Effective Field Theories

Andrey Grozin
A.G.Grozin@inp.nsk.su

Budker Institute of Nuclear Physics
Novosibirsk

Photonia

Photonia has imported a single electron from Qedland, and physicists are studying its interaction with soft photons (both real and virtual)

Photonia

Photonia has imported a single electron from Qedland, and physicists are studying its interaction with soft photons (both real and virtual)

The ground state (“vacuum”) — the electron at rest $\varepsilon = 0$

$$\varepsilon(\vec{p}) = \frac{\vec{p}^{\,2}}{2M}$$

Photonia

Photonia has imported a single electron from Qedland, and physicists are studying its interaction with soft photons (both real and virtual)
The ground state ("vacuum") — the electron at rest $\varepsilon = 0$

$$\varepsilon(\vec{p}) = \frac{\vec{p}^{\,2}}{2M}$$

The leading-order mass shell

$$\varepsilon(\vec{p}) = 0$$

Photonia

Photonia has imported a single electron from Qedland, and physicists are studying its interaction with soft photons (both real and virtual)
The ground state ("vacuum") — the electron at rest $\varepsilon = 0$

$$\varepsilon(\vec{p}) = \frac{\vec{p}^{\,2}}{2M}$$

The leading-order mass shell

$$\varepsilon(\vec{p}) = 0$$

Velocity

$$\vec{v} = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M} \to 0$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

$$i\partial_0 h = 0$$

Lagrangian

$$L = h^+ i\partial_0 h$$

equation of motion

$$i\partial_0 h = 0$$

Charge $-e$

$$\varepsilon = -eA_0$$

Lagrangian

$$L = h^+ i\partial_0 h$$

equation of motion

$$i\partial_0 h = 0$$

Charge $-e$

$$\varepsilon = -eA_0$$

Equation of motion

$$iD_0 h = 0$$

$$D_\mu = \partial_\mu - ieA_\mu$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

$$i\partial_0 h = 0$$

Charge $-e$

$$\varepsilon = -eA_0$$

Equation of motion

$$iD_0 h = 0$$
$$D_\mu = \partial_\mu - ieA_\mu$$

Lagrangian

$$L = h^{+} iD_0 h$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

$$i\partial_0 h = 0$$

Charge $-e$

$$\varepsilon = -eA_0$$

Equation of motion

$$iD_0 h = 0$$

$$D_\mu = \partial_\mu - ieA_\mu$$

Lagrangian

$$L = h^{+} iD_0 h$$

Not Lorentz-invariant

Lagrangian

+ Lagrangian of the photon field

$$\partial_\mu F^{\mu\nu} = j^\nu$$
$$j^0 = -eh^+h$$

The electron produces the Coulomb field

Spin symmetry

At the leading order in $1/M$, the electron spin does not interact with electromagnetic field

We can rotate it without affecting physics

In addition to the $U(1)$ symmetry $h \to e^{i\alpha} h$, also the $SU(2)$ spin symmetry

$$h \to U h$$

Spin symmetry

At the leading order in $1/M$, the electron spin does not interact with electromagnetic field

We can rotate it without affecting physics

In addition to the $U(1)$ symmetry $h \to e^{i\alpha} h$, also the $SU(2)$ spin symmetry

$$h \to U h$$

The electron magnetic moment $\vec{\mu} = \mu \vec{\sigma}$ interacts with magnetic field: $-\vec{\mu} \cdot \vec{B}$

By dimensionality $\mu \sim e/M$

(Bohr magneton $e/(2M)$ up to radiative corrections)

$$L_m = -\frac{e}{2M} h^+ \vec{B} \cdot \vec{\sigma} h$$

Violates the $SU(2)$ spin symmetry at the $1/M$ level

Spin-flavour symmetry

n_f flavours of heavy fermions

$$L = \sum_{i=1}^{n_f} h_i^+ iD_0 h_i$$

$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction

Spin-flavour symmetry

n_f flavours of heavy fermions

$$L = \sum_{i=1}^{n_f} h_i^+ i D_0 h_i$$

$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction
At the leading order in $1/M$, not only the spin direction but also its magnitude is irrelevant
We can, for example, switch the electron spin off:

$$L = \varphi^* i D_0 \varphi$$

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry
The superflavour $SU(3)$ symmetry:

- $\varphi \to e^{2i\alpha}\varphi$, $h \to e^{-i\alpha} h$
- $SU(2)$ spin rorations
- $$\delta \begin{pmatrix} \varphi \\ h \end{pmatrix} = i \begin{pmatrix} 0 & \varepsilon^+ \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ h \end{pmatrix}$$

  ε — an infinitesimal spinor

Broken at $1/M$

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry
The superflavour $SU(3)$ symmetry:

- $\varphi \to e^{2i\alpha}\varphi$, $h \to e^{-i\alpha} h$
- $SU(2)$ spin rorations
- $$\delta \begin{pmatrix} \varphi \\ h \end{pmatrix} = i \begin{pmatrix} 0 & \varepsilon^+ \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ h \end{pmatrix}$$

  ε — an infinitesimal spinor

Broken at $1/M$
We can consider, e. g., spins $\frac{1}{2}$ and 1
$SU(5)$ superflavour symmetry

Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

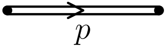
Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

The momentum-space free electron propagator



$$= i S_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

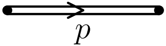
Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

The momentum-space free electron propagator

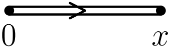


$$= iS_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

The coordinate-space propagator



$$= iS_0(x) \qquad S_0(x) = S_0(x_0)\delta(\vec{x}\,) \qquad S_0(t) = -i\theta(t)$$

Static electron does not move

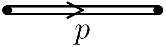
Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

The momentum-space free electron propagator

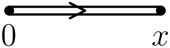


$$= i S_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

The coordinate-space propagator



$$= i S_0(x) \qquad S_0(x) = S_0(x_0)\delta(\vec{x}\,) \qquad S_0(t) = -i\theta(t)$$

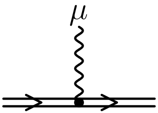
Static electron does not move

Solving the equation

$$i\partial_0 S_0(x) = \delta(x)$$

Feynman rules

Vertex

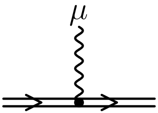


$$= ie_0 v^\mu$$

$$v^\mu = (1, \vec{0}\,)$$

Feynman rules

Vertex



$$= ie_0 v^\mu$$

$$v^\mu = (1, \vec{0}\,)$$

The static field φ_0 (or h_0) describes only particles, there are no antiparticles.
No loops formed by static-electron propagators.
The electron propagates only forward in time;
the product of θ functions for a loop vanishes.
In the momentum space: all poles of the propagators are in the lower p_0 half-plane;
closing the integration contour upwards, we get 0.

Residual momentum

The full-theory energy M is the HEET zero level

$$E = M + \varepsilon$$

ε — the residual energy

Residual momentum

The full-theory energy M is the HEET zero level

$$E = M + \varepsilon$$

ε — the residual energy

$$P^\mu = M v^\mu + p^\mu$$

- P^μ — 4-momentum of some state (containing a single electron) in the full theory
- p^μ — its momentum in HEET (the residual momentum)

v^μ — 4-velocity of a reference frame in which the electron always stays approximately at rest

Reparametrization invariance

HEET is applicable if there exists such v that

$$p^\mu \ll M \qquad p^\mu_{\gamma i} \ll M$$

Reparametrization invariance

HEET is applicable if there exists such v that

$$p^\mu \ll M \qquad p^\mu_{\gamma i} \ll M$$

This condition does not fix v uniquely: $v \to v + \delta v$, $\delta v \sim p/M$.

Effective theories corresponding to different choices of v must produce identical physical predictions: reparametrization invariance.

Relations between quantities at different orders in $1/M$.

Relativistic notation

Lagrangian

$$L = \varphi_0^* i v \cdot D \varphi_0 + (\text{light fields})$$

Free propagator

$$S_0(p) = \frac{1}{p \cdot v + i0}$$

Mass shell

$$p \cdot v = 0$$

Spin $\frac{1}{2}$

4-component spinor field

$$\not{v} h_v = h_v$$

Lagrangian

$$L = \bar{h}_{v0} i v \cdot D h_{v0} + (\text{light fields})$$

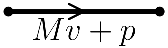
Propagator

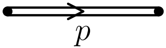
$$S_0(p) = \frac{1 + \not{v}}{2} \frac{1}{p \cdot v + i0}$$

Vertex $i e_0 v^\mu$

Qedland

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$

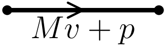


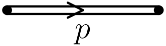
=


$+\, \mathcal{O}\left(\frac{p}{M}\right)$

Qedland

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$



$=$


$+\, \mathcal{O}\left(\frac{p}{M}\right)$

$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$$

We may insert the projectors $(1+\not{v})/2$ before $u(P_i)$ and after $\bar{u}(P_i)$, too, because

$$\not{v}u(Mv+p) = u(Mv+p) + \mathcal{O}\left(\frac{p}{M}\right)$$

Qedland

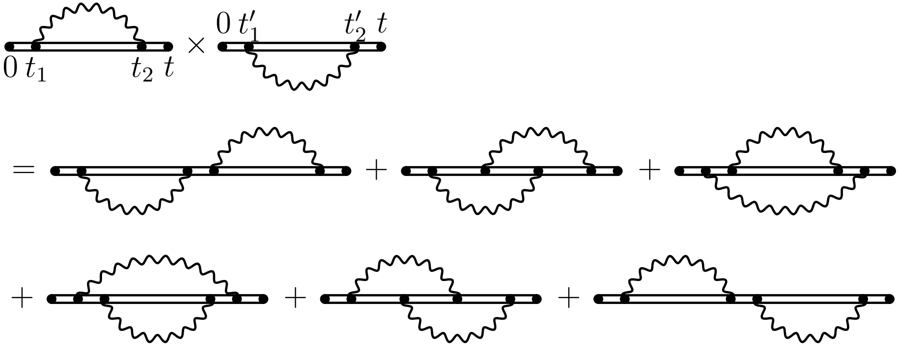
We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules.

Qedland

We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules.
We have thus proved that at the tree level any QED diagram is equal to the corresponding HEET diagram up to $\mathcal{O}(p/m)$ corrections. This is not true at loops, because loop momenta can be arbitrarily large. Renormalization properties of HEET (anomalous dimensions, etc.) differ from those in QED.

Exponentiation

1-loop correction to x-space propagator, multiply by itself
Integral in t_1, t_2, t'_1, t'_2 with $0 < t_1 < t_2 < t$, $0 < t'_1 < t'_2 < t$
Ordering of primed and non-primed t's can be arbitrary
6 regions corresponding to 6 diagrams



Exponentiation

This is 2× the 2-loop correction
1-loop correction cubed is 3!× the 3-loop correction, ...

$$S(t) = S_0(t)\,\exp w_1$$

$$w_1 = -\frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)\left(1+\frac{2}{d-3}-a_0\right)$$

In the d-dimensional Yennie gauge the exact propagator is free

Exponentiation

No corrections to the photon propagator $Z_A = 1$: $a = a_0$, $e = e_0$

$$Z_h = \exp\left[-(a-3)\frac{\alpha}{4\pi\varepsilon}\right]$$

$$\gamma_h = 2(a-3)\frac{\alpha}{4\pi}$$

exactly!

Operators

Full QED operators — series in $1/M$
via HEET operators

$$O(\mu) = C(\mu)\tilde{O}(\mu) + \frac{1}{2M}\sum_i B_i(\mu)\tilde{O}_i(\mu) + \cdots$$

Matching on-shell matrix elements

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

On-shell matrix elements

$$<0|\psi_0|e(p)> = \left(Z_\psi^{\rm os}\right)^{1/2} u(p)$$
$$<0|h_{v0}|e(p)> = \left(Z_h^{\rm os}\right)^{1/2} u_v(k)$$

Bare decoupling $Z_h^{\rm os} = 1$

$$z_0 = \frac{Z_\psi^{\rm os}(e_0^{(1)})}{Z_h^{\rm os}(e_0^{(0)})}$$

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

On-shell matrix elements

$$<0|\psi_0|e(p)> = \left(Z_\psi^{\mathrm{os}}\right)^{1/2} u(p)$$
$$<0|h_{v0}|e(p)> = (Z_h^{\mathrm{os}})^{1/2}\, u_v(k)$$

Bare decoupling $Z_h^{\mathrm{os}} = 1$

$$z_0 = \frac{Z_\psi^{\mathrm{os}}(e_0^{(1)})}{Z_h^{\mathrm{os}}(e_0^{(0)})}$$

Renormalized decoupling

$$z(\mu) = \frac{Z_h(\alpha^{(0)}(\mu), a^{(0)}(\mu))}{Z_\psi(\alpha_s^{(1)}(\mu), a^{(1)}(\mu))} z_0$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\; e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

$$\Delta(k) = \frac{a_0}{(k^2)^2} \qquad \tilde{\Delta}(0) = 0 \text{ in dim. reg.}$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

$$\Delta(k) = \frac{a_0}{(k^2)^2} \qquad \tilde{\Delta}(0) = 0 \text{ in dim. reg.}$$

Landau, Khalatnikov (1955)
Fradkin (1955)
Bogoliubov, Shirkov (1957)
Zumino (1960)

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon)$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon)$$

$$= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$
\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon) \\
&= \log Z_\psi + \sigma_r
\end{aligned}
$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$
\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon) \\
&= \log Z_\psi + \sigma_r \\
\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}
\end{aligned}
$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^\varepsilon \Gamma(-\varepsilon)$$

$$= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^\varepsilon e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)$$

$$= \log Z_\psi + \sigma_r$$

$$\log Z_\psi(\alpha, a) = \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}$$

$$\gamma_\psi(\alpha, a) = 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)$$

$d \log(a(\mu)\alpha(\mu)) / d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$
\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^{\varepsilon} \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^{\varepsilon} e^{\gamma_E \varepsilon} \Gamma(-\varepsilon) \\
&= \log Z_\psi + \sigma_r \\
\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon} \\
\gamma_\psi(\alpha, a) &= 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)
\end{aligned}
$$

$d \log(a(\mu)\alpha(\mu)) / d \log \mu = -2\varepsilon$ exactly
$\gamma_L(\alpha)$ starts from α^2
known to 5 loops

Gauge independence of $z(\mu)$ in QED

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant
- $\log Z_h = (3 - a^{(0)}) \dfrac{\alpha^{(0)}}{4\pi\varepsilon}$
  $\alpha^{(0)} = \alpha_{\text{os}} \approx 1/137$
- $\log Z_\psi = -a^{(1)}(\mu) \dfrac{\alpha^{(1)}(\mu)}{4\pi\varepsilon} + (\text{gauge invariant})$
- Decoupling $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
  Gauge dependence cancels in $\log(\tilde{Z}_\psi / Z_\psi)$

Result

$$z(M_{\text{os}}) = 1 - \frac{\alpha}{\pi} + \left(\pi^2 \log 2 - \frac{3}{2}\zeta_3 - \frac{55}{48}\pi^2 + \frac{5957}{1152}\right)\left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$P = (M + \omega)v \qquad \Sigma(P) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$P = (M + \omega)v \qquad \Sigma(P) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

$$\begin{aligned} S(P) &= \frac{1}{\not{p} - M_0 - \Sigma(p)} \\ &= \frac{1}{[M + \omega - \Sigma_1(\omega)]\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)} \end{aligned}$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$P = (M + \omega)v \qquad \Sigma(P) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

$$\begin{aligned} S(P) &= \frac{1}{\not{p} - M_0 - \Sigma(p)} \\ &= \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)} \end{aligned}$$

The denomunator

$$[M + \omega - \Sigma_1(\omega)]^2 - [M - \delta M + \Sigma_0(\omega) - \Sigma_1(\omega)]^2$$

should vanish at $\omega = 0$:

$$\delta M = \Sigma_0(0)$$

Electron propagator near the mass shell

$$S(P) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

Electron propagator near the mass shell

$$S(P) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M-\Sigma_1(0)+\omega-\Sigma_1(\omega)+\Sigma_1(0)]^2$$
$$-\,[M-\Sigma_1(0)+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)+\Sigma_1(0)]^2$$
$$\approx 2\,(M-\Sigma_1(0))\,[\omega-\Sigma_0(\omega)+\Sigma_0(0)]$$

Electron propagator near the mass shell

$$S(P) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

Electron propagator near the mass shell

$$S(P) = \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M+\omega-\Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

$$S(P) \approx \frac{\not{v}+1}{2}\,\frac{1}{\omega - \Sigma_0(\omega) + \Sigma_0(0)}$$

Heavy–heavy current

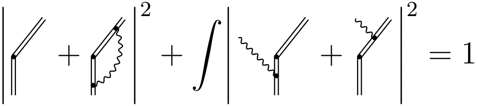
$$J_0 = \varphi^*_{v'0}\varphi_{v0} = Z_J(\alpha(\mu))J(\mu) \qquad \cosh\varphi = v\cdot v'$$

$$\Gamma(\vartheta) = \frac{d\log Z_J}{d\log\mu}$$

Exponentiation: 1-loop formula is exact

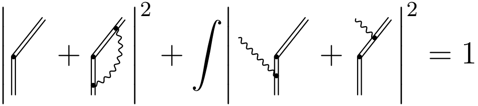
Cusp anomalous dimension

Unitarity



Cusp anomalous dimension

Unitarity



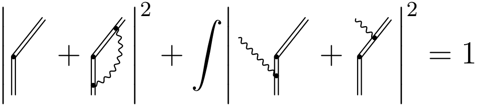
Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\, d\omega$$

$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega}$$

Cusp anomalous dimension

Unitarity



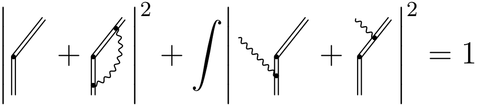
Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\, d\omega$$

$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}}$$

Cusp anomalous dimension

Unitarity



Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth\vartheta - 1)\, d\omega$$

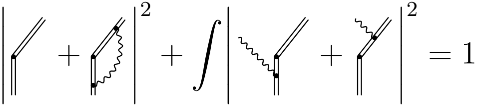
$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth\vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}}$$

$$F = 1 - \frac{1}{2}\int_\lambda^\infty \frac{e^2}{2\pi^2}(\vartheta \coth\vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}} = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta \coth\vartheta - 1)$$

$$\Gamma = 4\frac{\alpha}{4\pi}(\vartheta \coth\vartheta - 1)$$

Cusp anomalous dimension

Unitarity



Classical electrodynamics

$$dE = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\, d\omega$$

$$dw = \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}}$$

$$F = 1 - \frac{1}{2}\int_\lambda^\infty \frac{e^2}{2\pi^2}(\vartheta \coth \vartheta - 1)\frac{d\omega}{\omega^{1+2\varepsilon}} = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta \coth \vartheta - 1)$$

$$\Gamma = 4\frac{\alpha}{4\pi}(\vartheta \coth \vartheta - 1)$$

The Guiness Book of Records: the anomalous dimension known for the longest time ($>$ 100 years)

Limiting cases

$\vartheta \ll 1$ Series in ϑ^2

$$\Gamma(\vartheta) = \frac{\alpha}{3\pi}\vartheta^2 + \mathcal{O}(\vartheta^4)$$

$\vartheta \gg 1$ $\Gamma(\vartheta) = \Gamma_l \vartheta + \mathcal{O}(\vartheta^0)$

$$\Gamma_l = \frac{\alpha}{\pi}$$

Limiting cases

$\vartheta \ll 1$ Series in ϑ^2

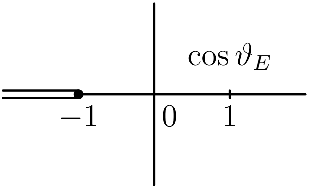
$$\Gamma(\vartheta) = \frac{\alpha}{3\pi}\vartheta^2 + \mathcal{O}(\vartheta^4)$$

$\vartheta \gg 1$ $\Gamma(\vartheta) = \Gamma_l \vartheta + \mathcal{O}(\vartheta^0)$

$$\Gamma_l = \frac{\alpha}{\pi}$$

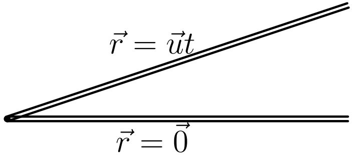
Euclidean space $\cos\vartheta_E = v \cdot v'$

$$\Gamma(\vartheta_E) = 4\frac{\alpha}{4\pi}(\vartheta_E \cot\vartheta_E - 1)$$



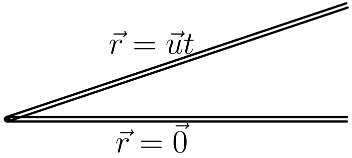
$\vartheta_E \to \pi$

Heavy-particle pair production



$\vartheta_E \to \pi$

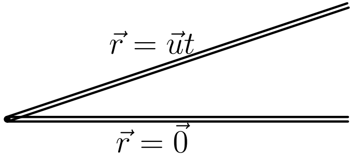
Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r}$$

$\vartheta_E \to \pi$

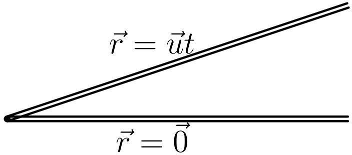
Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r^{1-2\varepsilon}}$$

$\vartheta_E \to \pi$

Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r^{1-2\varepsilon}}$$

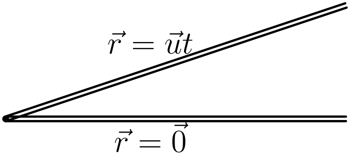
$$W = \exp\left[-i\int_0^T dt\,U(ut)\right] = \exp\left[i\frac{e^2}{4\pi}\frac{T^{2\varepsilon}}{2\varepsilon u^{1-2\varepsilon}}\right]$$

$$Z_J = \exp\left[i\frac{\alpha}{2\varepsilon u}\right]$$

$$\Gamma = -i\frac{\alpha}{u}$$

$\vartheta_E \to \pi$

Heavy-particle pair production



$$U(r) = -\frac{e^2}{4\pi}\frac{1}{r^{1-2\varepsilon}}$$

$$W = \exp\left[-i\int_0^T dt\, U(ut)\right] = \exp\left[i\frac{e^2}{4\pi}\frac{T^{2\varepsilon}}{2\varepsilon u^{1-2\varepsilon}}\right]$$

$$Z_J = \exp\left[i\frac{\alpha}{2\varepsilon u}\right]$$

$$\Gamma = -i\frac{\alpha}{u} \qquad u \Rightarrow i\delta \qquad \Gamma(\pi - \delta) = -\frac{\alpha}{\delta}$$

HQET

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 + \frac{C_m^0}{2M} O_m^0 + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$L_0 = h_0^+ i D_0 h_0$$

$$O_k^0 = h_0^+ \vec{D}^2 h_0 = Z_k(\alpha_s(\mu)) O_k(\mu)$$

$$O_m^0 = g_0 h_0^+ \vec{B}^a \cdot \vec{\sigma} t_a h_0 = Z_m(\alpha_s(\mu)) O_m(\mu)$$

HQET

$$L = L_0 + \frac{C_k^0}{2M} O_k^0 + \frac{C_m^0}{2M} O_m^0 + \mathcal{O}\left(\frac{1}{M^2}\right)$$

$$L_0 = h_0^+ i D_0 h_0$$

$$O_k^0 = h_0^+ \vec{D}^2 h_0 = Z_k(\alpha_s(\mu)) O_k(\mu)$$

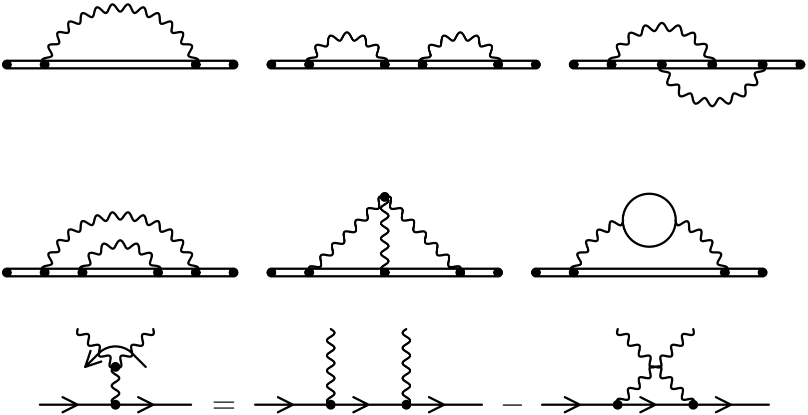
$$O_m^0 = g_0 h_0^+ \vec{B}^a \cdot \vec{\sigma} t_a h_0 = Z_m(\alpha_s(\mu)) O_m(\mu)$$

Reparametrization invariance

$$Z_k = 1 \qquad O_k = O_k^0$$

$$C_k^0 = 1 \qquad C_k(\mu) = Z_k^{-1} C_k^0 = 1$$

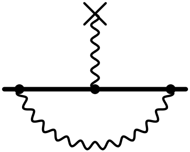
Propagator

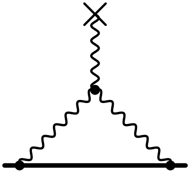


$$S(t) = S_0(t)\exp\Bigg[C_F \frac{g_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon} S$$

$$+ C_F \frac{g_0^4}{(4\pi)^d}\left(\frac{it}{2}\right)^{4\varepsilon}(C_A S_A + T_F n_l S_l)\Bigg]$$

Chromomagnetic interaction

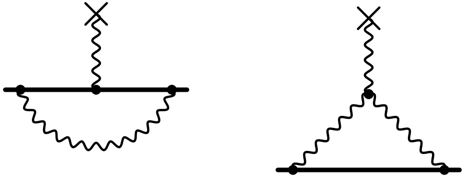




$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)}$$
$$\times \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

IR divergent (unlike QED)

Chromomagnetic interaction



$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)}$$
$$\times \left[2(d-4)(d-5)C_F - (d^2-8d+14)C_A\right]$$

IR divergent (unlike QED)

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A (17C_A - 13T_F n_l) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$
$$C_m(\mu) = 1 + 2\left(-C_A \log\frac{M}{\mu} + C_F + C_A\right) \frac{\alpha_s(M)}{4\pi} + \cdots$$

Mass splitting

$$M_{B^*}^2 - M_B^2 = \frac{4}{3} C_m^{(4)}(\mu) \mu_{G(4)}^2(\mu) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{M_b}\right)$$

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \left(\frac{\alpha_s^{(4)}(M_c)}{\alpha_s^{(4)}(M_b)}\right)^{-9/25} \left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\mathrm{QCD}}}{M_{b,c}}\right)\right]$$

In the past

Only renormalizable theories were considered well-defined: they contain a finite number of parameters, which can be extracted from a finite number of experimental results and used to predict an infinite number of other potential measurements. Non-renormalizable theories were rejected because their renormalization at all orders in non-renormalizable interactions involve infinitely many parameters, so that such a theory has no predictive power. This principle is absolutely correct, if we are impudent enough to pretend that our theory describes the Nature up to arbitrarily high energies (or arbitrarily small distances).

At present

We accept the fact that our theories only describe the Nature at sufficiently low energies (or sufficiently large distances). They are effective low-energy theories. Such theories contain all operators (allowed by the relevant symmetries) in their Lagrangians. They are necessarily non-renormalizable. This does not prevent us from obtaining definite predictions at any fixed order in the expansion in E/M, where E is the characteristic energy and M is the scale of new physics. Only if we are lucky and M is many orders of magnitude larger than the energies we are interested in, we can neglect higher-dimensional operators in the Lagrangian and work with a renormalizable theory.

Conclusion

Practically all physicists believe that the Standard Model is also a low-energy effective theory. But we don't know what is a more fundamental theory whose low-energy approximation is the Standard Model. Maybe, it is some supersymmetric theory (with broken supersymmetry); maybe, it is not a field theory, but a theory of extended objects (superstrings, branes); maybe, this more fundamental theory lives in a higher-dimensional space, with some dimensions compactified; or maybe it is something we cannot imagine at present.

Conclusion

The only model-independent method to search for physics beyond the Standard Model (without inventing arbitrary scenarios) is to use SMEFT: add operators having higher dimensions (5, 6) to the Standard Model Lagrangian with unknown coefficients, and to try to measure these coefficients experimantally. As soon as some coefficient(s) is proved to be non-zero, we know that the Standard Model is not exact. After measuring sufficiently many such coefficients we can start inventing a more fundamental theory which explains them.