0) = \bar{w}_n^{(1)}. \end{cases}$$

Extension to $t < 0$:

$$\mathcal{L}_n \tilde{w}_n(t) = \frac{\theta(t)}{a_1}\bar{f}_n + \bar{w}_n^{(0)}\delta'(t) + \left[\bar{w}_n^{(1)} + \frac{a_2}{a_1}\bar{w}_n^{(0)}\right]\delta(t) = \tilde{f}_n(t),$$

$$\tilde{w}_n(t) = \frac{1}{a_1}\int_0^t \bar{f}_n(\bar{t})\mathcal{W}_n(t - \bar{t})d\bar{t} + \bar{w}_n^{(0)}\mathcal{E}_n'(t) + \left[\bar{w}_n^{(1)} + \frac{a_2}{a_1}\bar{w}_n^{(0)}\right]\mathcal{E}_n(t).$$

$$u(z,t) = u_{g.h.}(z,t) + u_{p.i.}(z,t),$$

$$u(z,t) \sim Z(z)T(t),$$

$$\bar{z}L = z,\ p = bL^2 M_0 g, \bar{\lambda} = \lambda L^4.$$

$$u_{p.i.}(z,t) = \sum_{n=1}^{\infty} \bar{w}_n^{p.i.}(t) W_n(\bar{z})$$

Krylov functions $K_i(z)$

Fundamental solution

$$\mathcal{L}_n \mathcal{E}_n(t) = \delta(t),$$

$$\begin{cases} \mathcal{L}_n \mathcal{W}_n = 0, \\ \mathcal{W}_n(0) = 0, \mathcal{W}_n'(0) = 1, \end{cases}$$

$$\mathcal{W}_n = \frac{1}{\beta_n} e^{-\gamma t}\sin(\beta_n t),\ \mathcal{E}_n = \theta \mathcal{W}_n.$$

$$\tilde{w}_n(t) = \left(\mathcal{E}_n * \tilde{f}_n\right)(t),$$

Appendix: construction of the solution for 2.

Krylov functions $K_i(z)$:

$$Z_j(\bar{z}) = \sum_{s=1}^{4} C_s^j K_s(\alpha_j \bar{z}),\ C_1^1 = C_2^1 = 0,\ k_{j,p}^{(s)} = K_s\left(\alpha_j \bar{z}_p\right)$$

$$c = \left\{c_1, \ldots, c_{14}\right\}^T,\ C_s^j = c_{s-6+4j}.$$

Substitution into the boundary conditions:

$$\Lambda \vec{c} = \vec{\theta},\ \Lambda(\bar{\omega}) = \begin{pmatrix} A(\bar{\omega}) & B(\bar{\omega}) \\ C(\bar{\omega}) & D(\bar{\omega}) \end{pmatrix},\ A, B, C, D \in \mathbb{R}^{7\times 7},\quad \det \Lambda(\bar{\omega}) = 0 \Rightarrow \left\{\bar{\omega}_l\right\}_{l\in\mathbb{N}}.$$

Eigenfunction problem:

$$\bar{z} = 1 : \Phi^2(\alpha_4)\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = \vec{f}^{11-12}(c_{13}, c_{14}),\ \Phi^2(x) = \begin{pmatrix} K_3(x) & K_4(x) \\ K_2(x) & K_3(x) \end{pmatrix},\ \det\left(\Phi^2(x)\right) \neq 0,$$

$$\Rightarrow \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = c_{13}\vec{P}^{11-12} + c_{14}\vec{Q}^{11-12}.$$

Analogously for variables at the points $\bar{z} = \bar{z}_3$ *and* $\bar{z} = \bar{z}_2$:

$$\bar{z} = \bar{z}_3 : \Phi^4\left(\alpha_3 \bar{z}_3\right)\vec{c}^{7-10} = \vec{f}^{7-10}(c_{11}, \ldots, c_{14}) \Rightarrow \vec{c}^{7-10} = c_{13}\vec{P}^{7-10} + c_{14}\vec{Q}^{7-10},$$

$$\bar{z} = \bar{z}_2 : \Phi^4\left(\alpha_2 \bar{z}_2\right)\vec{c}^{3-6} = \vec{f}^{3-6}(c_7, \ldots, c_{10}) \Rightarrow \vec{c}^{3-6} = c_{13}\vec{P}^{3-6} + c_{14}\vec{Q}^{3-6}.$$

$$\bar{z} = \bar{z}_1 : M(\bar{\omega})(c_1, c_2, c_{13}, c_{14})^T = \vec{\theta},\ \det M = 0,\ \ c_{14} = 1,\ \Omega\left(\bar{\omega}_l\right)\vec{c}^{1,2,13} = \vec{f}^{1,2,13}.$$

Appendix: construction of the solution for 3.

$$\vec{\eta} = \left\{ u_1^{(r)}, ..., u_N^{(r)}, u_1^{(\varphi)}, ..., u_N^{(\varphi)}, p_1^{(r)}, ..., p_N^{(r)}, \tilde{p}_1^{(\varphi)}, ..., \tilde{p}_N^{(\varphi)} \right\}^T, \; u_j^{(r)} = r_j - R_j, \; u_s^{(\varphi)} = \varphi_s - \Theta_s,$$

$$where \left(R_j, \Theta_j \right) - \; equilibrium \; \; point, \; \tilde{p}_n^{(\varphi)} = p_n^{(\varphi)} \Big/ R_n .$$

Approximation of the system's Hamiltonian

$$H_a\left(u,p\right) = \sum_{n=1}^{N} \frac{1}{2m_n} \eta_{2N+n}^2 + \sum_{n=1}^{N} \frac{1}{2m_n} \eta_{3N+n}^2 + \frac{1}{2} \sum_{j,s=1}^{N} \kappa_{j,s}^{(1,1)} \eta_j \eta_s + \sum_{j,s=1}^{N} \kappa_{j,s}^{(1,2)} \eta_j \eta_{N+s} +$$

$$+ \frac{1}{2} \sum_{j,s=1}^{N} \kappa_{j,s}^{(2,2)} \eta_{N+j} \eta_{N+s},$$

$$where \; \; \kappa_{j,s}^{(1,1)} = \frac{\partial^2 U\left(R,\Theta\right)}{\partial r_j \, \partial r_s}, \; \; \kappa_{j,s}^{(1,2)} = \frac{\partial^2 U\left(R,\Theta\right)}{\partial r_j \, \partial \varphi_s}, \; \; \kappa_{j,s}^{(2,1)} = \kappa_{s,j}^{(1,2)}, \; \; \kappa_{j,s}^{(2,2)} = \frac{\partial^2 U\left(R,\Theta\right)}{\partial \varphi_j \partial \varphi_s}.$$

$$\dot{\vec{\eta}} = J \partial_\eta H_a = \mathrm{G} \vec{\eta}, \; \partial_\eta = \left\{ \partial_{\eta_1}, ..., \partial_{\eta_{4N}} \right\}^T,$$

$$\mathrm{G} = \begin{pmatrix} 0 & \mathrm{D} \\ -\mathrm{K} & 0 \end{pmatrix}, \; \mathrm{K} = \begin{pmatrix} \kappa^{(1,1)} & \kappa^{(1,2)} \\ \kappa^{(2,1)} & \kappa^{(2,2)} \end{pmatrix}, \; \mathrm{D} = \begin{pmatrix} d^{(1,1)} & 0 \\ 0 & d^{(2,2)} \end{pmatrix}, \; d^{(1,1)} = d^{(2,2)} = \mathrm{diag}\left(\frac{1}{m_1}, ..., \frac{1}{m_N} \right)$$

$$\vec{\eta} = \mathrm{M} \vec{\xi}, \; \det \mathrm{M} \neq 0 \; \Rightarrow \; \dot{\vec{\xi}} = \Lambda \vec{\xi}, \; \Lambda = \mathrm{M}^{-1} \mathrm{G} \, \mathrm{M},$$

Appendix: construction of the solution for 3.

$$L(\vec{r},\vec{v},t)=\left[\frac{mv^2}{2}-U(\vec{r})\right]e^{2\gamma t}\Rightarrow\ H(\vec{r},\vec{\mathrm{p}},t)=\frac{\mathrm{p}^2}{2m}e^{-2\gamma t}+U(\vec{r})e^{2\gamma t},\ \ where\ \ \vec{\mathrm{p}}\overset{\text{def}}{=}\nabla_v L=m\vec{v}e^{2\gamma t}=\vec{p}e^{2\gamma t}.$$

$$\vec{\eta}\overset{\text{def}}{=}\left\{u_1^{(r)},...,u_N^{(r)},u_1^{(\varphi)},...,u_N^{(\varphi)},\mathrm{p}_1^{(r)},...,\mathrm{p}_N^{(r)},\tilde{\mathrm{p}}_1^{(\varphi)},...,\tilde{\mathrm{p}}_N^{(\varphi)}\right\}^T.$$

$$\dot{\vec{\eta}}=\mathrm{G}_\gamma(t)\vec{\eta},\quad \mathrm{G}_\gamma(t)=\mathrm{G}\,\mathrm{E}(t),\quad \mathrm{E}(t)\overset{\text{def}}{=}\begin{pmatrix} e^{2\gamma t}\,\mathrm{I}_{2N} & 0\\ 0 & e^{-2\gamma t}\,\mathrm{I}_{2N}\end{pmatrix}.$$

Solution $$\vec{\eta}(t)=e^{-\gamma t}\mathcal{M}\left[\mathcal{E}(t)\vec{C}_+ +\mathcal{E}(-t)\vec{C}_-\right],$$

where $$\vec{C}^{\pm}\overset{\text{def}}{=}\begin{pmatrix}\vec{A}^{\pm}\\ \vec{B}^{\pm}\end{pmatrix},\ \mathcal{M}\overset{\text{def}}{=}\begin{pmatrix}\bar{\mathrm{M}} & 0\\ 0 & \bar{\mathrm{M}}\end{pmatrix},$$

$$\vec{A}^{\pm}=\frac{1}{2}\bar{\mathrm{M}}^{-1}\left\{\vec{\eta}'(0)\mp\Gamma'^{-1}\left[\gamma\vec{\eta}'(0)+\mathrm{D}\,\vec{\eta}''(0)\right]\right\},$$

$$\vec{B}^{\pm}=\frac{1}{2}\bar{\mathrm{M}}^{-1}\left\{\vec{\eta}''(0)\mp\Gamma''^{-1}\left[\gamma\vec{\eta}''(0)-\mathrm{K}\,\vec{\eta}'(0)\right]\right\},$$

$$\Gamma'\overset{\text{def}}{=}\bar{\mathrm{M}}'i\bar{\Omega}\bar{\mathrm{M}}'^{-1},\ \Gamma''\overset{\text{def}}{=}\bar{\mathrm{M}}''i\bar{\Omega}\bar{\mathrm{M}}''^{-1},$$

$$\Omega^2\overset{\text{def}}{=}\mathrm{M}'^{-1}(\mathrm{D}\,\mathrm{K})\mathrm{M}'=\mathrm{M}''^{-1}(\mathrm{K}\,\mathrm{D})\mathrm{M}''=\mathrm{diag}\left(\omega_1^2,...,\omega_{2N}^2\right),$$

$$\bar{\Omega}=\mathrm{diag}\left(\bar{\omega}_1,...,\bar{\omega}_{2N}\right),\ \bar{\omega}_k=\sqrt{\omega_k^2-\gamma^2}.$$