Effective Field Theories

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There is a high energy scale M where an effective theory breaks down. Its Lagrangian describes light particles ($m_i \ll M$) and their interactions at $p_i \ll M$ (distances $\gg 1/M$); physics at distances $\lesssim 1/M$ produces local interactions of these light fields.

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The Lagrangian contains all possible operators (allowed by symmetries). Coefficients of operators of dimension $n+4$ contain $1/M^n$. If M is much larger than energies we are interested in, we can retain only renormalizable terms (dimension 4), and, maybe, a power correction or two.

EFT in classical mechanics

- Slow motion – characteristic time $1/\omega$
- Fast motion – characteristic time $1/\Omega$

$$\Omega \gg \omega$$

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Average over fast oscillations
Effective Lagrangian describes slow motion
Poincaré, Krylov, Bogoliubov, Kapitza, . . .

EFT in classical mechanics

$$m\ddot{x} = -\frac{dU}{dx} + F \qquad F = F_0(x)\cos\Omega t$$

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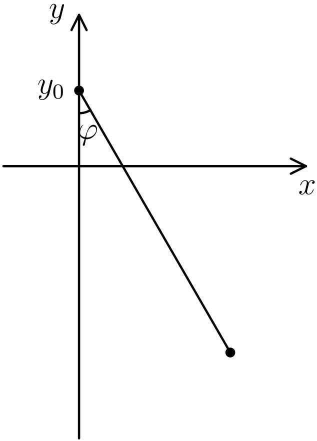
$$U_{\text{eff}} = U + \frac{1}{2m\Omega^2}\overline{F^2}$$

Kapitza pendulum



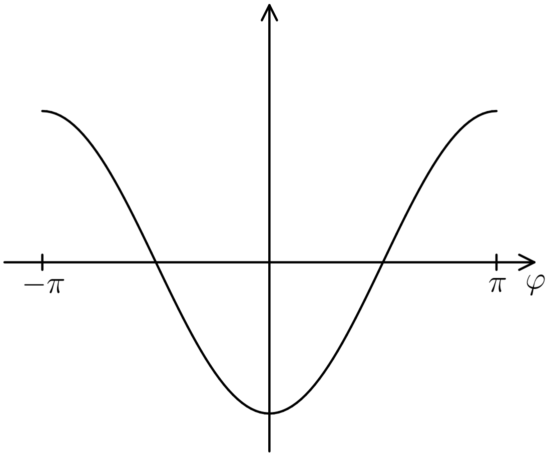
$$y_0 = a\cos\Omega t$$

$$\Omega \gg \sqrt{\frac{g}{l}}$$

$$\lambda = \frac{a^2\Omega^2}{2gl}$$

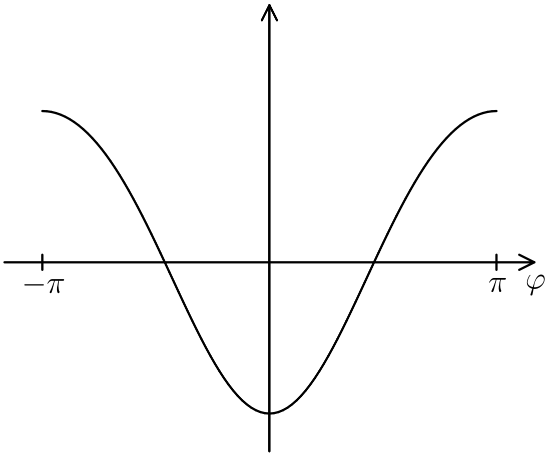
$$U_{\text{eff}} = mgl\left(-\cos\varphi + \frac{\lambda}{2}\sin^2\varphi\right)$$

Kapitza pendulum



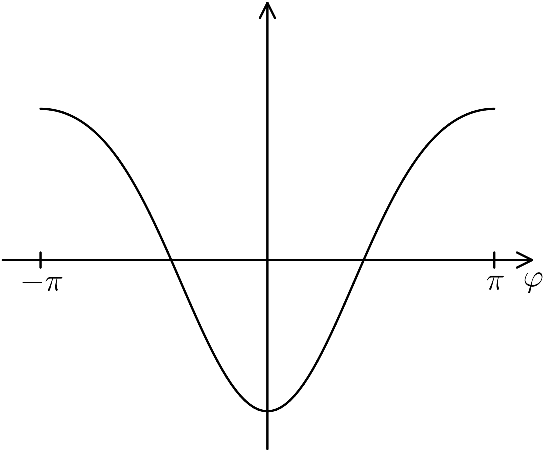
$\lambda = 0$

Kapitza pendulum



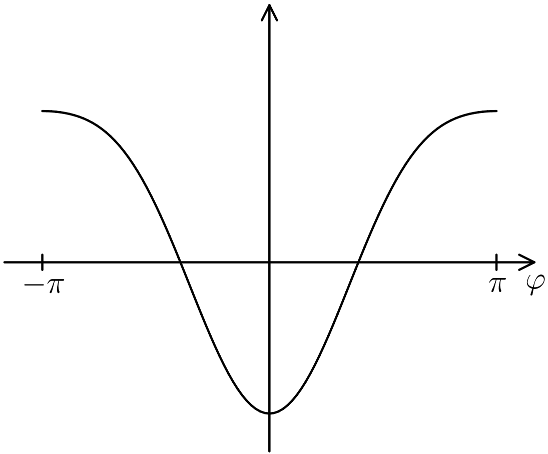
$\lambda = 0.25$

Kapitza pendulum



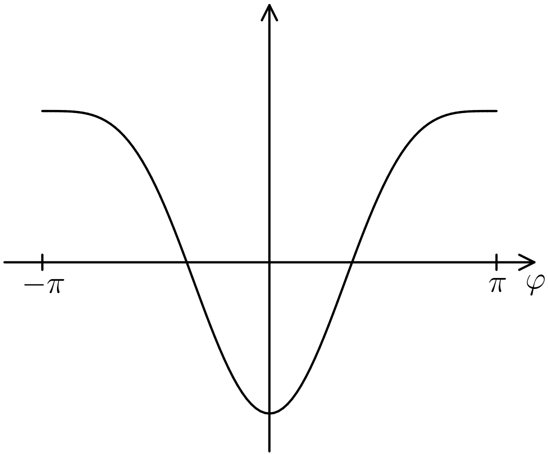
$\lambda = 0.5$

Kapitza pendulum



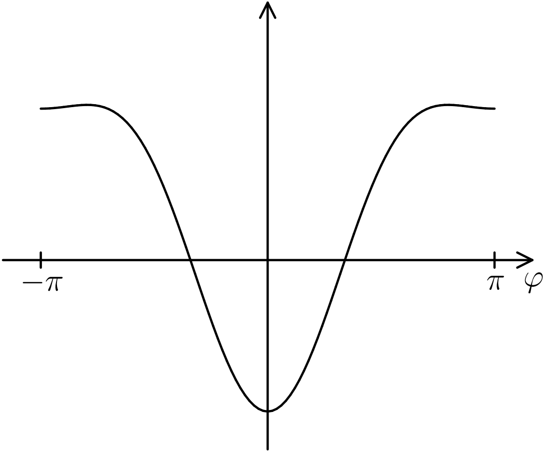
$\lambda = 0.75$

Kapitza pendulum



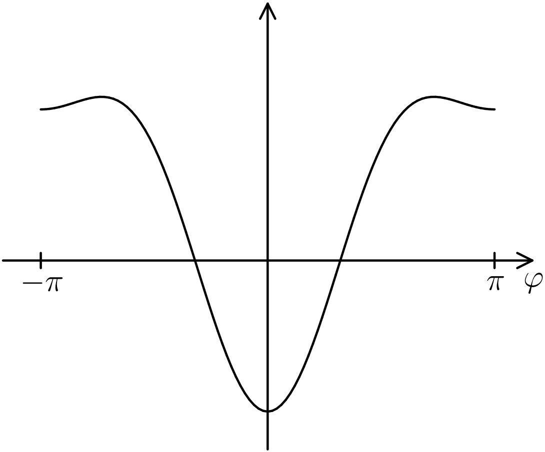
$\lambda = 1$

Kapitza pendulum



$\lambda = 1.25$

Kapitza pendulum



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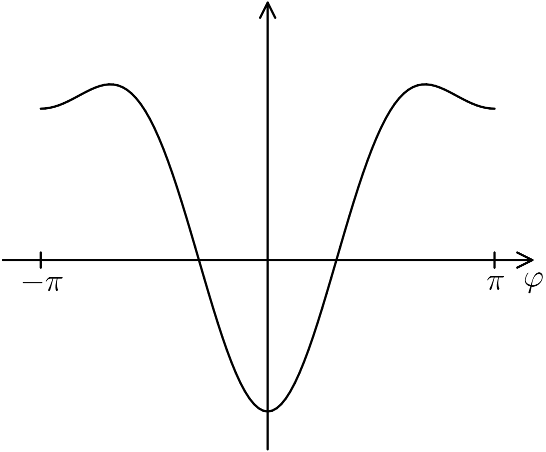

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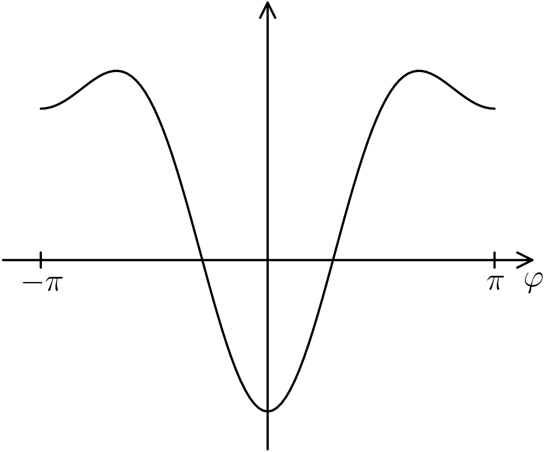
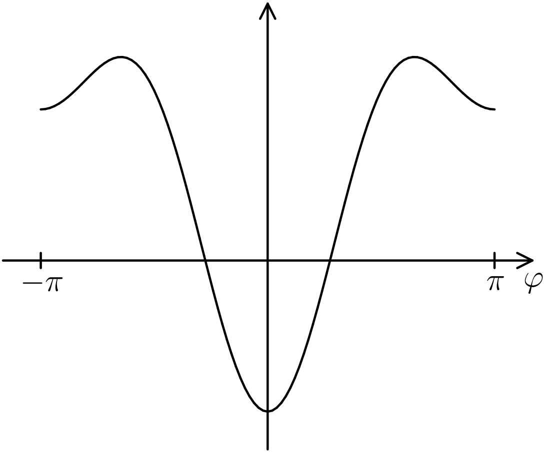
Kapitza pendulum



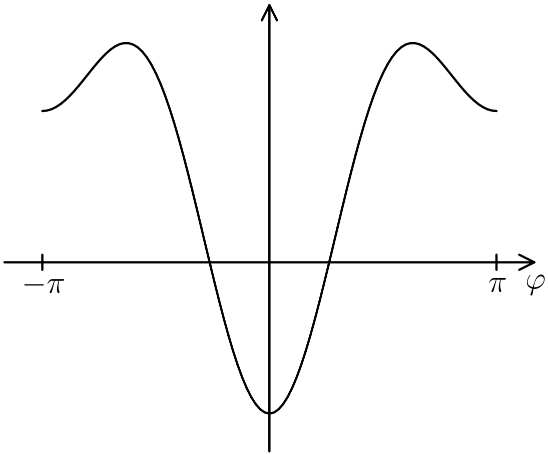
$\lambda = 2$

Kapitza pendulum



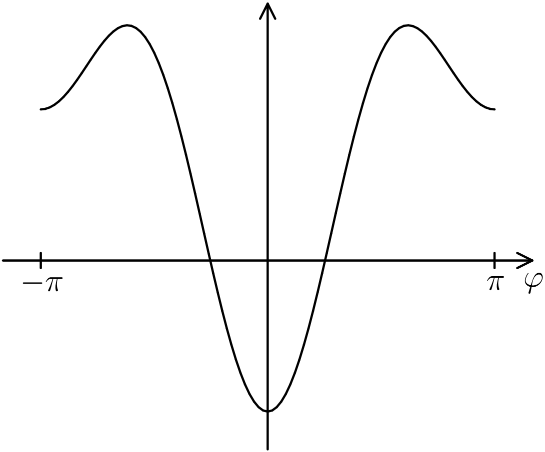
$\lambda = 2.25$

Kapitza pendulum



$\lambda = 2.5$

Kapitza pendulum



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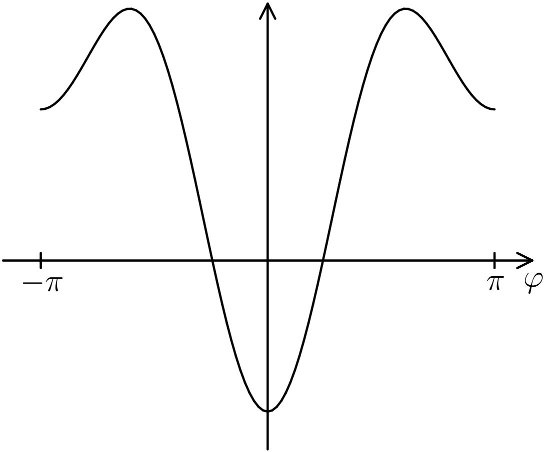

$\lambda = 3$

Photonia

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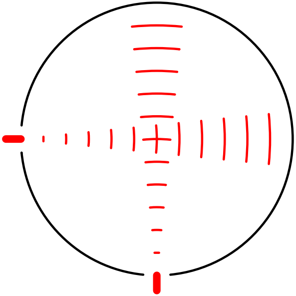
We indignantly refuse to discuss the question "What the experimantalists and their apparata are made of?" as irrelevant.

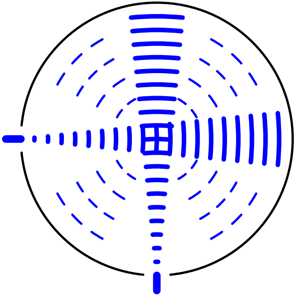
Photonia



Quantum PhotoDynamics (QPD)

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Photonia



Quantum PhotoDynamics (QPD)

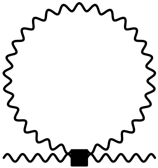
$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1 O_1 + c_2 O_2$$

$$O_1 = (F_{\mu\nu}F^{\mu\nu})^2 \qquad O_2 = F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} \qquad c_{1,2} \sim 1/M^4$$

Photonia

We work at the order $1/M^4$, there can be only 1 4-photon vertex

No corrections to the photon propagator



$= 0$

No renormalization of the photon field

No corrections to the 4-photon vertex
No renormalization of the operators $O_{1,2}$ and the couplings $c_{1,2}$

Qedland

Physicists in the neighboring Qedland are more advanced: in addition to photons, they know electrons and positrons, and investigate their interactions at energies $E \sim M$. They have constructed a nice theory, QED, which describes their experimental results.

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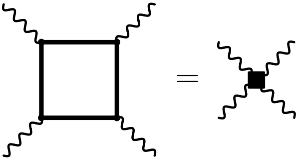
They understand that QPD (constructed in Photonia) is just a low-energy approximation to QED.

Matching

$c_{1,2}$ can be found by matching S-matrix elements



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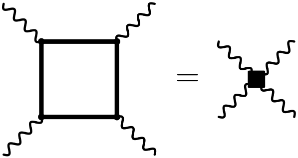

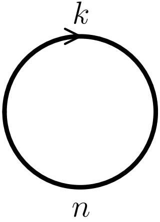

$$= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = M^{d-2n} V(n)$$

$$D = M^2 - k^2 - i0$$

$$V(n) = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}$$

Matching

$$T^{\mu_1\mu_2\nu_1\nu_2} = \frac{e_0^4 M^{-4-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{(d-4)(d-6)}{2880} \times \left(-5T_1^{\mu_1\mu_2\nu_1\nu_2} + 14T_2^{\mu_1\mu_2\nu_1\nu_2}\right)$$

Matching

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$$\times \left(-5T_1^{\mu_1\mu_2\nu_1\nu_2} + 14T_2^{\mu_1\mu_2\nu_1\nu_2}\right)$$

Heisengerg–Euler Lagrangian

$$L_1 = \frac{\pi\alpha^2}{180M^4} (-5O_1 + 14O_2)$$

Wilson line

Physicists in Photonia have some classical (infinitely heavy) charged particles and can manipulate them.

$$S_{\text{int}} = e \int_l dx^\mu A_\mu(x)$$

Feynman path integral: $\exp(iS)$ contains

$$W_l = \exp\left(ie \int_l dx^\mu A_\mu(x) \right)$$

The vacuum-to-vacuum transition amplitude is the vacuum average of the Wilson lines

Potential

Charges e and $-e$ stay at some distance $\vec{r}$ during a large time T: the vacuum amplitude $e^{-iU(\vec{r})T}$

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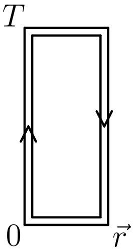
$T \gg r$



$= e^{-iU(\vec{r})T}$

Potential

Charges e and $-e$ stay at some distance $\vec{r}$ during a large time T: the vacuum amplitude $e^{-iU(\vec{r})T}$

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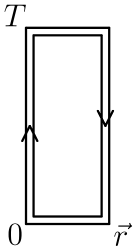


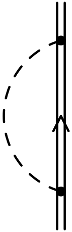
$$= e^{-iU(\vec{r})T}$$

Coulomb gauge

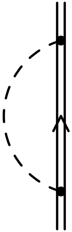
$$D^{00}(q) = -\frac{1}{\vec{q}^2}$$

$$D^{ij}(q) = \frac{1}{q^2 + i0}\left(\delta^{ij} - \frac{q^i q^j}{\vec{q}^2}\right)$$

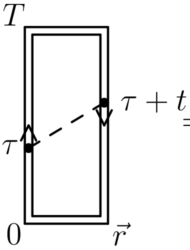
Wilson line

 $= 0$

Wilson line



$$= 0$$



$$= -i\, e^2\, T \int D^{00}(t, \vec{r})\, dt$$

$$= -i\, e^2\, T \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\, D^{00}(0, \vec{q})\, e^{i\, \vec{q}\cdot\vec{r}}$$

Coulomb potential

$$U(\vec{q}) = e^2 D^{00}(0, \vec{q}) = -\frac{e^2}{\vec{q}^2}$$

$$U(\vec{r}) = -\frac{\alpha}{r}$$

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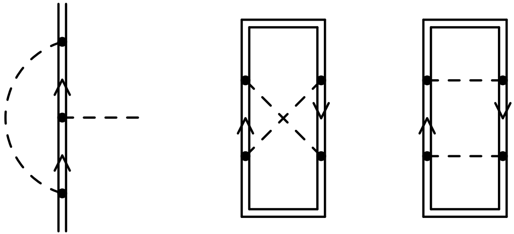

No corrections

Contact interaction

In the presence of sources

$$L_c = c\left(\partial^\mu F_{\lambda\mu}\right)\left(\partial_\nu F^{\lambda\nu}\right)$$

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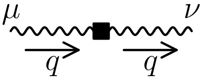

$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

Contact interaction

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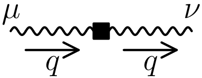
$$L_c = c\left(\partial^\mu F_{\lambda\mu}\right)\left(\partial_\nu F^{\lambda\nu}\right)$$



$$= 2icq^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

$$U_c(\vec{r}) = 2c\delta(\vec{r})$$

Qedland

$$D^{00}(\vec{q}) = -\frac{1}{\vec{q}^2}\frac{1}{1-\Pi(-\vec{q}^2)} \qquad U(\vec{q}) = e_0^2 D^{00}(\vec{q})$$

Qedland

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In macroscopic measurements $\vec{q} \to 0$

$$U(\vec{q}) \to -\frac{e_0^2}{\vec{q}^2}\frac{1}{1-\Pi(0)} = -\frac{e_{\mathrm{os}}^2}{\vec{q}^2}$$

On-shell renormalization scheme

$$e_0 = \left[Z_\alpha^{\mathrm{os}}\right]^{1/2} e_{\mathrm{os}} \qquad A_0 = \left[Z_A^{\mathrm{os}}\right]^{1/2} A_{\mathrm{os}}$$

$$D^{00}(\vec{q}) = Z_A^{\mathrm{os}} D_{\mathrm{os}}^{00}(\vec{q}) \qquad D_{\mathrm{os}}^{00}(\vec{q}) \to -\frac{1}{\vec{q}^2}$$

$$Z_\alpha^{\mathrm{os}} = \left[Z_A^{\mathrm{os}}\right]^{-1} = 1-\Pi(0)$$

$\overline{\text{MS}}$ renormalization scheme

$$e_0 = Z_\alpha^{1/2}(\alpha(\mu))e(\mu) \qquad A_0 = Z_A^{1/2}(\alpha(\mu))A(\mu)$$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$D^{00}(\vec{q}) = Z_A D^{00}(\vec{q};\mu) \qquad D^{00}(\vec{q};\mu) = \text{finite}$$

$$U(\vec{q}) = e^2(\mu) D^{00}(\vec{q};\mu) Z_\alpha Z_A = \text{finite} \qquad Z_\alpha = Z_A^{-1}$$

$$\frac{\alpha(\mu)}{4\pi} = \frac{e^2(\mu)\,\mu^{-2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$

RG equations

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\varepsilon - 2\beta(\alpha(\mu))$$

$$\beta(\alpha(\mu)) = \frac{1}{2}\frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu} \qquad \beta(\alpha) = \beta_0\frac{\alpha}{4\pi} + \beta_1\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\frac{dA(\mu)}{d\log\mu} = -\frac{1}{2}\gamma_A(\alpha(\mu))A(\mu)$$

$$\gamma_A = \frac{d\log Z_A(\alpha(\mu))}{d\log\mu} \qquad \gamma_A(\alpha) = \gamma_{A0}\frac{\alpha}{4\pi} + \gamma_{A1}\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\text{QED} \quad \beta(\alpha) = -\frac{1}{2}\gamma_A(\alpha)$$

Charge decoupling

QPD

$$e'_0 = e'_{\mathrm{os}} = e'(\mu)$$

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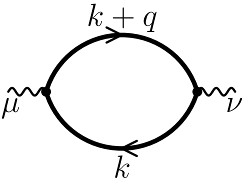
Macroscopically measured charge is the same in QED and QPD

$$e_{\text{os}} = e'_{\text{os}}$$

$$e_0 = \left[\zeta_\alpha^0\right]^{-1/2} e'_0 \qquad \zeta_\alpha^0 = \left[Z_\alpha^{\text{os}}\right]^{-1}$$

$$e(\mu) = \left[\zeta_\alpha(\mu)\right]^{-1/2} e'(\mu) \qquad \zeta_\alpha(\mu) = Z_\alpha \zeta_\alpha^0 = \frac{Z_\alpha}{Z_\alpha^{\text{os}}}$$

1 loop



$$= i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2)$$

$$\Pi(q^2) = -\frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1 - \frac{d-4}{10}\frac{q^2}{M_0^2} + \cdots\right)$$

1 loop

$$[\zeta_\alpha(\mu)]^{-1} = \frac{Z_\alpha^{\rm os}}{Z_\alpha} = Z_\alpha^{-1}\left[1 - \Pi(0)\right] = \text{finite}$$

$$Z_\alpha = 1 - \beta_0 \frac{\alpha}{4\pi\epsilon} + \cdots$$

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$$\beta_0 = -\frac{4}{3}$$

$$[\zeta_\alpha(\mu)]^{-1} = 1 + \frac{4}{3}\left[\left(\frac{\mu}{M}\right)^{2\varepsilon} e^{\gamma\epsilon}\Gamma(1+\varepsilon) - 1\right]\frac{\alpha(\mu)}{4\pi\varepsilon} + \cdots$$

1 loop

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$$\to 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi}L \qquad L = 2\log\frac{\mu}{M}$$

Full theory and effective low-energy theory

QED

$$L = \bar{\Psi}_0 \left(i \not{D}_0 - M_0 \right) \Psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} \left(\partial_\mu A_0^\mu \right)^2$$

QPD

$$L' = -\frac{1}{4} F'_{0\mu\nu} F_0'^{\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0'^\mu \right)^2 + \frac{1}{M_0^4} \sum_i C_i^0 O_i'^0 + \cdots$$

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QPD

$$L' = -\frac{1}{4} F'_{0\mu\nu} F_0^{\prime\mu\nu} - \frac{1}{2a'_0} \left(\partial_\mu A_0^{\prime\mu} \right)^2 + \frac{1}{M_0^4} \sum_i C_i^0 O_i^{\prime 0} + \cdots$$

Bare decoupling

$$A_0 = \left[\zeta_A^0 \right]^{-1/2} A'_0 + \cdots$$

$$a_0 = \left[\zeta_A^0 \right]^{-1} a'_0 \qquad e_0 = \left[\zeta_\alpha^0 \right]^{-1/2} e'_0$$

$\overline{\text{MS}}$ renormalization scheme

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$\overline{\text{MS}}$ renormalization scheme

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$$a_0 = Z_A(\alpha(\mu))\, a(\mu) \qquad e_0 = Z_\alpha^{1/2}(\alpha(\mu))\, e(\mu)$$

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{e^2(\mu)}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}$$

$\overline{\text{MS}}$ renormalization scheme

QED

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$$D_{\mu\nu}(p) = \frac{1}{p^2\left[1 - \Pi(p^2)\right]}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

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$$Z_A^{-1} D_{\mu\nu}(p) = D_{\mu\nu}(p;\mu)$$

QPD

$$Z'_A = 1 \qquad Z'_\alpha = 1$$

$\overline{\text{MS}}$ renormalization scheme

Renormalized decoupling

$$A(\mu) = \zeta_A^{-1/2}(\mu) A'(\mu)$$

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RG equations

$$\frac{d \log \zeta_A(\mu)}{d \log \mu} = \gamma_A(\alpha(\mu)) - \gamma'_A(\alpha'(\mu))$$

$$\frac{d \log \zeta_\alpha(\mu)}{d \log \mu} = 2 \left[\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))\right]$$

On-shell renormalization scheme

QED

$$A_0 = \left[Z_A^{\mathrm{os}}(e_0)\right]^{1/2} A_{\mathrm{os}}$$

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$$\text{At } p \to 0 \qquad D_\perp^{\rm os}(p^2) \to D_\perp^0(p^2) = \frac{1}{p^2}$$

$$Z_A^{\rm os}(e_0) = \frac{1}{1 - \Pi(0)}$$

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QPD

$$Z_A'^{\rm os} = 1 \qquad Z_\alpha'^{\rm os} = 1$$

Photon field decoupling

At $p^2 \to 0$, $D_\perp^{\rm os}(p) = D_\perp'^{\rm os}(p) = D_\perp^0(p)$

$$A^{\rm os} = A'^{\rm os}$$

$$\zeta_A^0(e_0) = \frac{Z_A'^{\rm os}(e_0')}{Z_A^{\rm os}(e_0)} = 1 - \Pi(0)$$

1 loop

$$\zeta_A^0 = \left[\zeta_\alpha^0\right]^{-1} = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

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$$= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \left(\frac{\mu}{M}\right)^{2\varepsilon} \Gamma(1+\varepsilon) e^{\gamma\varepsilon}$$

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$$\zeta_A(\mu) = Z_A \zeta_A^0 = \text{finite}$$

$$Z_A(\alpha) = Z_\alpha^{-1} = 1 - \frac{4}{3}\frac{\alpha}{4\pi\varepsilon}$$

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Mass renormalization

$$M_0 = Z_m(\alpha(\mu))M(\mu) = Z_m^{\text{os}} M_{\text{os}}$$

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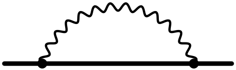
On-shell



$$M(n_1, n_2) = \frac{\Gamma(d - n_1 - 2n_2)\Gamma\left(n_1 + n_2 - \frac{d}{2}\right)}{\Gamma(n_1)\Gamma(d - n_1 - n_2)}$$

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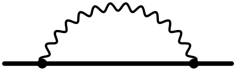

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$$Z_m^{\mathrm{os}} = 1 - \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} + \cdots$$

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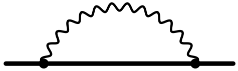

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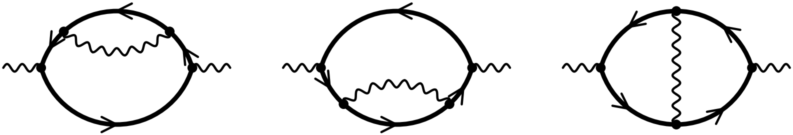
$$Z_m^{\rm os} = 1 - \frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-1}{d-3} + \cdots$$

$\overline{\rm MS}$

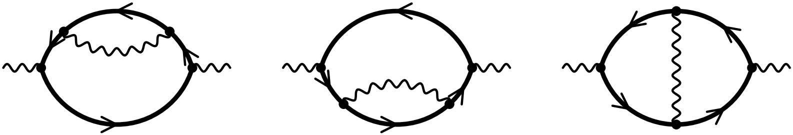
Both $M_{\rm os}$ and $M(\mu)$ are finite at $\varepsilon \to 0$

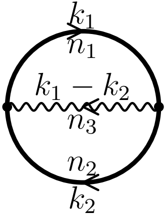
$$Z_m(\alpha) = 1 - 3\frac{\alpha}{4\pi\varepsilon} + \cdots$$

2 loops



2 loops





$$\frac{\Gamma\left(\frac{d}{2} - n_3\right)\Gamma\left(n_1 + n_3 - \frac{d}{2}\right)\Gamma\left(n_2 + n_3 - \frac{d}{2}\right)\Gamma(n_1 + n_2 + n_3 - d)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n_1)\Gamma(n_2)\Gamma(n_1 + n_2 + 2n_3 - d)}$$

A. Vladimirov (1980)

2 loops

$$\zeta_A^0 = \left[\zeta_\alpha^0\right]^{-1} = 1 - \Pi(0) = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

$$+ \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2$$

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$$= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}e^{L\varepsilon}\left(1 + \frac{\pi^2}{12}\varepsilon^2 + \cdots\right) Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu))$$

$$- \varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right)\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 e^{2L\varepsilon}$$

$$L = 2\log\frac{\mu}{M(\mu)}$$

2 loops

$$
\begin{aligned}
\zeta_A^0 = \left[\zeta_\alpha^0\right]^{-1} &= 1 - \Pi(0) = 1 + \frac{4}{3}\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) \\
&+ \frac{2}{3}\frac{(d-4)(5d^2-33d+34)}{d(d-5)}\left(\frac{e_0^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^2 \\
&= 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon} e^{L\varepsilon}\left(1 + \frac{\pi^2}{12}\varepsilon^2 + \cdots\right) Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \\
&- \varepsilon\left(6 - \frac{13}{3}\varepsilon + \cdots\right)\left(\frac{\alpha(\mu)}{4\pi\varepsilon}\right)^2 e^{2L\varepsilon} \\
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$$

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$$\zeta_A = Z_A \zeta_A^0 = \text{finite}$$

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Define $M(\bar{M}) = \bar{M}$, then $L = 0$

$$\zeta_A(\bar{M}) = \zeta_\alpha^{-1}(\bar{M}) = 1 + \frac{\pi^2}{9}\varepsilon\frac{\alpha(\bar{M})}{4\pi} + \frac{13}{3}\left(\frac{\alpha(\bar{M})}{4\pi}\right)^2$$

2 loops

Alternatively use $M_{\rm os}$

$$\frac{M(\mu)}{M_{\rm os}} = 1 - 6\left(\log\frac{\mu}{M_{\rm os}} + \frac{2}{3}\right)\frac{\alpha}{4\pi} \qquad L = 8\frac{\alpha}{4\pi}$$

$$\zeta_A(M_{\rm os}) = \zeta_\alpha^{-1}(M_{\rm os}) = 1 + \frac{\pi^2}{9}\varepsilon\frac{\alpha(M_{\rm os})}{4\pi} + 15\left(\frac{\alpha(M_{\rm os})}{4\pi}\right)^2$$

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For any $\mu = M(1 + \mathcal{O}(\alpha))$, $\zeta_\alpha = 1 + \mathcal{O}(\varepsilon)\alpha + \mathcal{O}(\alpha^2)$

Qedland

Physicists in Qedland suspect that QED is also a low-energy effective theory. They are right: muons exist (let's suppose that pions don't exist). Two ways to search for new physics:

- increase the energy of e^+e^- colliders to produce pairs of new particles
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- increase the energy of e^+e^- colliders to produce pairs of new particles
- performing high-precision experiments at low energies

We were lucky: the scale of new physics in QED is $M \gg m_e$, loops of heavy particles also suppressed by α^n. μ_e agrees with QED without non-renormalizable corrections to a good precision. Physicists expected the same for proton. No luck here.

Power counting

$$\lambda \sim \frac{p_i}{M}$$

$p \sim \lambda$, $x \sim 1/\lambda$, $\partial \sim \lambda$

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Soft photon

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Soft electron

$$<0|T\left\{\psi(x)\bar{\psi}(0)\right\}|0> \sim \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{\not{p} - m},$$

$\psi \sim \lambda^{3/2}$

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$\psi \sim \lambda^{3/2}$

Lagrangian: $F_{\mu\nu}F^{\mu\nu} \sim \lambda^4$, $\bar{\psi} i \not{D} \psi \sim \lambda^4$

Action: ~ 1

Corrections to the Lagrangian $\sim \lambda^6$, to the action $\sim \lambda^2$

We can add higher-dimensional contributions to the Lagrangian, with further unknown coefficients. To any finite order in $1/M$, the number of such coefficients is finite, and the theory has predictive power.

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For example, if we want to work at the order $1/M^4$, then either a single $1/M^4$ (dimension 8) vertex or two $1/M^2$ ones (dimension 6) can occur in a diagram. UV divergences which appear in diagrams with two dimension 6 vertices are compensated by dimension 8 counterterms. So, the theory can be renormalized.

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The usual arguments about non-renormalizability are based on considering diagrams with arbitrarily many vertices of nonrenormalizable interactions (operators of dimensions > 4); this leads to infinitely many free parameters in the theory.

QCD

- QED: effects of decoupling of muon loops are tiny; pion pairs become important at about the same energies as muon pairs
- QCD: decoupling of heavy flavours is fundamental and omnipresent; everybody using QCD with $n_f < 6$ uses an effective field theory (even if he does not know that he speaks prose)

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Full theory QCD with n_l massless flavours
and 1 flavour of mass M

Effective theory QCD with n_l massless flavours

QCD decoupling

$$\alpha_s^{(n_l+1)}(\mu) = \zeta_\alpha^{-1}(\mu)\alpha_s^{(n_l)}(\mu)$$

$$\zeta_\alpha(\bar{M}) = 1 - \left(\frac{13}{3}C_F - \frac{32}{9}C_A\right) T_F \left(\frac{\alpha_s(\bar{M})}{4\pi}\right)^2 + \cdots$$

RG equation

$$\frac{d\log\zeta_\alpha(\mu)}{d\log\mu} - 2\beta^{(n_l+1)}(\alpha_s^{(n_l+1)}(\mu)) + 2\beta^{(n_l)}(\alpha_s^{(n_l)}(\mu)) = 0$$

QCD