Estimating the radiative part of QED effects for systems with supercritical charge

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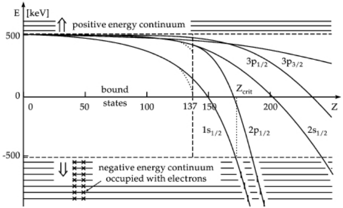
Non-perturbative vacuum effects for $Z > Z_{cr}$

- The most well-know effect predicted by QED for supercritical region ($Z > Z_{cr}$) is a vacuum positron emission caused by diving the discrete electronic levels into the lower continuum.[1]



Critical charge Z_{cr}:

$$E_{1s_{1/2}}(Z_{cr}) = -mc^2$$

$$Z_{cr} \simeq 170$$

- The recent non-perturbative computations show that E_{VP} demonstrates essentially non-linear behaviour for $Z > Z_{cr}$ and, under certain conditions, E_{VP} can compete with E_{Coul}.[2]

[1] S. S. Gershtein and Y. B. Zeldovich, Zh. Eksp. Teor. Fiz. **57**, 654–659 (1969), W. Pieper and W. Greiner, Z. Phys. **218**, 327–340 (1969).

[2] A. Davydov, K. Sveshnikov, and Y. Voronina, Int. J. Mod. Phys. A **32**, 1750054 (2017), arXiv:1709.04239 [hep-th], A. Davydov, K. Sveshnikov, and Y. Voronina, Int. J. Mod. Phys. A **33**, 1850004 (2018), arXiv:1712.02704 [hep-th].

Non-perturbative vacuum effects for $Z > Z_{cr}$

- Is it possible to compensate for non-perturbative vacuum effects (caused by fermion loops) by QED effects with virtual photon exchange?

Non-perturbative vacuum effects for $Z > Z_{cr}$

- Is it possible to compensate for non-perturbative vacuum effects (caused by fermion loops) by QED effects with virtual photon exchange?
- Full non-perturbative analysis of the radiative effects is too complicated

$$\Downarrow$$

The interaction of the electron magnetic anomaly with Coulomb field

ΔU_{AMM} is a component of the self-energy contribution to the total radiative shift of the levels.
ΔU_{AMM} is a local operator, which allows for a detailed non-perturbative analysis.

Effective interaction due to AMM

In the static limit

$$\Delta U^{(0)}_{AMM}(\vec{r}) = \frac{\Delta g}{2}\frac{e}{4m}\,\sigma^{\mu\nu}F_{\mu\nu}\,. \tag{1}$$

AMM is not an intrinsic property of the electron $\Rightarrow \Delta g \to \Delta g_{free}\, c(r)$.[3]

The one-loop vertex correction[4]

$$\Gamma^{\mu}(q^2) = \gamma^{\mu}F_1(q^2) + \frac{i}{2m}F_2(q^2)\sigma^{\mu\nu}q_{\nu}\,. \tag{2}$$

Hence, the effective potential of the interaction with an external electromagnetic field:

$$\Delta U_{AMM}(\vec{r}) = \frac{e}{2m}\,\sigma^{\mu\nu}\partial_{\mu}\mathcal{A}^{(cl)}_{\nu}(\vec{r}), \tag{3}$$

where

$$\mathcal{A}^{(cl)}_{\mu}(\vec{r}) = \frac{1}{(2\pi)^3}\int d\vec{q}\; e^{i\vec{q}\,\vec{r}}\,\tilde{A}^{(cl)}_{\mu}(\vec{q})F_2(-\vec{q}^{\,2})\,. \tag{4}$$

[3] K. Geiger, J. Reinhardt, B. Müller, et al., Z. Phys. A - Atomic Nuclei **329**, 77–88 (1988), A. O. Barut, en, Z. Phys. A - Atomic Nuclei **336**, 317–320 (1990).

[4] C. Itzykson and J.-B. Zuber, (McGraw-Hill, 1980).

Effective interaction due to AMM

Taking into account the dynamical screening of the electronic AMM at short distances, one obtains[5]:

$$\Delta U_{AMM}(\vec{r}) = i\,\lambda\,\vec{\gamma}\cdot\vec{\nabla}\left(\sum_i \frac{Z_i\,c(|\vec{r}-\vec{r_i}|)}{|\vec{r}-\vec{r_i}|}\right)\,, \tag{5}$$

where $\lambda = \alpha^2/4\pi m$, $F_2(0) = \Delta g_{free}/2 \simeq \alpha/2\pi$,

$$c(r) = 2\int_0^\infty q dq\;\sin qr\left(-\frac{1}{Ze}\,\tilde{\Phi}(q)\right)\frac{1}{\pi}\,\frac{F_2(-q^2)}{F_2(0)}\,, \tag{6}$$

and $\tilde{\Phi}(q)$ is a Fourier-transform of the nuclear Coulomb field $\Phi(r)$.

[5] A. Roenko and K. Sveshnikov, Int. J. Mod. Phys. A **32**, 1750130 (2017), arXiv:1608.04322 [physics.atom-ph], A. Roenko and K. Sveshnikov, Phys. Part. Nucl. Lett. **15**, 20–28 (2018).

Dynamical screening of AMM

For the extended nucleus in a form of the uniformly charged ball with raduis R the calculations give[6]

$$c_N(r) = 1 - \int\limits_{4m^2}^{\infty} \frac{dQ^2}{Q^2}\, \frac{3QR\cosh QR - 3\sinh QR}{R^3Q^3}\, e^{-Qr}\, \frac{1}{\pi}\, \frac{\mathrm{Im}\, F_2(Q^2)}{F_2(0)}, \qquad r > R,$$

$$c_N(r) = \frac{(3R^2 - r^2)}{2R^3}\, r - \frac{r}{2m^2R^3} + $$

$$+ \int\limits_{4m^2}^{\infty} \frac{dQ^2}{Q^2}\, \frac{3(QR+1)}{R^3Q^3}\, \sinh Qr\, e^{-QR}\, \frac{1}{\pi}\, \frac{\mathrm{Im}\, F_2(Q^2)}{F_2(0)}, \qquad r < R.$$

$$c(r) \to 0 \text{ for } r \to 0, \qquad c(r) \simeq 1 \text{ for } r \gg 1/m$$

[6] A. Roenko and K. Sveshnikov, Int. J. Mod. Phys. A **32**, 1750130 (2017), arXiv:1608.04322 [physics.atom-ph].

The Dirac equation with ΔU_{AMM}

The Dirac equation with an additional effective interaction ΔU_{AMM} has the form ($\hbar = c = m = 1$)

$$(\vec{\alpha}\vec{p} + \beta + W(r) + \Delta U_{AMM})\,\psi = \epsilon\psi\,. \tag{7}$$

In the spherically symmetric case (H-like atom):

$$\psi = \begin{pmatrix} i\varphi \\ \chi \end{pmatrix}, \qquad \varphi = f_\kappa(r)\,\Omega_{jlm_j}\,, \quad \chi = g_\kappa(r)\,\Omega_{jl'm_j}\,. \tag{8}$$

The system of equation for f_κ, g_κ has the form

$$\begin{aligned} \partial_r f_\kappa - \frac{Z\lambda\nu(r)}{r^2}\,f_\kappa + \frac{1+\kappa}{r}\,f_\kappa &= (\epsilon + 1 - W(r))g_\kappa\ , \\ \partial_r g_\kappa + \frac{Z\lambda\nu(r)}{r^2}\,g_\kappa + \frac{1-\kappa}{r}\,g_\kappa &= -(\epsilon - 1 - W(r))f_\kappa\ , \end{aligned} \tag{9}$$

where $\kappa = \pm(j + {}^1\!/_2)$, $\nu(r) = c(r) - rc'(r)$.
The potential ΔU_{AMM} is accounted non-perturbatively both in $Z\alpha$ and (partially) in α/π, since α/π enters as a factor in the coupling constant λ.

The two-center Dirac equation with ΔU_{AMM}

For a compact nuclear quasi-molecule ($d \lesssim 100$ fm) the expansion of the electronic wave-function under the spherical harmonics and the multipole expansion of the two-center potential may be used, $W(\vec{r}) = -\alpha\, U(\vec{r})$

$$\varphi = \sum_{\kappa=\pm 1}^{\pm N} f_\kappa\, X_{\kappa, m_j}\ , \qquad \chi = \sum_{\kappa=\pm 1}^{\pm N} g_\kappa\, X_{-\kappa, m_j}\ , \tag{10}$$

where $X_{-|\kappa|, m_j} \equiv \Omega_{jlm_j}$ и $X_{|\kappa|, m_j} \equiv (\vec{\sigma}\vec{n})\,\Omega_{jlm_j}$
As a result one obtain[7]

$$\begin{aligned} \partial_r f_\kappa + \frac{1+\kappa}{r} f_\kappa + \lambda \sum_{\bar{\kappa}} M_{\kappa;\bar{\kappa}}(r)\, f_{\bar{\kappa}} &= (1+\epsilon) g_\kappa + \alpha \sum_{\bar{\kappa}} N_{-\kappa;-\bar{\kappa}}(r)\, g_{\bar{\kappa}}\ , \\ \partial_r g_\kappa + \frac{1-\kappa}{r} g_\kappa - \lambda \sum_{\bar{\kappa}} M_{-\kappa;-\bar{\kappa}}(r)\, g_{\bar{\kappa}} &= (1-\epsilon) f_\kappa - \alpha \sum_{\bar{\kappa}} N_{\kappa;\bar{\kappa}}(r)\, f_{\bar{\kappa}}\ , \end{aligned} \tag{11}$$

[7] A. A. Roenko and K. A. Sveshnikov, Phys. Rev. A **97**, 012113 (2018), arXiv:1710.08494 [physics:atom-ph].

The two-center Dirac equation with ΔU_{AMM}

where

$$N_{\kappa;\bar\kappa}(r) = \sum_n U_n(r)\, W^{\varsigma}_{\bar\varsigma}(n; l_\kappa; l_{\bar\kappa}), \qquad (12)$$
$$M_{\kappa;\bar\kappa}(r) = \sum_n \left(\partial_r + \frac{\kappa - \bar\kappa}{r}\right) V_n(r)\, W^{\varsigma}_{\bar\varsigma}(n; l_\kappa; l_{\bar\kappa}),$$

$$\varsigma = \mathrm{sign}(-\kappa) = \begin{cases} -, & \kappa > 0\ , \\ +, & \kappa < 0\ , \end{cases} \qquad l_\kappa = \begin{cases} \kappa, & \kappa > 0\ , \\ |\kappa| - 1, & \kappa < 0\ , \end{cases}$$

and

$$W^{\varsigma}_{\bar\varsigma}(n; l_\kappa; l_{\bar\kappa}) \equiv \langle X_{\kappa,m_j} | P_n(\cos\vartheta) | X_{\bar\kappa,m_j} \rangle .$$

The Eq. (12) includes the multipole moments of the two-center potentials U_n and V_n :

$$U(\vec r) = \int d\vec r\,' \, \frac{\rho(\vec r\,)}{|\vec r - \vec r\,'|}\ , \quad \rho(\vec r\,) = \rho_0(\vec r - \vec a\,) + \rho_0(\vec r + \vec a\,) \qquad (13)$$

$$V(\vec r\,) = Z \left(\frac{c(|\vec r - \vec a\,|)}{|\vec r - \vec a\,|} + \frac{c(|\vec r + \vec a\,|)}{|\vec r + \vec a\,|} \right) . \qquad (14)$$

The general properties of the shifts due to ΔU_{AMM}

- The self-energy shift of the electronic levels for H-like atoms is usually represented in the form[8]

$$\Delta E_{nj}^{SE}(Z\alpha) = \frac{Z^4\alpha^5}{\pi n^3} F_{nj}(Z\alpha)\,. \tag{15}$$

In perturbative QED, $F_{nj}(Z\alpha)$ is found[9] for the lowest electronic levels of H-like atoms with the nucleus charge in the range $Z = 1 - 110$. For the level $1s_{1/2}$ the calculations with a precision of about 5% were performed up to $Z = 170$.[10]

- And although ΔE_{AMM} is not a dominant contribution to ΔE_{SE}, the behaviour F_{nj}^{AMM} (with accounting for the dynamical screening) for a number of the lower electronic levels qualitively reproduces[11] the behaviour of the F_{nj}.

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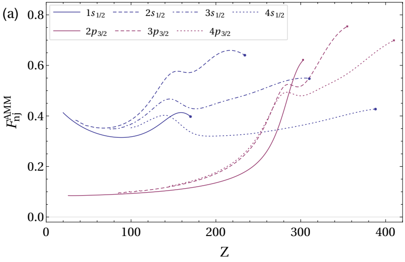
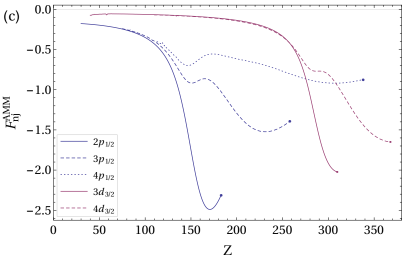
[8] P. J. Mohr, G. Plunien, and G. Soff, Phys. Rept. **293**, 227–369 (1998).
[9] V. A. Yerokhin and V. M. Shabaev, J. Phys. Chem. Ref. Data **44**, 033103 (2015).
[10] K. T. Cheng and W. R. Johnson, Phys. Rev. A **14**, 1943–1948 (1976), G. Soff, P. Schlüter, B. Müller, et al., Phys. Rev. Lett. **48**, 1465–1468 (1982).
[11] A. Roenko and K. Sveshnikov, Phys. Part. Nucl. Lett. **15**, 20–28 (2018).

The shift due to ΔU_{AMM}



(a)



(b)

Figure: The function F_{nj}^{AMM} for the contribution from ΔU_{AMM} as a function from the nuclear charge Z for levels with $n \leq 4$ and $j = {}^1/_2, {}^3/_2$.
The levels with $j = l + {}^1/_2$ (Fig. a) and $j = l - {}^1/_2$ (Fig. b) are shown.

The shift due to ΔU_{AMM} near $\epsilon = -1$

Table: The values $F_{nj}^{AMM}(Z_{cr}\alpha)$ for the shift due to ΔU_{AMM} for levels with different nlj near the threshold of the lower continuum in the range $Z_{cr,1s} < Z < 1000$.

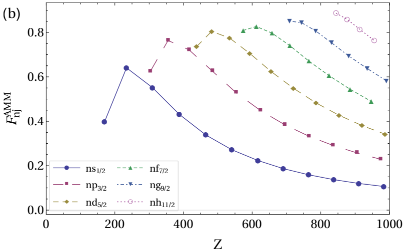
n	$ns_{1/2}$	$np_{3/2}$	$nd_{5/2}$	$nf_{7/2}$	$ng_{9/2}$	$nh_{11/2}$
1	0.398	0	0	0	0	0
2	0.641	0.624	0	0	0	0
3	0.551	0.764	0.736	0	0	0
4	0.431	0.722	0.805	0.809	0	0
5	0.338	0.626	0.775	0.827	0.851	0
6	0.272	0.529	0.706	0.797	0.844	0.886
7	0.223	0.449	0.624	0.740	0.807	0.859
8	0.187	0.384	0.549	0.671	0.754	0.814
9	0.159	0.333	0.483	0.606	0.695	0.763
10	0.137	0.292	0.427	0.544	0.638	–
11	0.120	0.258	0.381	0.490	0.583	–
12	0.106	0.229	0.342	–	–	–

The shift due to ΔU_{AMM} near $\epsilon = -1$

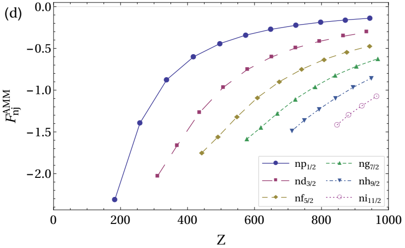
Table: The values $F_{nj}^{AMM}(Z_{cr}\alpha)$ for the shift due to ΔU_{AMM} for levels with different nlj near the threshold of the lower continuum in the range $Z_{cr,1s} < Z < 1000$.

n	$np_{1/2}$	$nd_{3/2}$	$nf_{5/2}$	$ng_{7/2}$	$nh_{9/2}$	$ni_{11/2}$
2	−2.309	0	0	0	0	0
3	−1.393	−2.031	0	0	0	0
4	−0.874	−1.669	−1.749	0	0	0
5	−0.601	−1.273	−1.559	−1.586	0	0
6	−0.442	−0.969	−1.318	−1.447	−1.486	0
7	−0.341	−0.754	−1.089	−1.278	−1.358	−1.414
8	−0.272	−0.604	−0.900	−1.107	−1.228	−1.296
9	−0.223	−0.496	−0.750	−0.957	−1.095	−1.186
10	−0.187	−0.416	−0.633	−0.823	−0.968	−1.071
11	−0.160	−0.355	−0.543	−0.714	−0.855	–
12	−0.138	−0.307	−0.472	−0.624	–	–

The shift due to ΔU_{AMM} near $\epsilon = -1$



(a)



(b)

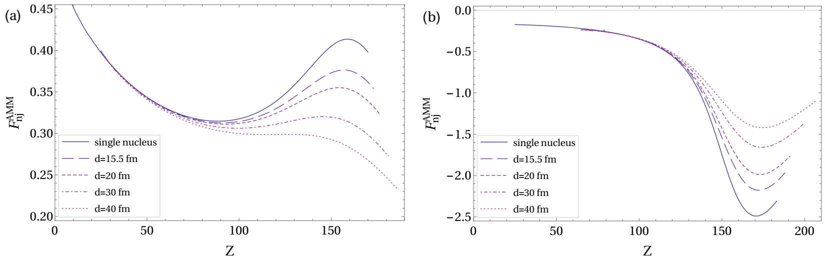
Figure: The shift of the levels due to ΔU_{AMM} in term of $F_{nj}^{AMM}(Z_{cr}\alpha)$ for levels with different nlj near the threshold of the lower continuum in the range $Z_{cr,1s} < Z < 1000$.
The separate trajectories correspond to the series of the levels with different parity and lj (Fig. a — $\kappa < 0\,,\ j = l + {}^{1}\!/_{2}$, Fig. b — $\kappa > 0\,,\ j = l - {}^{1}\!/_{2}$)

The shift due to ΔU_{AMM}

Table: The shift of the levels $1\sigma_g$ and $1\sigma_u$ due to ΔU_{AMM} for the different values $Z_\Sigma = 2Z$ and d. The shift of the levels $1s_{1/2}$ and $2p_{1/2}$ for H-like atom is depicted for comparison (in keV).

level	Z_Σ	H-like	$d = 15.5$ fm	$d = 20$ fm	$d = 30$ fm	$d = 40$ fm
	140	0.495	0.465	0.448	0.413	0.385
	150	0.690	0.635	0.603	0.545	0.500
	160	0.912	0.828	0.779	0.692	0.626
$1\sigma_g$	170	1.118	1.017	0.953	0.840	0.755
$(1s_{1/2})$	173	—	1.068	1.002	0.883	0.793
	176	—	—	1.047	0.924	0.830
	181	—	—	—	0.987	0.888
	186	—	—	—	—	0.942
	150	−0.373	−0.329	−0.304	−0.264	−0.234
	160	−0.632	−0.546	−0.497	−0.417	−0.361
	170	−0.875	−0.763	−0.696	−0.580	−0.498
	180	−1.052	−0.937	−0.861	−0.725	−0.625
$1\sigma_u$	183	−1.090	−0.978	−0.901	−0.763	−0.659
$(2p_{1/2})$	188	—	−1.034	−0.960	−0.819	−0.711
	191	—	—	−0.989	−0.848	−0.738
	195	—	—	—	−0.883	−0.773
	199	—	—	—	−0.912	−0.802
	206	—	—	—	—	−0.843

The shift due to ΔU_{AMM}



(a) (b)

Figure: The function F_{nj}^{AMM} for the contribution from ΔU_{AMM} as a function of total charge Z for the electronic level $1\sigma_g$ a and $1\sigma_u$ b in the diatomic quasi-molecule (for fixed internuclear distances $r = 15.5, 20, 30, 40$ fm).

Conclusions

- The shift of the electronic level caused by ΔU_{AMM} is considered within the nonperturbative approach for H-like atom and diatomic quasi-molecule. There appears a natural assumption that in the overcritical region, the decrease with the growing Z and the size of the system of Coulomb sources R should also take place for the total self-energy contribution to the levels' shift near the threshold of the lower continuum, and so for the other radiative QED effects with virtual photon exchange.
- Thus, one could expect, that the non-linear growth of the vacuum energy for $Z >> Z_{cr}$, where the contribution from the fermionic loop plays the main role, can not be compensated by the contribution from the radiative corrections.

Thanks for your attention!

In the static limit

$$\Delta U^{(0)}_{AMM}(\vec{r}) = \frac{\Delta g}{2}\,\frac{e}{4m}\,\sigma^{\mu\nu}F_{\mu\nu}\,. \tag{16}$$

The one-loop vertex correction[12]

$$\Gamma^{\mu}(q^2) = \gamma^{\mu}F_1(q^2) + \frac{i}{2m}F_2(q^2)\sigma^{\mu\nu}q_{\nu}\,. \tag{17}$$

Hence, the effective potential of the interaction with the external electromagnetic field:

$$\Delta U_{AMM}(\vec{r}) = \frac{e}{2m}\,\sigma^{\mu\nu}\partial_{\mu}\mathcal{A}^{(cl)}_{\nu}(\vec{r}), \tag{18}$$

where

$$\mathcal{A}^{(cl)}_{\mu}(\vec{r}) = \frac{1}{(2\pi)^3}\int d\vec{q}\;e^{i\vec{q}\,\vec{r}}\,\tilde{A}^{(cl)}_{\mu}(\vec{q})F_2(-\vec{q}^{\,2})\,. \tag{19}$$

[12] C. Itzykson and J.-B. Zuber, (McGraw-Hill, 1980).

Dynamical screening of AMM

For a spherically symmetric Coulomb field $A_\mu^{(cl)}(\vec{r}) = \delta_{0,\mu}\Phi(r)$

$$\boxed{\Delta U_{AMM}(r) = -i\,\lambda\,\vec{\gamma}\cdot\vec{\nabla}\left(-\frac{Zc(r)}{r}\right)\,,} \tag{20}$$

where $\lambda = \alpha^2/4\pi m$, $\alpha = e^2/4\pi$, $F_2(0) = \Delta g_{free}/2 \simeq \alpha/2\pi$,

$$c(r) = 2\int_0^\infty qdq\ \sin qr\left(-\frac{1}{Ze}\,\tilde{\Phi}(q)\right)\frac{1}{\pi}\,\frac{F_2(-q^2)}{F_2(0)}\,. \tag{21}$$

It should be noted, that Eq. (20) has the same structure as Eq. (16)

$$\Delta U_{AMM}^{(0)}(\vec{r}) = -i\,\lambda\,\vec{\gamma}\cdot\vec{\nabla}\left(\frac{4\pi\Phi(\vec{r})}{e}\right)\,. \tag{22}$$