Three-loop massive effective potential from differential equations

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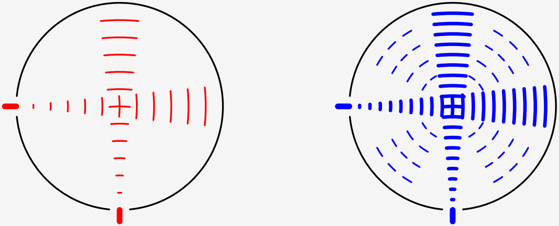
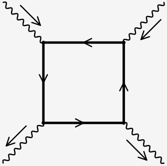
Effective theories

- Example from QED: [Grozin'09]

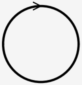


$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2 F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}$$

- But also we have interecting fermions term $\bar{\psi}\gamma^{\mu}\psi A_{\mu}$



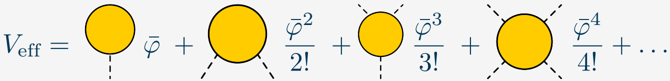
In the $m_e \to \infty$ limit lead to the massive tadpoles defining coefficients c_1 and c_2



Effective approach to the scalar potential

- Near phase transiotion point physics become very sensitive to input parameters and higher order effects e.g.
  - Phase transitions in various condensed matter systems described by the different variations of the $\mathcal{O}(N)\ \varphi^4$ theories
  - Standard Model near the point of the spontaneous symmetry breaking

- Effective potential approach is a way to account infinite sum of the higher dimensional interaction terms in lagrangian with higher order perturbative corrections



- From the QED example we learned how to account for the contributions with fixed dimension, but what about all posible cases?

Lagrangian parameters and mass scales

$$\mathcal{L}_S = \underbrace{\frac{{m_H}^2}{2}H^2 + \frac{{m_G}^2}{2}G_i^2}_{\text{mass terms}} + \underbrace{\frac{\tau_0}{6}H^3 + \frac{\tau_i}{6}HG_i^2}_{\text{triple interaction}} + \underbrace{\frac{\lambda_0}{24}H^4 + \frac{\lambda_i}{12}H^2G_i^2 + \frac{\lambda_{ij}}{24}G_i^2G_j^2}_{\text{quartic interaction}}$$

- $O(N)$ symmetric scalar φ^4 theory, $\langle\varphi_1\rangle = v \neq 0$ and all other $\langle\varphi_i\rangle = 0$

$$\mathcal{L} = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{24}(\varphi^2)^2$$

$$\tau_0 = \tau_i = \lambda v, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda \quad m_H^2 = m^2 + \frac{\lambda}{2}v^2, m_G^2 = m^2 + \frac{\lambda}{6}v^2$$

- Standard Model in the broken phase

$$\mathcal{L} = m^2\Phi^\dagger\Phi + \frac{\lambda}{6}(\Phi^\dagger\Phi)^2, \quad \Phi = \frac{1}{\sqrt{2}}(v + H + iG_0, G_r + iG_i)^T$$

$$\tau_0 = \tau_i = \lambda v, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda, \quad m_H^2 = m^2 + \frac{\lambda}{2}v^2, m_G^2 = m^2 + \frac{\lambda}{6}v^2$$

Known results for the effective potential

2-loop Analytically SM [Ford,Jack,Jones'93], general theory [Martin'01]

3-loop Numerically general theory [Martin'17]

$$D(m_H) = \text{———} = \frac{1}{p^2 + m^2 + \frac{\lambda v^2}{2}}, \quad D(m_G) = \text{- - - - -} = \frac{1}{p^2 + m^2 + \frac{\lambda v^2}{6}}$$

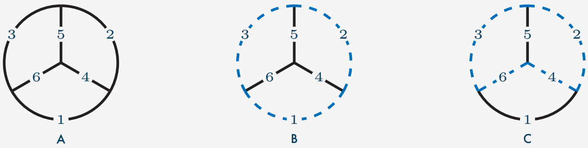
Three-loop analytically known results:

- Massless broken $\mathcal{O}(N)$ symmetric φ^4 theory (only $\mathcal{O}(\frac{1}{\varepsilon})$ part) $m_G^2 = 3m_H^2$
  [Chung,Chung'97;Kotikov'98]
- Single component masive φ^4 theory only m_H
  [Chung,Chung'99]

Our goal:

General case $m \neq 0$ with two scales $m_G \neq m_H$

Three-loop topologies for vacuum integrals



- Topology **A** is a single scale, reduction using `MATAD` [Steinhauser'00] master integrals known up to the weight 6 [Kniehl,AP,Veretin'17]

- Topologies **B** and **C** have 11 master integrals each depending on a single variable $x = \frac{m_G^2}{m_H^2}$, reduction using `LiteRed` [Lee'14]

- Differentiating in x and reducing back to the set of the mater integrals we obtain closed system of 11 differential equations:

$$\partial_x J_a(\varepsilon, x) = M_{ab}(\varepsilon, x) J_b(\varepsilon, x)$$

- We are looking for the solution as a series expansion in $\varepsilon = 2 - d/2$

Iterated integrals

$$\int_0^x dz_1 f_1(z_1) \int_0^{z_1} dz_2 f_2(z_2) \int_0^{z_2} dz_3 f_3(z_3) \cdots \int_0^{z_{n-1}} dz_n f_n(z_n)$$

- Harmonic polylogarithms(HPL), include Li_n and $S_{n,p}$

$$f_{-1}(z) = \frac{1}{z-1}, \quad f_0(z) = \frac{1}{z}, \quad f_1(z) = \frac{1}{z+1}$$

- Generalized polylogarithms(GPL), include HPL

$$f_a(z) = \frac{1}{z-a}$$

- Cyclotomic polylogarithms, after factorization over $\mathbb{C}$ and partial fractioning can be reexpressed through GPL

$$f_a^b(z) = \frac{z^b}{\Phi_a(z)}, f_0^0(z) = 1/z, \quad \Phi_n(z) = \prod_{\gcd(k,n)=1} \left(z - e^{2\pi i \frac{k}{n}}\right)$$

Differential equations and canonical basis

- Set of the master integrals is not unique, we are looking for the basis, which coefficients of ε-expansion have constant transcendental weight
- Differentiation reduces transcendental weight by one, if we assign weight one to ε, DE for integrals in a new basis would have following form [Henn'13]:

$$\partial_x g_a(x) = \varepsilon M_{ab}(x) g_b(x)$$

- For the coefficients of ε-expansion system decouple and solution can be written explicitly, upto a constant for each integration:

$$g_a\{\varepsilon^n\}(y) = \int dy\, M_{ab}(y)\, g_b\{\varepsilon^{n-1}\}(y) + \mathcal{C}_{a,n}$$

- For system solvable in terms of GPL, algorithmic ways of canonical basis construction exists [Lee'14] and [Meyer'16] with public implementations `Fuchsia` [Gituliar,Magerya'17], `epsilon` [Prausa'17] and `CANONICA` [Meyer'17]
- Rational transformation can be constructed only after apropriate variable change

$$x = \frac{y^2}{(1+y^2)^2}$$

Cyclotomic polylogarithms integration

- System in canonical basis can be easily decomposed into the form, where $B_{a,b}$ and $C_{a,b}$ are pure numeric matrices and all y dependence is inside functions f_a^b known how to integrate using definition of CPL:

$$
\begin{aligned}
B(y) = \big(& f_0^0 B_{0,0} + f_1^0 B_{1,0} + f_2^0 B_{2,0} + f_3^0 B_{3,0} + f_3^1 B_{3,1} \\
&+ f_4^1 B_{4,1} + f_6^0 B_{6,0} + f_6^1 B_{6,1} + f_{12}^1 B_{12,1} + f_{12}^3 B_{12,3}\big) \\
C(y) = \big(& f_0^0 C_{0,0} + f_1^0 C_{1,0} + f_2^0 C_{2,0} + f_3^0 C_{3,0} + f_3^1 C_{3,1} \\
&+ f_4^1 C_{4,1} + f_6^0 C_{6,0} + f_6^1 C_{6,1} + f_8^3 C_{8,3}\big)
\end{aligned}
$$

- Integration constants fixed from the finite number of terms in small m_G mass expansion ($y \to 0$) of integrals and expansion of result in terms of CPL using **HarmonicSums** package [Ablinger'13] and our own implementation

- Finite parts of the three-loop integrals are expressible through the cyclotomic polylogarithms up to the weight four

Numerical evaluation and transformations

- Cyclotomic polylogarithms are easy to evaluate with high precision as a series expansion near zero
- Comparing to HPL and even GPL lack of transformation rules like $x \to 1 - x$
- Differential equations with initial conditions are known for CPL
- Possible to construct terms of series expansion in different regions using expansion around singular points [Lee,Smirnov,Smirnov'17]

Constructing result from the diagrams

$V_3 = \frac{\lambda_0^2}{16}$  $+ \frac{2}{3}\frac{\lambda_0\lambda_i}{16}$ 

$+ \frac{1}{9}\frac{\lambda_i^2}{16}$ 

$+ \frac{1}{9}\frac{\lambda_i\lambda_j}{16}$ 

$+ \frac{2(1+2\delta_{ij})}{9}\frac{\lambda_i\lambda_{ij}}{16}$ 

$+ \frac{(1+2\delta_{ij})(1+2\delta_{jk})}{9}\frac{\lambda_{ij}\lambda_{jk}}{16}$ 

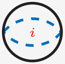
$+ \frac{\lambda_0\tau_0^2}{8}$  $+ \frac{1}{3}\frac{\lambda_i\tau_0^2}{8}$ 

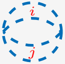
$+ \frac{2}{27}\frac{\lambda_i\tau_i^2}{8}$ 

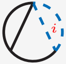
$+ \frac{2(1+2\delta_{ij})}{27}\frac{\lambda_{ij}\tau_j^2}{8}$ 

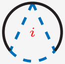
$+ \frac{1}{9}\frac{\lambda_0\tau_i^2}{8}$ 

$+ \frac{1}{27}\frac{\lambda_i\tau_j^2}{8}$ 

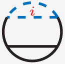
$+ \frac{\lambda_0^2}{48}$  $+ \frac{2}{3}\frac{\lambda_i^2}{48}$ 

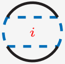
$+ \frac{1+2\delta_{ij}}{3}\frac{\lambda_{ij}^2}{48}$ 

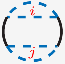
$+ \frac{\lambda_0\tau_0^2}{8}$  $+ \frac{2}{9}\frac{\tau_0\tau_i\lambda_i}{8}$ 

$+ \frac{4}{27}\frac{\lambda_i\tau_i^2}{8}$ 

$+ \frac{1+2\delta_{ij}}{27}\frac{\lambda_{ij}\tau_i\tau_j}{8}$ 

$+ \frac{\tau_0^4}{16}$  $+ \frac{2}{9}\frac{\tau_0^2\tau_i^2}{16}$ 

$+ \frac{4}{81}\frac{\tau_i^4}{16}$ 

$+ \frac{1}{81}\frac{\tau_i^2\tau_j^2}{16}$ 

$+ \frac{\tau_0^4}{24}$  $+ \frac{4}{9}\frac{\tau_0\tau_i^3}{24}$ 

$+ \frac{1}{9}\frac{\tau_i^4}{24}$ 

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Conclusion

1. We have calculated closed analytical expression for the three-loop effective potential in general massive scalar theory in the broken phase.

2. New set of the two-mass three-loop tadpole integrals calculated. Result can find its application in coure of calculation in more complicated theories.

3. New set of functions from the iterated integrals class used to represent results of the calculations. To calculate them numerically prepared efficient code. Their properties need further investigation.

Thank you for attention!